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Exact travelling wave solutions of the (3+1)-dimensional mKdV-ZK equation and the (1+1)-dimensional compound KdVB equation using the new approach of generalized (G'/G)-expansion method

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Abstract. In this paper, the new generalized (G'/G)-expansion method is executed to find the travelling wave solutions of the (3+1)-dimensional mKdV-ZK equation and the (1+1)-dimensional compound KdVB equation. The efficiency of this method for finding exact and travelling wave solutions has been demonstrated. It is shown that the new approach of generalized (G'/G)-expansion method is a straightforward and effective mathematical tool for solving nonlinear evolution equations in applied mathematics, mathematical physics and engineering. Moreover, this procedure reduces the large volume of calculations.

Keywords. The new generalized (G'/G)-expansion method; the (3+1)-dimensional mKdV-ZK equation and the (1+1)-dimensional compound KdVB equation; nonlinear partial differential equation; travelling wave solutions.

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1. Introduction

Now-a-days nonlinear evolution equations (NLEEs) have been the subject of extensive studies in various branches of nonlinear science. A special class of analytical solutions, named travelling wave solutions for NLEEs, are very important, because most of the phenomena that arise in mathematical physics and engineering fields can be described by NLEEs. NLEEs are frequently used to describe several problems of protein chemistry, chemically reactive materials, in ecology (most of the population models), in physics (the heat flow and the wave propagation phenomena), quantum mechanics, fluid mechanics, plasma physics, propagation of shallow water waves, optical fibres, biology, solid-state physics, chemical kinematics, geochemistry, meteorology, electricity etc. Therefore, investigation of travelling wave solutions is becoming more and more interesting in nonlinear sciences day-by-day. However, not all equations posed by these models are solvable. As a result, many new techniques have been successfully developed by diverse groups of mathematicians and physicists, such as the exp-function method [1–3], the generalized Riccati equation [4], the Miura transformation [5], the Jacobi elliptic function expansion method [6,7], the Hirota's bilinear method [8], the sine–cosine method [9], the tanh-function method [10], the extended tanh-function method [11,12], the homogeneous balance method [13], the modified exp-function method [14], the (G'/G)-expansion method [15–22], the improved (G'/G)-expansion method [23], the modified simple equation method [24–27], the inverse scattering transform [28] and so on.

Recently, Naher and Abdullah [29] established a highly effective extension of the (G'/G)-expansion method, called the new generalized (G'/G)-expansion method to obtain exact travelling wave solutions of NLEEs. The objective of this paper is to apply the new generalized (G'/G) expansion method to construct exact solutions for nonlinear evolution equations in mathematical physics through the $(3+1)$-dimensional mKdV-ZK equation and the $(1+1)$-dimensional compound KdVB equation.

This paper is organized as follows: in §2, the new generalized (G'/G)-expansion method is discussed. In §3, this method is applied to the nonlinear evolution equations. Section 4 has physical explanations while in §5 the comparisons and in §6 conclusions are given.

2. Material and method

In this section, we describe the new generalized (G'/G)-expansion method for finding travelling wave solutions of nonlinear evolution equations. Let us consider a general nonlinear partial differential equation (PDE) in the form

$$P(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \ldots) = 0, \tag{1}$$

where $u = u(x,t)$ is an unknown function, P is a polynomial in $u(x,t)$ and its derivatives in which, highest-order derivatives and nonlinear terms are involved and the subscripts stand for the partial derivatives.

Step 1: We combine the real variables x and t by a complex variable Φ

$$u(x,t) = u(\Phi), \qquad \Phi = x + y + z \pm Vt, \tag{2}$$

where V is the speed of the travelling wave. The travelling wave transformation (2) converts eq. (1) into an ordinary differential equation (ODE) for $u = u(\Phi)$:

$$Q(u, u', u'', u''', \ldots) = 0, \tag{3}$$

where Q is a polynomial of u and its derivatives and the superscripts indicate ordinary derivatives with respect to Φ.

Step 2: According to a possibility, eq. (3) can be integrated term by term one or more times, which yields constant(s) of integration. The integral constant may be zero, for simplicity.

