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Some exact solutions of magnetized viscous model in string cosmology

C P SINGH
Department of Applied Mathematics, Delhi Technological University
(Formerly Delhi College of Engineering), Bawana Road, Delhi 110 042, India
E-mail: cpsphd@rediffmail.com

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Abstract. In this paper, we study anisotropic Bianchi-V Universe with magnetic field and bulk viscous fluid in string cosmology. Exact solutions of the field equations are obtained by using the equation of state (EoS) for a cloud of strings, and a relationship between bulk viscous coefficient and scalar expansion. The bulk viscous coefficient is assumed to be inversely proportional to the expansion scalar. It is interesting to examine the effects of magnetized bulk viscous string model in early and late stages of evolution of the Universe. This paper presents different string models like geometrical (Nambu string), Takabayasi (p-string) and Reddy string models by taking certain physical conditions. We discuss the nature of classical potential for viscous fluid with and without magnetic field. The presence of bulk viscosity stops the Universe from becoming empty in its future evolution. It is observed that the Universe expands with decelerated rate in the presence of viscous fluid with magnetic field whereas, it expands with marginal inflation in the presence of viscous fluid without magnetic field. The other physical and geometrical aspects of each string model are discussed in detail.

Keywords. Bianchi models; string cosmology; bulk viscosity.

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1. Introduction

Recently, the string cosmology has received considerable attention in the framework of general relativity mainly because of its possible role in the early Universe. String theory is particularly relevant to the initial singularity problem whose solution has been thought to require a quantum theory of gravity, for which string theory seems to be the most promising candidate. The concept of string cosmology was developed to describe the events at the very early stages of the evolution of the Universe. The presence of the strings during the early Universe can be explained using grand unified theories [1–3]. In the early stages of the evolution of the Universe, it was expected that topological defects could have formed naturally during the phase transitions followed by spontaneous broken symmetries. Cosmic strings, which might have played important roles in structure formation, are linear topological defects, having very interesting properties. These cosmic strings have stress energy and are coupled to the gravitational field.

Letelier [4] and Stachel [5] have studied the gravitational effects of strings in general relativity. Letelier [6] first used this idea for obtaining some relativistic cosmological solutions of cloud formed by massive strings in Bianchi type-I and Kantowski–Sachs space-times. In these models, each massive string was formed by a geometric string with particles attached along its extension. In principle, the string can be eliminated and ended up with the cloud of particles. This desirable property of the model of a string cloud can be used in cosmology. Matraverse [7] presented a class of exact solutions of Einstein field equations with a two-parameter family of classical strings as the source of gravitational field. Exact solutions of string cosmology in Bianchi type II, VI_0, VIII and IX space-times were studied by Krori *et al* [8]. Yavuz and Tarhan [9], Bali and Dave [10,11], Bali and Upadhaya [12], Bali and Singh [13], Bali and Pradhan [14], Pradhan and Chouhan [15] and Mahanto *et al* [16] have investigated Bianchi-type string cosmological models in general relativity.

The study of magnetic field provides an effective way to understand the initial phases of cosmic evolution. Primordial magnetic field of cosmological origin was discussed by Asseo and Sol [17] and Madsen [18]. Wolfe *et al* [19], Kulsrud *et al* [20] and Barrow [21] have studied the cosmological models with magnetic field and have pointed out its importance in the early evolution of the Universe. Matraverse and Tsagas [22] have found that the interaction of the cosmological magnetic field with the space-time geometry could affect the expansion of the Universe. Banerjee *et al* [23], Chakraborty [24], Tikekar and Patel [25,26] have studied Bianchi type-I and type-III string cosmological models with and without source-free magnetic field. ShriRam and Singh [27] have obtained some new exact solutions of string cosmology with and without a source-free magnetic field in Bianchi type I space-time in different basic forms. Patel and Maharaj [28] and Singh and Singh [29] have studied string cosmology with magnetic field in anisotropic models. Singh and ShriRam [30] have presented a technique to generate new exact Bianchi type III string cosmological solutions with magnetic field. Kilinç and Yavuz [31], Pradhan *et al* [32], Pradhan [33], Bali and Jain [34], Saha and Visinescu [35], Pradhan *et al* [36,37], Saha *et al* [38], Pradhan *et al* [39], Amirhashchi *et al* [40] and Rikhvitsky *et al* [41] have investigated string cosmological models in the presence and absence of magnetic field.

The observations indicate an accelerated expansion and it is generally interpreted in terms of perfect fluid. However, for a realistic treatment of the problem one has to consider material distributions other than the perfect fluid. It is known that when neutrino decoupling occurred, the matter behaved like a viscous fluid in the early stage of the Universe. From a hydrodynamics point of view this is somewhat surprising, as there are several situations in fluid mechanics, even in homogeneous space without boundaries, where the two viscosity coefficients, the shear coefficient and the bulk coefficient, come into play. This means a deviation from thermal equilibrium to the first order [42]. Therefore, to have the realistic cosmological models we should consider the presence of material distributions other than a perfect fluid. It is found that the bulk viscosity plays a very important role in the history of the Universe. Also, we find that viscosity causes the expansion of the Universe to accelerate. The possibility of particular interest for the present work, however, is that of bulk viscosity within the cosmic fluid. Such a term resists the cosmic expansion and therefore acts as negative pressure. Indeed, it has been shown that for the right viscosity coefficient, an accelerating cosmology can be achieved in the absence of a cosmological constant. Therefore, dissipative processes are thought to be present in any realistic theory of the evolution of the Universe. Earlier, as we did not know the nature of the content of the Universe (dark matter and dark energy components) very clearly, assuming the bulk viscosity was reasonable and practical. In order to study the evolution of the Universe, many authors [43–51] have investigated cosmological models with a fluid containing viscosity. Yadav [52], Mohanty and Gauranga [53] are some of the authors who have studied string cosmological models with viscous fluid.

