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Inclined periodic homoclinic breather and rogue waves for the (1+1)-dimensional Boussinesq equation

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Abstract. A new method, homoclinic (heteroclinic) breather limit method (HBLM), for seeking rogue wave solution to nonlinear evolution equation (NEE) is proposed. (1+1)-dimensional Boussinesq equation is used as an example to illustrate the effectiveness of the suggested method. Rational homoclinic wave solution, a new family of two-wave solution, is obtained by inclined periodic homoclinic breather wave solution and is just a rogue wave solution. This result shows that rogue wave originates by the extreme behaviour of homoclinic breather wave in (1+1)-dimensional nonlinear wave fields.

Keywords. Homoclinic breather limit; rational homoclinic wave; rogue wave; Boussinesq equation.

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1. Introduction

In recent years, rogue waves, a special type of solitary waves, has triggered much interest in various physical branches. Rogue wave is a type of wave that seems abnormal, which is first observed in the deep ocean. Its amplitude is always two to three times higher than its surrounding waves and generally forms in a short time so that people think that it comes from nowhere. Rogue waves have been the subject of intensive research in oceanography [1,2], optical fibres [3–6], superfluids [7], Bose–Einstein condensates [8], financial markets [9] and other related fields. The first-order rational solution of the self-focussing nonlinear Schödinger equation (NLS) was first found by Peregrine to describe the rogue wave phenomenon [10]. Recently, rogue wave solutions in complex systems such as Hirota equation, Sasa–Satsuma equation, Davey–Stewartson equation, coupled Gross–Pitaevskii equation, coupled NLS Maxwell–Bloch equation, coupled Schrödinger–Boussinesq equation and so on, were obtained [11–17].

In this work, we propose a homoclinic (heteroclinic) breather limit method for seeking rogue wave solution to NEE. We consider a general nonlinear partial differential equation in the form

$$P(u, u_t, u_x, \ldots) = 0,$$

where P is a polynomial in its arguments. To determine $u(t, x)$ explicitly, we take the following four steps:

Step 1

By Painlevé analysis, a transformation

$$u = T(f),$$

is made for some new and unknown function f.

Step 2

By using the transformation in Step 1, original equation can be converted into Hirota's bilinear form

$$G(D_t, D_x; f) = 0,$$

where the D-operator [18] is defined by

$$D_t^n D_x^m f(x,t) \cdot g(x,t) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \times [f(x,t) g(x', t')]|_{x'=x, t'=t}.$$

Step 3

Solve the above equation to get homoclinic (heteroclinic) breather wave solution by using extended homoclinic test approach (EHTA) [19].

Step 4

Let the period of the periodic wave go to infinity in homoclinic (heteroclinic) breather wave solution. We shall obtain a rational homoclinic (heteroclinic) wave and this wave is just a rogue wave.

Now we consider (1+1)-dimensional Boussinesq equation. It is of considerable importance in both physics and mathematics. It is well known that many attempts have been made to study the nonlinear evolution equations governing wave motions in media with damping mechanism. One of the typical examples is the Boussinesq (Bq) equation

$$u_{tt} - u_{xx} - 3(u^2)_{xx} + \sigma u_{xxxx} = 0, \quad -\infty < x < \infty, \quad t \in R, \tag{1}$$

where $\sigma > 0$ is called the 'good' Bq equation and $\sigma < 0$ the 'bad' Bq equation. Bq equation can be used to describe many real-world processes such as the propagation of long waves in shallow water and the oscillations of nonlinear elastic strings. The 'good'

Bq equation describes the two-dimensional irrotational flow of an inviscid liquid in a uniform rectangular channel. There are known results due to local well-posed, global existence and blow-up of some solutions [20–22]. The 'bad' Bq equation is used to describe two-dimensional flow of shallow water waves having small amplitudes. There is a dense connection to the so-called Fermi–Pasta–Ulam (FPU) problem. The existence of Lax pair, Backlund transformation and some soliton-type solutions were also discussed [22]. Recently, Dai and his group further investigated the Bq equation [23–26] by Hirota technique and its exact homoclinic orbit, homoclinic breather and bifurcation of solution were obtained.