Step 3: Suppose the travelling wave solution of eq. (3) can be expressed as follows:

$$u(\Phi) = \sum_{i=0}^{N} a_i (d + H)^i + \sum_{i=1}^{N} b_i (d + H)^{-i}, \tag{4}$$

where either a_N or b_N may be zero, but both a_N or b_N could be zero at a time, $a_i (i = 0, 1, 2, \ldots, N)$ and $b_i (i = 1, 2, \ldots, N)$ and d are arbitrary constants to be determined later and $H(\Phi)$ is

$$H(\Phi) = \left(G'/G\right), \tag{5}$$

where $G = G(\Phi)$ satisfies the following auxiliary ordinary differential equation:

$$AGG'' - BGG' - EG^2 - C(G')^2 = 0, \tag{6}$$

where the prime stands for derivative with respect to Φ. A, B, C and E are real parameters.

Step 4: To determine the positive integer N, take a homogeneous balance between the highest-order nonlinear terms and the derivatives of the highest order appearing in eq. (3).

Step 5: By substituting eqs (4), (5) and (6) in eq. (3) with the value of N obtained in Step 4, we obtain polynomials in $(d+H)^N (N = 0, 1, 2, \ldots)$ and $(d+H)^{-N} (N = 0, 1, 2, \ldots)$. Then, we collect each coefficient of the resulted polynomial to zero, which yields a set of algebraic equations for $a_i (i = 0, 1, 2, \ldots, N)$ and $b_i (i = 1, 2, \ldots, N)$, d and V.

Step 6: Suppose the value of the constants $a_i (i = 0, 1, 2, \ldots, N)$, $b_i (i = 1, 2, \ldots, N)$, d and V can be found by solving the algebraic equations obtained in Step 5. As the general solution of eq. (6) is well known to us by inserting the values of $a_i (i = 0, 1, 2, \ldots, N)$, $b_i (i = 1, 2, \ldots, N)$, d and V in eq. (4), we obtain more general type and new exact travelling wave solutions of the nonlinear partial differential eq. (1). Using the general solution of eq. (6), we have the following solutions of eq. (5):

Family 1: When $B \neq 0$, $\psi = A - C$ and $\Omega = B^2 + 4E(A - C) > 0$,

$$H(\Phi) = \left(\frac{G'}{G}\right) = \frac{B}{2\psi} + \frac{\sqrt{\Omega}}{2\psi} \frac{C_1 \sinh\left(\frac{\sqrt{\Omega}}{2A}\Phi\right) + C_2 \cosh\left(\frac{\sqrt{\Omega}}{2A}\Phi\right)}{C_1 \cosh\left(\frac{\sqrt{\Omega}}{2A}\Phi\right) + C_2 \sinh\left(\frac{\sqrt{\Omega}}{2A}\Phi\right)}. \tag{7}$$

Family 2: When $B \neq 0$, $\psi = A - C$ and $\Omega = B^2 + 4E(A - C) < 0$,

$$H(\Phi) = \left(\frac{G'}{G}\right) = \frac{B}{2\psi} + \frac{\sqrt{-\Omega}}{2\psi} \frac{-C_1 \sin\left(\frac{\sqrt{-\Omega}}{2A}\Phi\right) + C_2 \cos\left(\frac{\sqrt{-\Omega}}{2A}\Phi\right)}{C_1 \cos\left(\frac{\sqrt{-\Omega}}{2A}\Phi\right) + C_2 \sin\left(\frac{\sqrt{-\Omega}}{2A}\Phi\right)}. \tag{8}$$

Family 3: When $B \neq 0$, $\psi = A - C$ and $\Omega = B^2 + 4E(A - C) = 0$,

$$H(\Phi) = \left(\frac{G'}{G}\right) = \frac{B}{2\psi} + \frac{C_2}{C_1 + C_2\Phi}. \tag{9}$$

Family 4: When $B = 0$, $\psi = A - C$ and $\Delta = \psi E > 0$,

$$H(\Phi) = \left(\frac{G'}{G}\right) = \frac{\sqrt{\Delta}}{\psi} \frac{C_1 \sinh\left(\frac{\sqrt{\Delta}}{A}\Phi\right) + C_2 \cosh\left(\frac{\sqrt{\Delta}}{A}\Phi\right)}{C_1 \cosh\left(\frac{\sqrt{\Delta}}{A}\Phi\right) + C_2 \sinh\left(\frac{\sqrt{\Delta}}{A}\Phi\right)}. \tag{10}$$