As we know, the standard Big Bang cosmological model has various drawbacks in explaining many observations such as the acceleration of the Universe and the anisotropy of the cosmic microwave background radiation. This led to the pursuit of more general models in which the geometry and matter content change when compared to the standard Friedmann–Robertson-Walker (FRW) cosmologies. As pointed out by Ellis [54], the anisotropic Bianchi-type cosmologies are worthy of attention even if the current observations indicate that our Universe does not rule out the possibility of dominant anisotropic effects in the early Universe.

Among the anisotropic Bianchi models, Bianchi type-I cosmological models are the simplest anisotropic Universe models which are the generalization of flat FRW models but the contraction or expansion rates are direction-dependent. Bianchi type-V cosmological models are the natural generalization of FRW models with negative curvature. These open models are favoured by the available evidences for low-density Universe. In these anisotropic models, it is possible to accommodate the presence of cosmic strings and magnetic fields. Also, in Bianchi V model we find one more field equation compared to Bianchi I model which makes the field equations very complicated to solve exactly. Therefore, these geometrical and physical structures make Bianchi V models interesting to study the dynamics of the Universe in early and later times of the evolution.

Coley [55], Singh *et al* [56–58], Singh and Beesham [59], Singh [60] have studied Bianchi V models in general relativity in many physical contexts. Chakraborty and Chakraborty [61] have studied string cosmological models with magnetic fields in Bianchi V model and have found solutions in some cases. Bali [62] has studied string Bianchi V magnetized dust string cosmological model. Some authors [63,64] have investigated anisotropic models with magnetic field and bulk viscosity in Bianchi I and VI string cosmological models where energy density is equal to string tension density. Recently, Sharif and Waheed [65] and Singh [66] have studied Bianchi I magnetized viscous fluid model in string cosmology and have discussed the effect of viscous fluid and magnetic field on the classical potential.

In [43–53,62,63], the authors have studied string cosmological models in Bianchi V with viscous fluid only and have obtained the solution for dust string model. Motivated by these works, the author has studied Bianchi V cosmological model with viscous fluid and magnetic field in string cosmological models to investigate the effects of viscous fluid with and without magnetic field for three different string models.

Therefore, the aim of this paper is to find some exact solutions in Bianchi V model with magnetized viscous fluid in string cosmology and discuss their effects in early and late time evolution of the Universe. The equation of state for a cloud of strings, and the inverse relation between expansion scalar and bulk viscous coefficient are used to solve the field equations exactly. The present work finds solutions for three different string models and discusses the physical and geometrical behaviours during early and late time evolution of the Universe. It is observed that the presence of bulk viscous fluid prevents the Universe from being empty in its late time evolution with or without magnetic field. This paper is organized as follows.

In §2 the field equations for Bianchi V model with bulk viscosity and magnetic field in string cosmology are presented. Section 3 deals with the solution of field equations where a general quadrature form of volume scale factor of the model is given. Three different exact string models are presented in §4. We discuss the various physical parameters in each case. Finally, we present a summary of the results in §5.

2. Model and basic equations

We consider the homogeneous and anisotropic Bianchi-V metric in the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + A^2\mathrm{d}x^2 + \mathrm{e}^{2x}\left(B^2\mathrm{d}y^2 + C^2\mathrm{d}z^2\right), \tag{1}$$

where A, B, and C are scale factors in anisotropic background and are functions of cosmic time t.

The energy–momentum tensor for bulk viscous string dust with magnetic field is given by [6,67]

$$T_i^j = \rho u_i u^j - \lambda x_i x^j - \xi u^l_{;l}(g_i^j + u_i u^j) + E_i^{\ j}, \tag{2}$$

where ρ is the proper energy density for a cloud of strings with particles attached to them and λ is the string tensor density and is related by $\rho = \rho_\mathrm{p} + \lambda$, where ρ_p is the particle energy density. The unit time-like vector u^i describes the four-velocity of the particle and unit space-like vector x^i denotes the direction of the string which must be taken in either of the three directions x-, y- or z-axes. Let us choose the string direction along x-axis, i.e.,

$$x^i = (A^{-1}, 0, 0, 0). \tag{3}$$

In a co-moving coordinate system, we have

$$u^i = (0, 0, 0, 1). \tag{4}$$

Therefore, we have

$$u_i u^i = -x_i x^i = -1, \qquad u_i x^i = 0. \tag{5}$$

In eq. (2), ξ is the coefficient of bulk viscosity, $u^l_{;l} = \theta$ is the expansion scalar and $E_i^{\ j}$ is the electromagnetic field tensor which is given by (see [68])

$$E_i^{\ j} = \bar{\mu}\left[|h|^2\left(u_i u^j + \frac{1}{2}g_i^{\ j}\right) - h_i h^j\right], \tag{6}$$

where $\bar{\mu}$ is the magnetic permeability and h_i is the magnetic flux vector defined by

$$h_i = \frac{1}{\bar{\mu}}\bar{F}_{ij}u^j. \tag{7}$$

The dual electromagnetic field tensor $\bar{F}_{ij}$ is defined as

$$\bar{F}_{ij} = \frac{\sqrt{-g}}{2}\epsilon_{ijkl}F^{kl}, \tag{8}$$

where F^{kl} is the electromagnetic field tensor and ϵ_{ijkl} is the Levi–Civita tensor density.

We assume that the magnetic field is generated in yz-plane as its source is the electric current that flows in x-direction. Therefore, the magnetic flux vector has only one non-zero component h_1, i.e., $h_1 \neq 0$, $h_2 = 0 = h_3 = h_4$. Moreover, the assumption of infinite electrical conductivity [69] along with finite current leads to $F_{14} = 0 = F_{24} = F_{34}$.