This work focusses on rational homoclinic wave solution of (1+1)-D Boussinesq equation. Applying HBLM to (1+1)-D 'bad' Bq equation, we obtain homoclinic breather solution and rational homoclinic solution. It is interesting to note that the rational homoclinic solution obtained here is just a rogue wave. This is the new physical phenomenon found out till now.

2. Inclined periodic homoclinic breather and rogue waves

We consider 'bad' Bq equation

$$u_{tt} - u_{xx} - 3(u^2)_{xx} - u_{xxxx} = 0, \quad -\infty < x < \infty, \quad t \in R. \tag{2}$$

It is easy to see that eq. (1) has an equilibrium solution u_0 which is an arbitrary constant. It was shown that u_0 is a homoclinic fixed point, when $u_0 \neq -(1/6)$ [23]. We suppose

$$u = u_0 + 2(\ln f)_{xx}, \tag{3}$$

where $f(x,t)$ are unknown real functions. Substituting (3) into (2), we obtain the following bilinear equation [19]:

$$(D_t^2 - (1+6u_0)D_x^2 - D_x^4)(f \cdot f) = 0, \tag{4}$$

where $D_x^2 f \cdot f = 2(f_{xx}f - f_x^2)$. In this case, we choose extended homoclinic test function

$$f = \mathrm{e}^{-p_1(x-\alpha t)} + b_1\cos(p(x+\beta t)) + b_2\mathrm{e}^{p_1(x-\alpha t)}, \tag{5}$$

where p_1, p, α, β, b_1 and b_2 are real constants to be determined.

Computing $D_t^2(f\cdot f)$, $D_x^2(f\cdot f)$ and $D_x^4(f\cdot f)$, we get

$$\begin{aligned} D_t^2(f\cdot f) = {} & 2[b_1(p_1^2\alpha^2 - p^2\beta^2)\cos(p(x+\beta t))\mathrm{e}^{-p_1(x-\alpha t)} \\ & + b_1b_2(p_1^2\alpha^2 - p^2\beta^2)\cos(p(x+\beta t))\mathrm{e}^{p_1(x-\alpha t)} \\ & + 4b_2p_1^2\alpha^2 - b_1^2p^2\beta^2 + 2b_1pp_1\alpha\beta\sin(p(x+\beta t))\mathrm{e}^{-p_1(x-\alpha t)} \\ & - 2b_1b_2pp_1\alpha\beta\sin(p(x+\beta t))\mathrm{e}^{p_1(x-\alpha t)}], \end{aligned} \tag{6}$$

$$\begin{aligned} D_x^2(f\cdot f) = {} & 2[b_1(p_1^2 - p^2)\cos(p(x+\beta t))\mathrm{e}^{-p_1(x-\alpha t)} \\ & + b_1b_2(p_1^2 - p^2)\cos(p(x+\beta t))\mathrm{e}^{p_1(x-\alpha t)} + 4b_2p_1^2 - b_1^2p^2 \\ & - 2b_1pp_1\sin(p(x+\beta t))\mathrm{e}^{-p_1(x-\alpha t)} \\ & + 2b_1b_2pp_1\sin(p(x+\beta t))\mathrm{e}^{p_1(x-\alpha t)}] \end{aligned} \tag{7}$$

and

$$D_x^4(f\cdot f) = 2[b_1(p_1^4+p^4-6p_1^2p^2)\cos(p(x+\beta t))\mathrm{e}^{-p_1(x-\alpha t)} \\ +b_1b_2(p_1^4+p^4-6p_1^2p^2)\cos(p(x+\beta t))\mathrm{e}^{p_1(x-\alpha t)} \\ +4b_1^2p^4+16b_2p_1^4-4b_1(p_1^3p-p^3p_1)\sin(p(x+\beta t))\mathrm{e}^{-p_1(x-\alpha t)} \\ +4b_1b_2(p_1^3p-p^3p_1)\sin(p(x+\beta t))\mathrm{e}^{p_1(x-\alpha t)}]. \tag{8}$$