Family 5: When $B = 0$, $\psi = A - C$ and $\Delta = \psi E < 0$,

$$H(\Phi) = \left(\frac{G'}{G}\right) = \frac{\sqrt{-\Delta}}{\psi} \frac{-C_1 \sin\left(\frac{\sqrt{-\Delta}}{A}\Phi\right) + C_2 \cos\left(\frac{\sqrt{-\Delta}}{A}\Phi\right)}{C_1 \cos\left(\frac{\sqrt{-\Delta}}{A}\Phi\right) + C_2 \sin\left(\frac{\sqrt{-\Delta}}{A}\Phi\right)}. \tag{11}$$

3. Applications

In this section, we shall apply the new generalized (G'/G)-expansion method to find exact solutions and the solitary wave solutions of the following two nonlinear evolution equations.

3.1 *The* $(3+1)$*-dimensional mKdV-ZK equation*

In this section we will exploit the new generalized (G'/G)-expansion method to find exact solutions and then the solitary wave solutions of the $(3+1)$-dimensional mKdV-ZK equation in the form

$$u_t + \alpha u^2 u_x + \left(u_{xx} + u_{yy} + u_{zz}\right)_x = 0. \tag{12}$$

The wave transformation equation $u(\Phi) = u(x, y, z, t)$, $\Phi = x + y + z - Vt$. Reduce eq. (12) into the following ODE:

$$-Vu' + \alpha u^2 u' + (3u'')' = 0, \tag{13}$$

where the superscripts stand for the derivatives with respect to Φ.

Integrating eq. (13) with respect to Φ, we get the following ODE:

$$P - Vu + \frac{1}{3}\alpha u^3 + 3u'' = 0, \tag{14}$$

where P is an integral constant that can be determined later.

Equating u^3 with u'' yields $N = 1$. Consequently, eq. (19) has the formal solution

$$u(\Phi) = a_0 + a_1(d + H) + b_1(d + H)^{-1}, \tag{15}$$

where a_0, a_1, b_1 and d are constants to be determined.

Substituting eq. (15) together with eqs (5) and (6) into eq. (14), the left-hand side is converted into polynomials of $(d + H)^N$ $(N = 0, 1, 2, \ldots)$ and $(d + H)^{-N}$ $(N = 1, 2, \ldots)$. We collect each coefficient of these resulted polynomials to zero, which yields a set of simultaneous algebraic equations (for minimalism the equations are not displayed) for a_0, a_1, b_1, d and V. Solving these algebraic equations with the help of a symbolic computation software, such as *Maple*, we obtain the following:

$$a_0 = -\frac{3(2d\psi + B)}{\pm A\alpha\sqrt{\frac{-2}{\alpha}}}, \qquad b_1 = 0, \qquad d = d, \qquad a_1 = \pm\frac{3\psi}{A}\sqrt{\frac{-2}{\alpha}},$$
$$V = -\frac{3}{2A^2}\left(B^2 + 4E\psi\right), \qquad P = 0, \tag{16}$$

where $\psi = A - C$, A, B, C and E are free parameters.

Substituting eq. (16) in eq. (15), along with eq. (7) and simplifying, yields the following travelling wave solutions (if $C_1 = 0$ but $C_2 \neq 0$ and $C_2 = 0$ but $C_1 \neq 0$), respectively:

$$u_1(\Phi) = \pm\frac{3\sqrt{\Omega}}{A\sqrt{-2\alpha}}\coth\left(\frac{\sqrt{\Omega}}{2A}\Phi\right),$$
$$u_2(\Phi) = \pm\frac{3\sqrt{\Omega}}{A\sqrt{-2\alpha}}\tanh\left(\frac{\sqrt{\Omega}}{2A}\Phi\right),$$

where

$$\Phi = x - \left\{-\frac{3}{2A^2}\left(B^2 + 4E\psi\right)\right\}t.$$

Substituting eq. (16) in eq. (15), along with eq. (8) and simplifying, the obtained exact solutions become (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$), respectively:

$$u_3(\Phi) = \pm\frac{3i\sqrt{\Omega}}{A\sqrt{-2\alpha}}\cot\left(\frac{\sqrt{-\Omega}}{2A}\Phi\right),$$
$$u_4(\Phi) = \pm\frac{3i\sqrt{\Omega}}{A\sqrt{-2\alpha}}\tan\left(\frac{\sqrt{-\Omega}}{2A}\Phi\right).$$