Using Maxwell's equations

$$F_{ij;k} + F_{jk;i} + F_{ki;j} = 0 \qquad \text{and} \qquad F^{ij}_{;k} = 0, \tag{9}$$

we find

$$F_{23} = I = \text{constant}. \tag{10}$$

Hence, the non-zero component of the magnetic flux vector is

$$h_1 = \frac{AI}{\bar{\mu}BC}. \tag{11}$$

As $|h|^2 = h_l h^l = h_1 h^1 = g^{11}(h_1)^2$,

$$|h|^2 = \frac{I^2}{\bar{\mu}^2 B^2 C^2}\ . \tag{12}$$

Using eqs (11) and (12) into (6), the components of E_i^j are given by

$$E_1^1 = -\frac{I^2}{2\bar{\mu}^2 B^2 C^2} = -E_2^2 = -E_3^3 = E_4^4. \tag{13}$$

The Einstein's field equations (in gravitational units $c = 8\pi G = 1$) read as

$$R_i^j - \frac{1}{2}g_i^j R = -T_i^j, \tag{14}$$

where R_i^j is the Ricci tensor and $R = g^{ij}R_{ij}$ is the Ricci scalar.

The field equations (14) with (1) and (2) subsequently lead to the following system of equations:

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA} - \frac{3}{A^2} = \rho + \frac{I^2}{2\bar{\mu}B^2C^2}, \tag{15}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} - \frac{1}{A^2} = -\frac{I^2}{2\bar{\mu}B^2C^2} + \xi\theta, \tag{16}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{1}{A^2} = -\frac{I^2}{2\bar{\mu}B^2C^2} + \xi\theta, \tag{17}$$

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} - \frac{1}{A^2} = \lambda + \frac{I^2}{2\bar{\mu}B^2C^2} + \xi\theta, \tag{18}$$

$$2\frac{\dot{A}}{A}-\frac{\dot{B}}{B}-\frac{\dot{C}}{C}=0, \tag{19}$$

where the over dots indicate ordinary differentiation with respect to t.

Let us define the volume scale factor τ as

$$\tau = ABC. \tag{20}$$

Let us consider the various important physical quantities such as expansion scalar θ, anisotropy parameter Δ and shear scalar σ^2, which describes the dynamics of the Universe during the early and late time evolution. They are defined, respectively as

$$\theta = u^l_{;l} = \frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}=\frac{\dot{\tau}}{\tau}, \tag{21}$$

$$\Delta = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2, \tag{22}$$

$$\sigma^2 = \frac{1}{2}\,\sigma_{ij}\sigma^{ij} = \frac{1}{3}\Delta H^2. \tag{23}$$

Here, $H = \theta/3$ is the mean Hubble parameter and $H_1 = \dot{A}/A$, $H_2 = \dot{B}/B$ and $H_3 = \dot{C}/C$ are the directional Hubble parameters along of x-, y- and z-axes, respectively.

The energy conservation equation $T^j_{i;j} = 0$ takes the form

$$\dot{\rho}+\frac{\dot{\tau}}{\tau}\rho-\frac{\dot{A}}{A}\lambda=\xi\frac{\dot{\tau}^2}{\tau^2}. \tag{24}$$

3. Solution of the field equations

To solve the field equations (15)–(19), we follow the method used by Saha and Visinescu [35]. From (16) and (17) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{C}}{C}-\frac{\dot{B}}{B}\right)+\left(\frac{\dot{C}}{C}-\frac{\dot{B}}{B}\right)\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)=0. \tag{25}$$

Using (20) into (25), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{C}}{C}-\frac{\dot{B}}{B}\right)+\left(\frac{\dot{C}}{C}-\frac{\dot{B}}{B}\right)\frac{\dot{\tau}}{\tau}=0, \tag{26}$$

which on integration, gives

$$C = d_1 B \exp\left(k_1\int\frac{\mathrm{d}t}{\tau}\right), \tag{27}$$

where d_1 and k_1 are constants of integration.

From (19), we get

$$A^2 = d_2 BC, \tag{28}$$

where d_2 is a constant of integration, which is taken as unity without the loss of generality.

From (15)–(18) and (28), we obtain

$$\frac{\ddot{\tau}}{\tau} = \frac{1}{2}\left[(3\rho + \lambda) + \frac{A^2 I^2}{\bar{\mu}\tau^2}\right] + \frac{3}{2}\xi\theta + \frac{6}{\tau^{2/3}}. \tag{29}$$

From eqs (20), (27) and (28), we find the following forms of metric functions in terms of τ:

$$A = \tau^{1/3}, \tag{30}$$

$$B = \frac{1}{\sqrt{d_1}}\,\tau^{1/3}\exp\left[-\frac{k_1}{2}\int\frac{\mathrm{d}t}{\tau}\right], \tag{31}$$

$$C = \sqrt{d_1}\,\tau^{1/3}\exp\left[\frac{k_1}{2}\int\frac{\mathrm{d}t}{\tau}\right]. \tag{32}$$

Now, eqs (24) and (29) take the forms

$$\dot{\rho} + \left(\rho - \frac{\lambda}{3}\right)\frac{\dot{\tau}}{\tau} = \xi\frac{\dot{\tau}^2}{\tau^2} \tag{33}$$

and

$$\ddot{\tau} = \frac{1}{2}(3\rho + \lambda)\tau + \frac{k}{2}\,\tau^{-1/3} + 6\tau^{1/3} + \frac{3}{2}\xi\theta\tau, \tag{34}$$

where $k = I^2/\bar{\mu}$.