Substituting (6)–(8) into (4) yield an algebraic equation of $\mathrm{e}^{p_1(x-\alpha t)}$. Equating all coefficients of different powers of $\mathrm{e}^{jp(x-\alpha t)}$ ($j=-1,0,1$) to zero, we get

$$\begin{cases} b_1(p_1^2\alpha^2-p^2\beta^2)-b_1(1+6u_0)(p_1^2-p^2)-b_1(p_1^4+p^4-6p_1^2p^2)=0, \\ b_1b_2(p_1^2\alpha^2-p^2\beta^2)-b_1b_2(1+6u_0)(p_1^2-p^2)-b_1b_2(p_1^4+p^4-6p_1^2p^2)=0, \\ 4b_2p_1^2\alpha^2-b_1^2p^2\beta^2-(1+6u_0)(4b_2p_1^2-b_1^2p^2)-4b_1^2p^4-16b_2p_1^4=0, \\ b_1pp_1\alpha\beta+b_1pp_1(1+6u_0)-b_1(p_1^3p-p^3p_1)=0, \\ b_1b_2pp_1\alpha\beta+b_1b_2pp_1(1+6u_0)-b_1b_2(p_1^3p-p^3p_1)=0. \end{cases} \tag{9}$$

Solving eq. (9) and taking $p_1=p$ yield

$$\begin{cases} p^2=\dfrac{1}{4}(\beta^2-\alpha^2), \\ b_1=\pm 2\dfrac{\sqrt{(2\alpha^2-(1+6u_0)-\beta^2)b_2}}{\sqrt{2\beta^2-(1+6u_0)-\alpha^2}}, \\ \alpha\beta=-(1+6u_0). \end{cases} \tag{10}$$

Choosing $u_0\neq-(1/6)$ and $b_2>0$, we get from (10)

$$|\beta|>|\alpha|,\ \alpha=-\frac{1+6u_0}{\beta}$$

and

$$\alpha^2>1+6u_0,\quad u_0<-\frac{1}{6}\quad\text{or}\quad >-\frac{1}{6} \tag{11}$$

or

$$\alpha^2<1+6u_0,\quad u_0>-\frac{1}{6}.$$

Substituting (10) and (11) into (3) and taking positive and negative signs in eq. (10) yield the solutions of 'bad' Bq equation as follows:

$$u_1(x,t)=u_0+\frac{p^2\left[h_0+2h_1\sin(p(x+\beta t))\sinh\left(p\left(x+\dfrac{1+6u_0}{\beta}t\right)+\gamma\right)\right]}{\left(\cosh\left(p\left(x+\dfrac{1+6u_0}{\beta}t\right)+\gamma\right)+h_1\cos(p(x+\beta t))\right)^2} \tag{12}$$

and

$$u_2(x,t)=u_0+\frac{p^2\left[h_0-2h_1\sin(p(x+\beta t))\sinh\left(p\left(x+\dfrac{1+6u_0}{\beta}t\right)+\gamma\right)\right]}{\left(\cosh\left(p\left(x+\dfrac{1+6u_0}{\beta}t\right)+\gamma\right)-h_1\cos(p(x+\beta t))\right)^2}, \tag{13}$$

where

$$h_0 = \frac{3(\beta^2-\alpha^2)}{2\beta^2-(1+6u_0)-\alpha^2}, \quad h_1 = \frac{\sqrt{2\alpha^2-(1+6u_0)-\beta^2}}{\sqrt{2\beta^2-(1+6u_0)-\alpha^2}} < 1,$$

$$\gamma = \ln\sqrt{b_2}, \quad p = \pm\frac{\sqrt{\beta^2-\alpha^2}}{2}, \quad \beta,\alpha \in R.$$