Substituting eq. (16) in eq. (15) together with eq. (9) and simplifying, we obtain

$$u_5(\Phi) = \pm\frac{6\psi}{A\sqrt{-2\alpha}}\left(\frac{C_2}{C_1 + C_2\Phi}\right).$$

Substituting eq. (16) in eq. (15), along with eq. (10) and simplifying, we obtain the following travelling wave solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$), respectively:

$$u_6(\Phi) = \pm\frac{1}{A\sqrt{-2\alpha}}\left(3B - 6\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{A}\Phi\right)\right),$$
$$u_7(\Phi) = \pm\frac{1}{A\sqrt{-2\alpha}}\left(3B - 6\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{A}\Phi\right)\right).$$

Substituting eq. (16) in eq. (15), together with eq. (11) and simplifying, our obtained exact solutions become (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$), respectively:

$$u_8(\Phi) = \pm\frac{1}{A\sqrt{-2\alpha}}\left(3B - 6i\sqrt{\Delta}\cot\left(\frac{\sqrt{-\Delta}}{A}\Phi\right)\right),$$
$$u_9(\Phi) = \pm\frac{1}{A\sqrt{-2\alpha}}\left(3B + 6i\sqrt{\Delta}\tan\left(\frac{\sqrt{-\Delta}}{A}\Phi\right)\right).$$

3.2 *The* (1 + 1)*-dimensional compound KdVB equation*

In this section, we shall utilize the new generalized (G'/G)-expansion method to find exact solutions and then the solitary wave solutions of the (1+1)-dimensional compound KdVB equation in the form

$$u_t + \alpha u u_x + \beta u^2 u_x + \gamma u_{xx} - \delta u_{xxx} = 0, \tag{17}$$

where α, β, γ and δ are nonzero constants.

The wave transformation equation $u(\Phi) = u(x, t)$, $\Phi = x - Vt$. Reduce eq. (17) into the following ODE:

$$-Vu' + \alpha u u' + \beta u^2 u' + \gamma u'' - \delta u''' = 0, \tag{18}$$

where the superscripts stand for derivatives with respect to Φ.

Integrating eq. (18) with respect to Φ, we get the following ODE:

$$P - Vu + \frac{1}{2}\alpha u^2 + \frac{1}{3}\beta u^3 + \gamma u' - \delta u'' = 0, \tag{19}$$

where P is an integral constant that can be determined later.

Taking the homogeneous balance between u^3 and u'' in eq. (19), we obtain $N = 1$. Therefore, the solution of eq. (19) is of the form

$$u(\xi) = a_0 + a_1(d + H) + b_1(d + H)^{-1}, \tag{20}$$

where a_0, a_1, b_1 and d are constants to be determined.

Substituting eq. (20) together with eqs (5) and (6) in eq. (19), the left-hand side is converted into polynomials in $(d + H)^N$ $(N = 0, 1, 2, \ldots)$ and $(d + H)^{-N}$ $(N = 1, 2, \ldots)$. We collect each coefficient of these resulted polynomials to zero, which yields a set of simultaneous algebraic equations (for minimalism the equations are not displayed) for a_0, a_1, b_1, d and V. Solving these algebraic equations with the help of a symbolic computation software, such as *Maple*, we obtain the following:

$$a_0 = \frac{m_2}{2A\beta m_1}, \qquad a_1 = \frac{\psi m_1}{A} \qquad b_1 = 0, \qquad d = d,$$
$$V = -\frac{1}{12A^2\beta\delta}(3A^2\delta\alpha^2 - 2A^2\beta\gamma^2 - 24A\delta^2 E\beta - 6\delta^2 B^2\beta + 24\delta^2 CE\beta),$$
$$P = \frac{1}{72\delta\beta^2 A^2 m_1}(-6A^2\alpha m_1\beta\gamma^2 + 3A^2\alpha^3 m_1\delta + 8A^2\beta\gamma^3 - 72A\alpha m_1\delta^2 E\beta$$
$$-288A\beta\gamma\delta^2 E + 72\alpha m_1\delta^2 CE\beta - 72\delta^2\gamma B^2\beta + 288\beta\gamma C\delta^2 E$$
$$-18\alpha m_1\delta^2 B^2\beta), \tag{21}$$