In [6,70–74], we have the equation of state (EoS) for a cloud of string models

$$\rho = \alpha\lambda, \tag{35}$$

where the constant α is defined as

$$\begin{aligned}\alpha &= 1 \qquad \text{(geometric or Nambu string)},\\ &= (1+\omega) \qquad \text{(p-string or Takabayasi string)},\\ &= -1 \qquad \text{(Reddy string)},\end{aligned} \tag{36}$$

where ω is a positive constant. It is pointed out by Letelier [6] that λ can be taken as negative or positive. Therefore, inspired by EoS (35) and assuming that λ is negative, to get the determinate solution of the highly non-linear field equations, Reddy [70,71] proposed that the sum of energy density and string tension density may be zero, i.e., $\rho + \lambda = 0$. Reddy [70,71] proposed this condition for the first time, therefore, it is called as Reddy string model, although Letelier [6] did not propose this possibility in his work.

We further assume that the coefficient of bulk viscosity is inversely proportional to the expansion scalar [75], which means that the rate of cosmic expansion decreases as viscosity increases, i.e.,

$$\xi\theta = k_2, \tag{37}$$

where k_2 is a positive constant. From (33) and (35), we get

$$\frac{\dot{\rho}}{(1 - (1/3\alpha))\rho - k_2} = -\frac{\dot{\tau}}{\tau}, \tag{38}$$

which on integration, gives

$$\rho = \frac{3\alpha}{(3\alpha - 1)}\left[k_2 + k_3\tau^{-((3\alpha-1)/3\alpha)}\right], \tag{39}$$

where k_3 is a constant of integration. For $\rho > 0$, we must have either $\alpha > 1/3$ or $\alpha < 0$. It means that all the above three string models (36) may be described by assuming relation (37). Further, inserting ρ from (39) into (34), we find

$$\ddot{\tau} = \frac{3\,(3\alpha+1)\,k_3}{2\,(3\alpha-1)}\tau^{1/3\alpha} + \frac{k}{2}\tau^{-1/3} + 6\tau^{1/3} + \frac{1}{2}\left(\frac{18\alpha}{3\alpha-1}\right)k_2\tau, \tag{40}$$

whose solution is

$$\dot{\tau}^2 = \left(\frac{9\alpha k_3}{3\alpha-1}\right)\tau^{(3\alpha+1)/3\alpha} + \frac{3}{2}k\tau^{2/3} + 9\tau^{4/3} + \left(\frac{9\alpha k_2}{3\alpha-1}\right)\tau^2 + k_4, \tag{41}$$

where k_4 is a constant of integration. Now, eq. (41) can be rewritten as

$$\dot{\tau} = \sqrt{\left(\frac{9\alpha k_3}{3\alpha-1}\right)\tau^{(3\alpha+1)/3\alpha} + \frac{3}{2}k\tau^{2/3} + 9\tau^{4/3} + \left(\frac{9\alpha k_2}{3\alpha-1}\right)\tau^2 + k_4}. \tag{42}$$

Taking into account that the energy and string tension densities obey the EoS (35), we conclude that ρ and λ, i.e., the right-hand side of eq. (34) is a function of τ only, i.e.,

$$\ddot{\tau} = F(\tau). \tag{43}$$

From the mechanical point of view, eq. (43) can be interpreted as EoS of a single particle with unit mass under force $F(\tau)$. Then, the following first integral exists:

$$\dot{\tau} = \sqrt{2[\epsilon - u(\tau)]}, \tag{44}$$

where ϵ can be viewed as energy level and $u(\tau)$ is the potential of the force F. A comprehensive description concerning the potential is found in [76]. Comparing (42) and (44), we find $\epsilon = k_4/2$ and

$$u(\tau) = -\frac{1}{2}\left[\left(\frac{9\alpha k_3}{3\alpha-1}\right)\tau^{(3\alpha+1)/3\alpha} + \frac{3}{2}k\tau^{2/3} + 9\tau^{4/3} + \left(\frac{9\alpha k_2}{3\alpha-1}\right)\tau^2\right]. \tag{45}$$

Finally, we write eq. (42) in a general quadrature form as

$$\int \frac{\mathrm{d}\tau}{\sqrt{\left(\frac{9\alpha k_3}{3\alpha-1}\right)\tau^{(3\alpha+1)/3\alpha} + \frac{3}{2}k\tau^{2/3} + 9\tau^{4/3} + \left(\frac{9\alpha k_2}{3\alpha-1}\right)\tau^2 + k_4}} = t + t_0, \tag{46}$$

where the integration constant t_0 can be taken as zero for simplicity.

4. Solution of various string models

We observe that it is too difficult to solve (46), in general. Therefore, we present the following three string models depending on the values of α as defined in (36).

4.1 *Geometric string model* ($\alpha = 1$)

Let us find the solution for viscous fluid with and without magnetic field in the following sections.

4.1.1 *Viscous fluid solution with magnetic field.*

For $k_4 = 0$, eq. (46) reduces to

$$\int \frac{\mathrm{d}\tau}{\sqrt{\frac{9}{2}(k_3+2)\,\tau^{4/3} + \frac{3}{2}k\tau^{2/3} + \frac{9}{2}k_2\tau^2}} = t. \tag{47}$$

For $3(k_3+2)^2 < 4kk_2$, on integration eq. (47) gives

$$\tau = \left[\frac{1}{2k_2}\sqrt{\frac{4kk_2 - 3(k_3+2)^2}{3}}\ \sinh\left(\sqrt{2k_2}\ t\right) - \frac{k_3+2}{2k_2}\right]^{3/2}, \qquad k_2 \neq 0, \tag{48}$$

and when $3(k_3+2)^2 > 4kk_2$, eq. (47) gives

$$\tau = \left[\frac{1}{2k_2}\sqrt{\frac{3(k_3+2)^2 - 4kk_2}{3}}\ \cosh\left(\sqrt{2k_2}\ t\right) - \frac{k_3+2}{2k_2}\right]^{3/2}, \qquad k_2 \neq 0. \tag{49}$$

For small t, we have $\sinh\left(\sqrt{2k_2}\ t\right) \approx \sqrt{2k_2}\ t$. Therefore, eq. (48) can be written as

$$\tau = (P_1 t - Q_1)^{3/2}, \tag{50}$$

where

$$P_1 = \sqrt{\frac{4kk_2 - 3(k_3+2)^2}{6k_2}} \qquad \text{and} \qquad Q_1 = \frac{k_3+2}{2k_2}.$$

At $t = 0$, τ becomes imaginary. Thus, for the model to be real, t must satisfy $t > (Q_1/P_1)$.