The solution $u_1(x,t)$ (respectively, $u_2(x,t)$) shows a new family of two-wave, inclined periodic homoclinic breatherwave, which is a homoclinic wave, homoclinic to a fixed point u_0 of eq. (2), when $t \to \pm\infty$ [23–25], and meanwhile it is a periodic wave whose amplitude periodically oscillates with the evolution of time. It shows elastic interaction between a left-propagation (backward direction) periodic wave with speed β and homoclinic wave of different direction with speed $(1+6u_0)/\beta$. The trajectories of these solutions are defined explicitly by

$$p\left(x+\frac{1+6u_0}{\beta}t\right)+\gamma = 0.$$

That is, these solutions evolve periodically along a straight line with a certain angle with x-axis and t-axis. So, we can see that this solution is not only a space-periodic breather or an Akhmediev breather, but also a time-periodic breather or Ma soliton [27]. From figure 1, it is observed that the spatial structures of the functions $u_1(x,t)$ and $u_2(x,t)$ are similar to the structure of breather-type solutions [27] and single homoclinic orbit [28]. At the same time, we can see that an obvious feature of these solutions u_1 and u_2 is that it is a singular breather and describes a single wave in localized space and time in each periodic unit. Meanwhile, it is periodic in $p\left(x+\frac{1+6u_0}{\beta}t\right)+\gamma = 0$ with period $2\pi/p$. So this solution is called the inclined periodic homoclinic breather solution.

Using eq. (10) and taking $b_2 = 1$, $\gamma = \ln(\sqrt{b_2}) = 0$ in u_2. So, solution u_2 can be rewritten as follows:

$$u_2(x,t) = u_0 + \frac{p^2\left[h_0 - 2h_1\sin(p\,(x+\beta t))\sinh\left(p\left(x+\frac{1+6u_0}{\beta}t\right)\right)\right]}{\left(\cosh\left(p\left(x+\frac{1+6u_0}{\beta}t\right)\right)\right) - h_1\cos(p(x+\beta t))^2}, \tag{14}$$

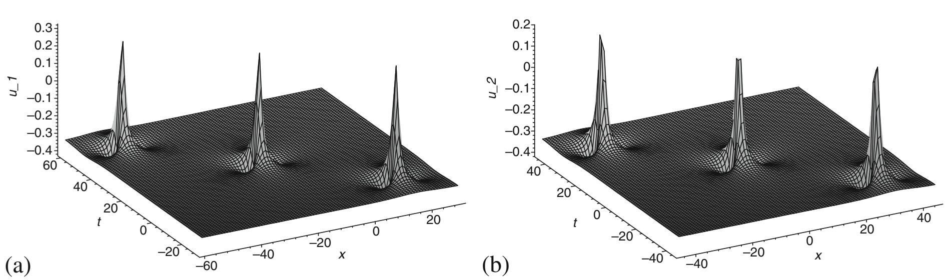

Figure 1. Spatial–temporal structures of the inclined periodic homoclinic breather waves: (**a**) $u_1(x,t)$ and (**b**) $u_2(x,t)$ with $p = 0.1$, $\beta = -1$, $u_0 = -1/3$.

where

$$h_0 = \frac{12p^2}{4p^2 + \beta^2 - 1 - 6u_0},$$
$$h_1 = \frac{\sqrt{\alpha^2 - 1 - 6u_0 - 4p^2}}{\sqrt{4p^2 - 1 - 6u_0 + \beta^2}}.$$

Now, we consider a limit behaviour of u_2 as the period $2\pi/p$ of periodic wave $\cos(p(x + \beta t))$ goes to infinity, i.e. $p \to 0$. By computing, we obtain the following result:

$$U_{\text{rogue wave}} = u_0 + \frac{8\left(A - 2\left(x + \frac{(1+6u_0)t}{\beta}\right)(x + \beta t)\right)}{\left(\left(x + \frac{(1+6u_0)t}{\beta}\right)^2 + (x + \beta t)^2 + A\right)^2}, \tag{15}$$