where

$$\psi = A - C, \qquad m_1 = \pm\sqrt{\frac{6\delta}{\beta}},$$
$$m_2 = -(A\alpha m_1 - 2\gamma A + 12\delta dA - 12\delta Cd + 6\delta B)$$

and A, B, C and E are free parameters.

Substituting eq. (21) in eq. (20), along with eq. (7) and simplifying, yields the following travelling wave solutions (if $C_1 = 0$ but $C_2 \neq 0$ and $C_2 = 0$ but $C_1 \neq 0$), respectively:

$$u_{10}(\Phi) = \frac{1}{2A\beta m_1}\left\{m_2 + \beta m_1^2\left(2\psi d + B + \sqrt{\Omega}\coth\left(\frac{\sqrt{\Omega}}{2A}\Phi\right)\right)\right\},$$
$$u_{11}(\Phi) = \frac{1}{2A\beta m_1}\left\{m_2 + \beta m_1^2\left(2\psi d + B + \sqrt{\Omega}\tanh\left(\frac{\sqrt{\Omega}}{2A}\Phi\right)\right)\right\},$$

where

$$\Phi = x - \left\{-\frac{1}{12A^2\beta\delta}\left(3A^2\delta\alpha^2 - 2A^2\beta\gamma^2\right.\right.$$
$$\left.\left.-24A\delta^2 E\beta - 6\delta^2 B^2\beta + 24\delta^2 CE\beta\right)\right\}t.$$

Substituting eq. (21) in eq. (20), along with eq. (8) and simplifying, the obtained exact solutions become (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$), respectively:

$$u_{12}(\Phi) = \frac{1}{2A\beta m_1}\left\{m_2 + \beta m_1^2\left(2\psi d + B + i\sqrt{\Omega}\cot\left(\frac{\sqrt{-\Omega}}{2A}\Phi\right)\right)\right\},$$
$$u_{13}(\Phi) = \frac{1}{2A\beta m_1}\left\{m_2 + \beta m_1^2\left(2\psi d + B - i\sqrt{\Omega}\tan\left(\frac{\sqrt{-\Omega}}{2A}\Phi\right)\right)\right\}.$$

Substituting eq. (21) in eq. (20) together with eq. (9) and simplifying, we obtain

$$u_{14}(\Phi) = \frac{1}{2A\beta m_1}\left\{m_2 + \beta m_1^2\left(2\psi d + B + 2\psi\left(\frac{C_2}{C_1 + C_2\Phi}\right)\right)\right\}.$$

Substituting eq. (21) in eq. (20), along with eq. (10) and simplifying, we obtain the following travelling wave solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$), respectively:

$$u_{15}(\Phi) = \frac{1}{2A\beta m_1}\left\{m_2 + 2\beta m_1^2\left(\psi d + \sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{A}\Phi\right)\right)\right\},$$
$$u_{16}(\Phi) = \frac{1}{2A\beta m_1}\left\{m_2 + 2\beta m_1^2\left(\psi d + \sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{A}\Phi\right)\right)\right\}.$$

Substituting eq. (21) in eq. (20), together with eq. (11) and simplifying, our obtained exact solutions become (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{17}(\Phi) = \frac{1}{2A\beta m_1}\left\{m_2 + 2\beta m_1^2\left(\psi d + i\sqrt{\Delta}\cot\left(\frac{\sqrt{-\Delta}}{A}\Phi\right)\right)\right\},$$
$$u_{18}(\Phi) = \frac{1}{2A\beta m_1}\left\{m_2 + 2\beta m_1^2\left(\psi d - i\sqrt{\Delta}\tan\left(\frac{\sqrt{-\Delta}}{A}\Phi\right)\right)\right\}.$$

4. Physical explanation

In this section we shall put forth the physical significances and graphical representations of the obtained results of the (3+1)-dimensional mKdV-ZK and (1+1)-dimensional compound KdVB equations.