Using (50), eqs (30)–(32) take the forms

$$A = \sqrt{P_1 t - Q_1}, \tag{51}$$

$$B = \frac{1}{\sqrt{d_1}}\sqrt{P_1 t - Q_1}\ \exp\left[\frac{k_1}{P_1\sqrt{P_1 t - Q_1}}\right], \tag{52}$$

$$C = \sqrt{d_1}\sqrt{P_1 t - Q_1}\ \exp\left[-\frac{k_1}{P_1\sqrt{P_1 t - Q_1}}\right]. \tag{53}$$

The directional Hubble parameters along x-, y- and z-axes are, respectively given by

$$H_1 = \frac{P_1}{2(P_1 t - Q_1)}, \qquad H_2 = \frac{P_1}{2(P_1 t - Q_1)} - \frac{k_1}{2(P_1 t - Q_1)^{3/2}},$$
$$H_3 = \frac{P_1}{2(P_1 t - Q_1)} + \frac{k_1}{2(P_1 t - Q_1)^{3/2}}. \tag{54}$$

The average Hubble parameter in terms of cosmic time t becomes

$$H = \frac{P_1}{2(P_1 t - Q_1)}. \tag{55}$$

The anisotropic parameter and shear scalar, respectively have the following expressions:

$$\Delta = \frac{2k_1^2}{3P_1^2(P_1 t - Q_1)}, \quad \sigma^2 = \frac{k_1^2}{4(P_1 t - Q_1)^3}. \tag{56}$$

We observe that the scale factors increase with time for $t > (Q_1/P_1)$. The other physical parameters (54)–(56) diverse at $t = Q_1/P_1$ and tend to be zero as $t \to \infty$. From (55) and (56), we get

$$\frac{\sigma}{\theta} = \frac{k_1}{3P_1\sqrt{(P_1 t - Q_1)}}, \tag{57}$$

which is time-dependent and tends to zero as $t \to \infty$, which shows that the model becomes isotropic in late time. The deceleration parameter $q = -\tau\ddot{\tau}/\dot{\tau}^2$ gives $q = 1$, i.e., a positive constant. Therefore, the present model expands with the decelerated rate throughout the evolution which means the strings dominate over the particles.

The energy density and string tension density are given by

$$\rho = \lambda = \frac{3}{2}\left(k_2 + \frac{k_3}{P_1 t - Q_1}\right), \tag{58}$$

which shows that the matter behaves as a cloud of geometric strings. The particle density ρ_p remains zero throughout the evolution. We observe that ρ and λ remain positive throughout the evolution and become infinite at the initial epoch at $t = Q_1/P_1$. However, ρ and λ decrease with time for $t > Q_1/P_1$ and approach a constant value, $\frac{3}{2}k_2$ as $t \to \infty$. We observe that this constant value of ρ is due to the viscous term, k_2. It means that the viscosity parameter stops the Universe from becoming empty at late times of its evolution.

The classical potential (45), in terms of t takes the form

$$u(t) = -\frac{3}{4}\left[3k_2(P_1 t - Q_1)^3 + 3(k_3 + 2)(P_1 t - Q_1)^2 + k(P_1 t - Q_1)\right]. \tag{59}$$

4.1.2 *Viscous fluid solution without magnetic field.* For $\alpha = 1$ and in the absence of magnetic field ($k = 0$), eq. (46) becomes

$$\int \frac{\mathrm{d}\tau}{\sqrt{\frac{9}{2}(k_3 + 2)\,\tau^{4/3} + \frac{9}{2}k_2\tau^2}} = t, \tag{60}$$

which on integration gives

$$\tau = \left[\frac{(k_3 + 2)}{2k_2}\left(\cosh(\sqrt{2k_2}\,t) - 1\right)\right]^{3/2}, \qquad k_2 \neq 0. \tag{61}$$

We find that the solution (49) also gives the same solution (61) in the absence of magnetic field, i.e., $k = 0$. Therefore, the solution (49) may be considered for viscous fluid without magnetic field.

For small t, eq. (61) gives

$$\tau = \left(\frac{k_3+2}{2}\right)^{3/2} t^3. \tag{62}$$

Therefore, the scale factors in terms of t have the following solutions:

$$A = \sqrt{\left(\frac{k_3+2}{2}\right)}\, t, \tag{63}$$

$$B = \frac{1}{\sqrt{d_1}}\sqrt{\left(\frac{k_3+2}{2}\right)}\, t \exp\left[\frac{k_1}{4}\left(\frac{2}{k_3+2}\right)^{3/2} \frac{1}{t^2}\right], \tag{64}$$

$$C = \sqrt{d_1}\sqrt{\left(\frac{k_3+2}{2}\right)}\, t \exp\left[-\frac{k_1}{4}\left(\frac{2}{k_3+2}\right)^{3/2} \frac{1}{t^2}\right]. \tag{65}$$

The directional Hubble parameters along x-, y- and z-axes are, respectively given by

$$H_1 = \frac{1}{t}, \qquad H_2 = \frac{1}{t} - \frac{k_1}{4}\left(\frac{2}{k_3+2}\right)^{3/2} \frac{1}{t},$$
$$H_3 = \frac{1}{t} + \frac{k_1}{4}\left(\frac{2}{k_3+2}\right)^{3/2} \frac{1}{t}. \tag{66}$$

The average Hubble parameter is

$$H = \frac{1}{t}. \tag{67}$$

The anisotropy parameter and shear scalar, respectively take the form

$$\Delta = \frac{k_1^2}{3(k_3+2)^3}, \qquad \sigma^2 = \frac{k_1^2}{9(k_3+2)^3}\,\frac{1}{t^2}. \tag{68}$$

From the above solutions, we observe that the Universe starts with $t = 0$ and expands in all directions for $t > 0$. The rate of expansion in each direction diverse at $t = 0$ and tends to zero as $t \to \infty$. The anisotropy parameter becomes constant throughout the evolution of the Universe whereas the shear scalar varies inversely with the square of time. The model remains anisotropic throughout the evolution as the ratio of $\sigma/\theta = 1/9(k_3+2)^{3/2}$, i.e., a constant.