where

$$A = -\frac{6}{1 + 6u_0} \quad \text{and} \quad \beta^2 = -(1 + 6u_0).$$

From eq. (15), this family of solution is valid, when $u_0 < -(1/6)$. Under this condition, the denominator in eq. (15) is clearly non-singular. From figure 2a, it is observed that it has one upper dominant peak and two small holes. The spatial structure of the function $U_{\text{rogue wave}}$ has similar structure of the rogue waves, which is a point of hot discussion recently [29,30]. The maximum amplitude of the rogue wave solution, $U_{\text{rogue wave}}$, occurs at point (0,0) and the maximum amplitude of this rogue wave solution is equal to $-7u_0 - (4/3)$. The amplitude of $U_{\text{rogue wave}}$ is minimum at two points $(t = 0, x = \pm 3\sqrt{-(1 + 6u_0)^{-1}})$, and the minimum amplitude of this rogue wave solution is equal to $2u_0 + (1/6)$. $U_{\text{rogue wave}}$ contains two waves with different velocities and directions. It is easy to verify that $U_{\text{rogue wave}}$ is a rational solution of 'bad' Boussinesq equation. Moreover, we can show that $U_{\text{rogue wave}}$ also is a homoclinic solution. In fact, $U_{\text{rogue wave}} \to u_0$ for fixed x as $t \to \pm\infty$. So, $U_{\text{rogue wave}}$ is not only a rational homoclinic solution, but also a rogue wave solution which has two to three times amplitude higher than its surrounding waves and generally forms in a short time (see figure 2a). It is a new discovery that the rogue wave solution can come from homoclinic breather solution for Boussinesq equation (see figures 1a and 1b). One may think whether the energy collection and superposition of homoclinic breather wave in many periods lead to a rogue wave or not. Moreover, it follows from figure 2b, that the amplitude of $U_{\text{rogue wave}}$ becomes more and more short as time goes, and finally approaches a non-zero constant background. It is shown that the rogue wave arises from the non-zero constant background and then disappears into the non-zero constant background again.

Remark

Applying HBLM to 'good' Bq equation and by following the same procedure of dealing with eq. (2), we can obtain similar results.

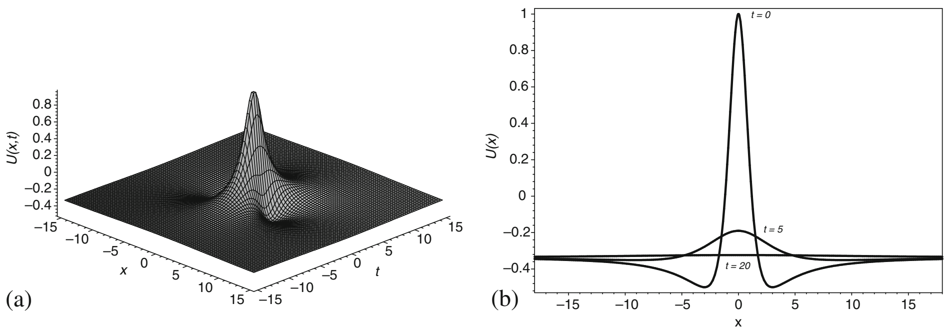

Figure 2. (**a**) Spatial–temporal structure and (**b**) plot of $U_{\text{rogue wave}}$ for time $t = 0\,\text{s}, 5\,\text{s}, 20\,\text{s}$ with $u_0 = -(1/3)$.

3. Conclusions

In this paper, we proposed a new method for seeking rogue wave, i.e. homoclinic (heteroclinic) breather variation method (HBLM). Applying this method to 'bad' Bq equation, we obtained a family of inclined periodic homoclinic breather solution and rational homoclinic solution. Furthermore, rational homoclinic solution obtained here is just a rogue wave solution. In future, we intend to study the interaction between breather wave and solitary wave. We intend to get answers to the following questions: Can we obtain similar results to other integrable or non-integrable system with homoclinic or heteroclinic breather wave? How can one use the homoclinic breather wave to obtain rogue wave under certain conditions?

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