4.1 *Results and discussion*

Solutions $u_1(\Phi)$, $u_2(\Phi)$, $u_6(\Phi)$, $u_7(\Phi)$, $u_{10}(\Phi)$, $u_{11}(\Phi)$, $u_{15}(\Phi)$ and $u_{16}(\Phi)$ are hyperbolic function solutions. Solutions $u_1(\Phi)$, $u_6(\Phi)$, $u_{10}(\Phi)$ and $u_{15}(\Phi)$ are the single soliton solutions. Figure 1 shows the shape of the exact single soliton solution (shows only the shape of solution of $u_{10}(\Phi)$ for $A = 4$, $B = 1$, $C = 1$, $E = 1$, $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $\delta = 1$, $d = 1$ with $-10 \leq x, t \leq 10$). The shape of the figure of solutions $u_1(\Phi)$, $u_6(\Phi)$ and $u_{15}(\Phi)$ are similar to the shape of the figure of solution $u_{10}(\Phi)$. Solutions $u_2(\Phi)$ and $u_7(\Phi)$ are singular soliton solutions. Figure 2 shows the shape of the exact singular soliton solution (shows only the shape of solution of $u_7(\Phi)$ for $A = 2$, $B = 0$, $C = 1$, $E = 1$, $\alpha = 1$ with $-10 \leq x, t \leq 10$). The shape of the figure of solution $u_2(\Phi)$ is similar to the shape of the figure of solution $u_7(\Phi)$. Solutions $u_{11}(\Phi)$ and $u_{16}(\Phi)$ are the kink solutions. Figure 3 shows the shape of the exact kink solution (shows only the shape of solution $u_{11}(\Phi)$ for $A = 2$, $B = 0$, $C = 1$, $E = 1$, $\alpha = 1$ with $-10 \leq x, t \leq 10$). The disturbance with $y = z = 0$ represented by $u_{11}(\Phi)$ moves in the positive x-direction. Solutions $u_3(\Phi)$, $u_4(\Phi)$, $u_8(\Phi)$, $u_9(\Phi)$, $u_{12}(\Phi)$, $u_{13}(\Phi)$, $u_{17}(\Phi)$ and $u_{18}(\Phi)$ are complex periodic travelling wave solutions. Solutions $u_3(\Phi)$, $u_{12}(\Phi)$ and $u_{17}(\Phi)$ are the single soliton solutions. The shape of the figure of solutions $u_3(\Phi)$, $u_{12}(\Phi)$ and $u_{17}(\Phi)$ are similar to the shape of the figure of solution $u_{10}(\Phi)$. Solution $u_{13}(\Phi)$ is the kink solution. The shape of the figure of solution $u_{13}(\Phi)$ is similar to the

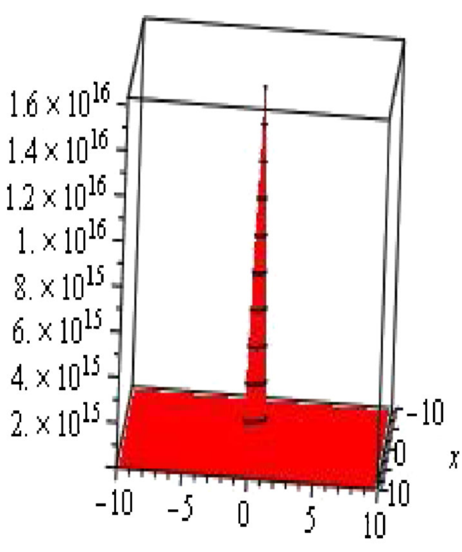

Figure 1. Single soliton.