The deceleration parameter q is zero which shows margin inflation during the expansion. Therefore, we can say that the particles dominate over the strings in this case. It is worth mentioning that the astronomical observations predict that there is no direct evidence of strings in the present day Universe.

The energy and string tension densities are given by

$$\rho = \lambda = \frac{3}{2}\left(k_2 + \frac{2k_3}{k_3+2}\,\frac{1}{t^2}\right). \tag{69}$$

We find that the energy density remains positive throughout the evolution of the Universe. It is infinite at initial epoch and decreases with time, and ultimately attains

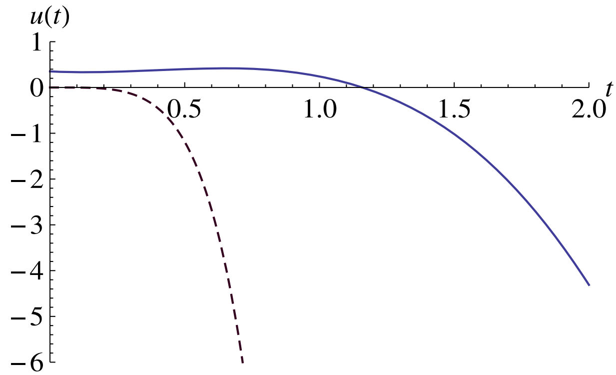

Figure 1. Classical potential vs. time for viscous fluid with (solid line) and without (dashed line) magnetic field.

a constant value, $3k_2/2$ in late time expansion of the Universe. Therefore, the bulk viscosity prevents the Universe to be empty during late time of evolution. From (58) and (69) we find that the energy density attains the same constant value with and without magnetic field in late time of evolution.

The classical potential (45) in terms of t takes the form

$$u(t) = -\frac{9(k_3+2)^3}{32}(2+k_2t^2)t^4. \tag{70}$$

Figure 1 plots the potential with respect to time in the presence of viscous fluid and magnetic field (solid line) and with only viscous fluid (dashed line). We have used the numerical values of various constants as $k_2 = 2$, $k_3 = 1$, $I = 2$ or 0 and $\bar{\mu} = 1.00001$. We observe that $\mu(t)$ shows positive and negative nature with respect to time t in magnetized viscous fluid. However, it always has negative value and decreases rapidly with time in the presence of viscous fluid, which may be acceptable for our model.

4.2 *Reddy string model ($\alpha = -1$)*

Let us find the solution for viscous fluid with and without magnetic field in the following sections for Reddy string model.

4.2.1 *Viscous fluid solution with magnetic field.*

For $k_4 = 0$, eq. (46) becomes

$$\int \frac{\mathrm{d}\tau}{\sqrt{9\tau^{4/3} + \frac{3}{4}(3k_3+2k)\tau^{2/3} + \frac{9}{4}k_2\tau^2}} = t. \tag{71}$$

For $k_2\,(3k_3+2k) < 12$, on integration (71), we obtain

$$\tau = \left[\frac{1}{k_2}\sqrt{\frac{12-k_2(3k_3+2k)}{3}}\,\cosh\left(\sqrt{k_2}\,t\right) - \frac{2}{k_2}\right]^{3/2} \tag{72}$$

and for $k_2\,(3k_3+2k) > 12$, eq. (71) gives

$$\tau = \left[\frac{1}{k_2}\sqrt{\frac{k_2(3k_3+2k)-12}{3}}\,\sinh\left(\sqrt{k_2}\,t\right) - \frac{2}{k_2}\right]^{3/2}. \tag{73}$$

For small t, we have $\sinh\left(\sqrt{k_2}\, t\right) \approx \sqrt{k_2}\, t$, and therefore, (73) takes the form

$$\tau = (P_2 t - Q_2)^{3/2}, \tag{74}$$

where

$$P_2 = \sqrt{\frac{1}{3k_2}[k_2(3k_3 + 2k) - 12]} \qquad \text{and} \qquad Q_2 = \frac{2}{k_2}.$$

At $t = 0$, τ becomes imaginary. For reality of the model, t must satisfy $t > Q_2/P_2$. From eqs (30)–(32) and (74), we find

$$A = \sqrt{P_2 t - Q_2}, \tag{75}$$

$$B = \frac{1}{\sqrt{d_1}}\sqrt{P_2 t - Q_2}\ \exp\left[\frac{k_1}{P_2\sqrt{P_2 t - Q_2}}\right], \tag{76}$$

$$C = \sqrt{d_1}\sqrt{P_2 t - Q_2}\ \exp\left[-\frac{k_1}{P_2\sqrt{P_2 t - Q_2}}\right]. \tag{77}$$

The directional Hubble parameters along x-, y- and z-axes are, respectively given by

$$H_1 = \frac{P_2}{2(P_2 t - Q_2)}, \qquad H_2 = \frac{P_2}{2(P_2 t - Q_2)} - \frac{k_1}{2(P_2 t - Q_2)^{3/2}},$$
$$H_3 = \frac{P_2}{2(P_2 t - Q_)} + \frac{k_1}{2(P_2 t - Q_2)^{3/2}}. \tag{78}$$

The average Hubble parameter in terms of cosmic time t is

$$H = \frac{P_2}{2(P_2 t - Q_2)}. \tag{79}$$

The anisotropic parameter and shear scalar, respectively take the forms

$$\Delta = \frac{2k_1^2}{3P_2^2(P_2 t - Q_2)}, \qquad \sigma^2 = \frac{k_1^2}{4(P_2 t - Q_2)^3}. \tag{80}$$

The above physical parameters in eqs. (78)–(80) diverge at $t = Q_2/P_2$ and asymptotically tend to zero as $t \to \infty$. From (79) and (80), we get

$$\frac{\sigma}{\theta} = \frac{k_1}{3P_2\sqrt{(P_2 t - Q_2)}} \tag{81}$$

which is time-dependent and tends to zero as $t \to \infty$. Therefore, this string model also becomes isotropic for large t. The energy, string tension and particle densities are, respectively given by

$$\rho = \frac{3}{4}\left[k_2 + \frac{k_3}{(P_2 t - Q_2)^2}\right], \quad \lambda = -\frac{3}{4}\left[k_2 + \frac{k_3}{(P_2 t - Q_2)^2}\right],$$
$$\rho_{\rm p} = \frac{3}{2}\left[k_2 + \frac{k_3}{(P_2 t - Q_2)^2}\right]. \tag{82}$$

We find that ρ and $\rho_{\rm p}$ are infinite at $t = Q_2/P_2$ and tend to constant values, $3k_2/4$ and $3k_2/2$, respectively as $t \to \infty$. The constant values of ρ and $\rho_{\rm p}$ in the later stages of evolution are due to the bulk viscosity. The string tension density λ remains negative and gradually increases with time and finally it approaches a constant value as $t \to \infty$. We observe that the ratio of strings and particles in the Universe, i.e., $\rho_{\rm p}/|\lambda|$ turns out to be $\rho_{\rm p} = 2|\lambda|$.

The potential in terms of t is written as

$$u(t) = -\frac{3}{8}\left[3k_2(P_2 t - Q_2)^3 + 12(P_2 t - Q_2)^2 + (3k_3 + 2k)(P_2 t - Q_2)\right]. \tag{83}$$

The nature of different density parameters and $u(t)$ are shown in figures 2 and 3, respectively.

4.2.2 *Viscous fluid solution without magnetic field.* For $\alpha = -1$ and in the absence of magnetic field ($k = 0$), eq. (46) gives

$$\int \frac{{\rm d}\tau}{\sqrt{9\tau^{4/3} + (9k_3/4)\tau^{2/3} + (9/4)k_2\tau^2}} = t. \tag{84}$$

For $k_2 k_3 < 4$, on integration (84), we obtain

$$\tau = \left[\frac{1}{k_2}\sqrt{4 - k_2 k_3}\ \cosh\left(\sqrt{k_2}\ t\right) - \frac{2}{k_2}\right]^{3/2} \tag{85}$$

and for $k_2 k_3 > 4$, eq. (84) gives

$$\tau = \left[\frac{1}{k_2}\sqrt{k_2 k_3 - 4}\ \sinh\left(\sqrt{k_2}\ t\right) - \frac{2}{k_2}\right]^{3/2}. \tag{86}$$

One may observe that eqs (85) and (86) are the same as the ones directly obtained by putting $k = 0$ in (72) and (73), respectively. Therefore, the solution of various physical parameters and their physical significance of this viscous solution may be discussed in §4.2.1 by taking $k = 0$.

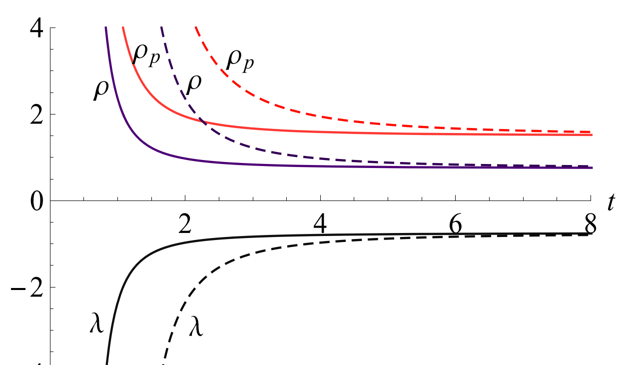

Figure 2. Density parameters vs. time for viscous fluid with (solid lines) and without (dashed lines) magnetic field.

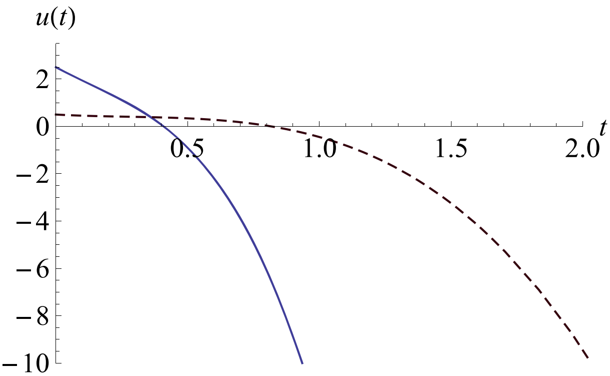

Figure 3. Potential vs. time for viscous fluid with (solid line) and without (dashed line) magnetic field.

For small t, eq. (86) gives

$$\tau = \left[\sqrt{\frac{k_2k_3-4}{k_2}}\,t - \frac{2}{k_2}\right]^{3/2}, \tag{87}$$

which is the same as the solution (74) in the absence of magnetic field ($k = 0$).