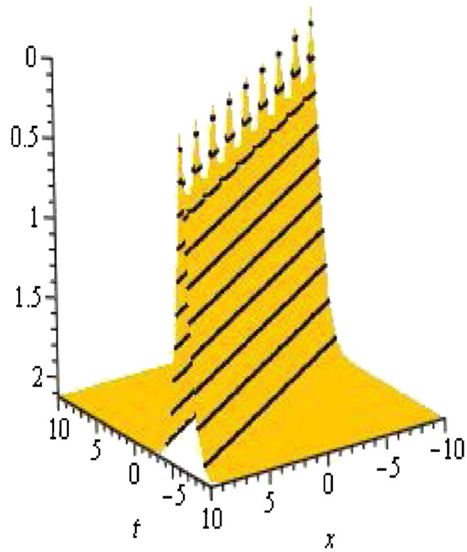

Figure 2. Singular soliton.

shape of the figure of solution $u_{11}(\Phi)$. Solutions $u_4(\Phi)$, $u_8(\Phi)$, $u_9(\Phi)$ and $u_{18}(\Phi)$ are the exact periodic travelling wave solutions. Figure 4 shows the multiple cuspon obtained from solution $u_9(\Phi)$ and graph of the solution $u_9(\Phi)$, for $A = 1$, $B = 0$, $C = 2$, $E = 2$, $\alpha = 1$, $d = 1$, $y = 0$, $z = 0$ has been plotted within $-1 \leq x, t \leq 1$. For convenience

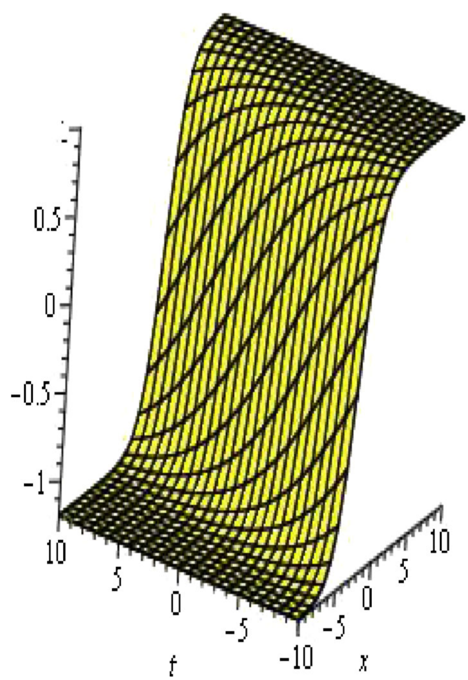

Figure 3. Kink solution.

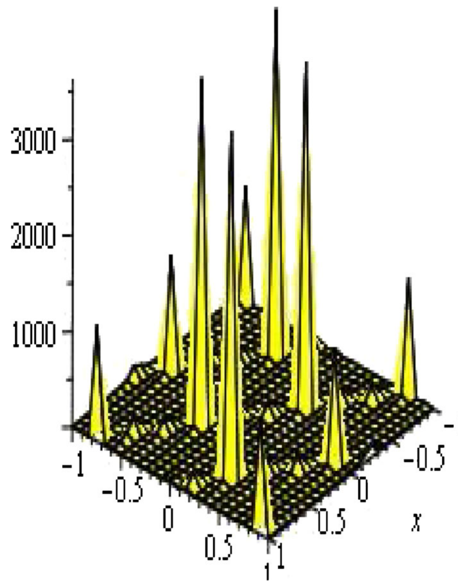

Figure 4. Periodic solutions.

this figure is omitted. Solutions $u_5(\Phi)$ and $u_{14}(\Phi)$ are complex rational travelling wave solutions. Figure 5 shows the shape of the exact soliton solution (shows only the shape of solution of $u_5(\Phi)$ for $A = 1$, $B = 2$,$C = 2$, $E = 1$, $\alpha = 1$, $d = 1$, $C_1 = 2$, $C_2 = 1$, $y = 0$, $z = 0$ with $-10 \le x, t \le 10$). Figure 6 shows the shape of the exact singular kink

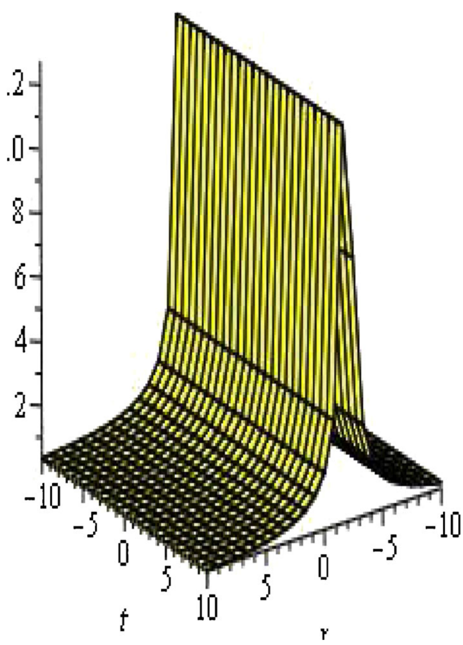

Figure 5. Soliton solution.

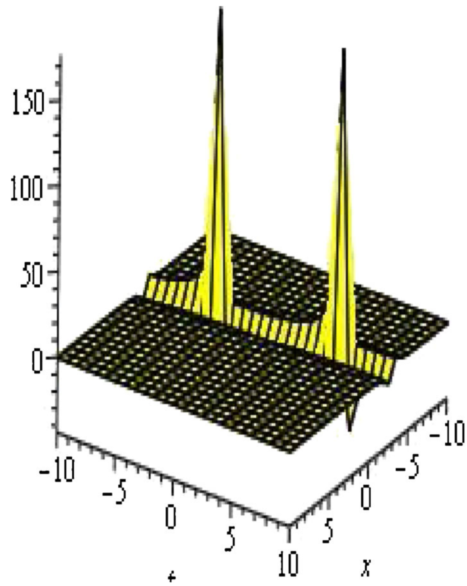

Figure 6. Singular kink solution.

solution (shows only the shape of solution of $u_{14}(\Phi)$ for $A = 1$, $B = 2$, $C = 2$, $E = 1$, $\alpha = 1$, $d = 1$, $C_1 = 2$, $C_2 = 1$, $\beta = 1$, $y = 1$, $\delta = 1$ with $-10 \leq x, t \leq 10$).

4.2 *Graphical representation*

The graphical depictions of the obtained solutions for particular values of the arbitrary constants are shown in figures 1–6 with the aid of the commercial software *Maple*.

5. Comparison

Zayed [30] investigated exact solutions of the (3+1)-dimensional mKdV-ZK equation by using the (G'/G)-expansion method and obtained only five solutions (A1)–(A5) (see Appendix). Moreover, in this paper nine solutions of the (3+1)-dimensional mKdV-ZK equation are constructed by applying the new approach of generalized (G'/G)-expansion method. But, by means of the new approach of generalized (G'/G)-expansion method, we obtained solutions which are different from Zayed [30] solutions. These solutions are new and were not obtained by Zayed [30]. On the other hand, the auxiliary equation used in this paper is different, and so the solutions obtained also different. Similarly, for any nonlinear evolution equation it can be shown that the new approach of generalized (G'/G)-expansion method is much easier than the other methods.

6. Conclusions

The new generalized (G'/G)-expansion method in this paper was applied to the (3+1)-dimensional mKdV-ZK and (1+1)-dimensional compound KdVB equations for finding exact solutions and the solitary wave solutions of these equations which attracted the attention of many mathematicians. The obtained solutions showed that the new generalized (G'/G)-expansion method is more effective and more general than the other methods (e.g., the (G'/G)-expansion method), because it gives more new solutions. Consequently, this simple and powerful method can be more successfully applied to study nonlinear partial differential equations, which frequently occur in engineering sciences, mathematical physics and other scientific real-time application fields.

Appendix. Zayed solutions

Zayed [30] examined the exact solutions of the nonlinear (3+1)-dimensional mKdV-ZK equation by using the (G'/G)-expansion method. He found the following five solutions of the form:

$$u(\xi) = -3\sqrt{\frac{-2}{\alpha}}i\,\sec h\xi, \tag{A.1}$$

$$u(\xi) = 3\sqrt{\frac{-2}{\alpha}}i\,\sec\xi, \tag{A.2}$$

$$u(\xi) = \pm 3\sqrt{\frac{-2}{\alpha}}(\coth\xi - \tanh\xi), \tag{A.3}$$

$$u(\xi) = \pm 3\sqrt{\frac{-2}{\alpha}}(\cot\xi + \tan\xi), \tag{A.4}$$

$$u(\xi) = \pm 3\sqrt{\frac{-2}{\alpha}}\left(\frac{B}{B\xi + c_1}\right). \tag{A.5}$$

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