Figure 2 plots the graph of energy density, energy tension density and particle density with respect to time in the presence of magnetized viscous fluid (solid lines) and non-magnetized viscous fluid only (dashed lines) for $k_2 = 1$, $k_3 = 2$, $I = 2$ and $\bar{\mu} = 1.00001$. The figure shows that the energy and particle densities are positive throughout the evolution and decrease with time whereas the energy tension density is negative, increases with time and attains a constant value in late time in both cases. It is interesting to note that these parameters attain the same value, respectively in both the cases in late time evolution. The behaviour of classical potential with time in both cases is shown in figure 3 for $k_2 = 3$, $k_3 = 2$, $I = 2$ or 0 and $\bar{\mu} = 1.00001$. The figure shows that $\mu(t)$ is positive and negative in nature in both the cases but decreases rapidly in the former case.

4.3 *Takabayasi string model* ($\alpha = 1 + \omega$)

For $k_4 = 0$, eq. (46) reduces to

$$\int \frac{\mathrm{d}\tau}{\sqrt{\frac{9(1+\omega)k_3}{3\omega+2}\tau^{(3\omega+4)/(3\omega+3)} + \frac{3k}{2}\tau^{2/3} + 9\tau^{4/3} + \frac{9(1+\omega)k_2}{3\omega+2}\tau^2}} = t. \tag{88}$$

One can observe that it is very difficult to find a general solution of τ in terms of t. Therefore, we express ρ, λ and ρ_{p} in terms of τ as

$$\rho = \frac{3(1+\omega)}{3\omega+2}\left[k_2 + k_3\tau^{-(2+3\omega)/3(1+\omega)}\right], \tag{89}$$

$$\lambda = \frac{3}{3\omega+2}\left[k_2 + k_3\tau^{-(2+3\omega)/3(1+\omega)}\right], \tag{90}$$

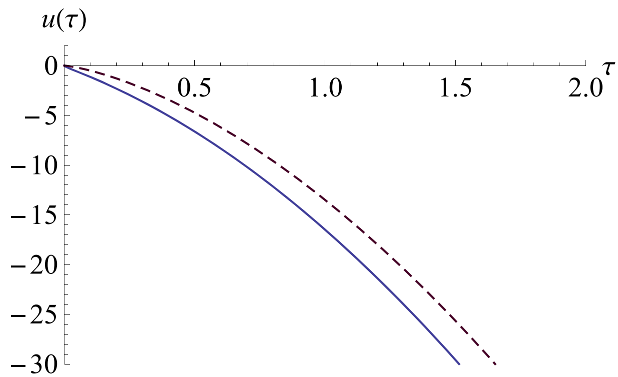

Figure 4. Potential vs. time for viscous fluid solution with (solid line) and without (dashed line) magnetic field.

$$\rho_p = \frac{3\omega}{3\omega+2}\left[k_2 + k_3\tau^{-(2+3\omega)/3(1+\omega)}\right]. \tag{91}$$

As $\omega > 0$, we find that ρ, λ and ρ_{p} remain positive throughout the evolution of the Universe. These physical parameters are decreasing functions of τ and become constant for large t due to the bulk viscosity. The geometrical string model may be recovered for $\omega = 0$ as discussed in §4.1. The classical potential is given by

$$u(\tau) = -\frac{3}{2}\left[\frac{3k_3(1+\omega)}{3\omega+2}\tau^{(4+3\omega)/3(1+\omega)} + \frac{k}{2}\tau^{2/3} + 3\tau^{4/3} + \frac{3k_2(1+\omega)}{3\omega+2}\tau^2\right]. \tag{92}$$

Figure 4 illustrates the behaviour of potential $\mu(t)$ with respect to τ for $k_2 = 3$, $k_3 = 2$, $I = 2$, or 0, $\bar{\mu} = 1.00001$ and $\omega = 1$. It is clear that the potential remains negative and decreases rapidly throughout the evolution of the Universe for viscous fluid with and without magnetic field. It decreases rapidly due to the viscous fluid only.

5. Conclusions

We have studied anisotropic Bianchi-V string cosmological model with viscous fluid and magnetic field in general relativity by taking certain physical assumptions. As viscous fluid and magnetic field have cosmological origin, it is interesting to discuss the viscous and magnetic field effects on the expansion history of the Universe in early and late stages of evolution in string cosmology. The Einstein's field equations have been solved exactly for geometrical and Reddy string models for viscous fluid with and without magnetic field whereas a general quadrature form of the average scale factor has been found in Takabayasi string model. The solutions present interesting features in the presence of viscous fluid and magnetic field and in the presence of viscous fluid only. We have found that the viscous term affects the solutions much rapidly compared to magnetic field. We have also analysed the behaviour of various physical parameters graphically in the presence of magnetized viscous fluid and bulk viscous fluid without magnetic field.

As we know, Hubble parameter and the deceleration parameter are some of the observational cosmological parameters which describe the dynamics of the Universe during early and late time evolution. As mentioned in the Introduction, since strings are not observed at the present time of evolution of the Universe, in principle we can eliminate the strings and end up with a cloud of particles. In other words, we can say that the particles dominate over the strings at the present time of the evolution of the Universe. The result is summarized as follows.

In geometrical string model, we have observed that proper energy density remains positive throughout the evolution and attains the finite constant value during late time for viscous fluid with and without magnetic field. Hence, the presence of viscous term prevents to be empty in its future evolution. The classical potential changes its behaviour rapidly due to the bulk viscous term. It is negative throughout the evolution in the absence of magnetic field but it is positive for some finite time and after that it shows the negative nature during late time.

In Reddy string model, the proper energy density and particle density remain positive throughout the evolution and attains some finite constant value during late time for viscous fluid with and without magnetic field. The string tension density is always negative and increases with time, and turns out to be finite during late time. At initial epoch, these physical parameters are infinite while they will be finite in late time due to bulk viscous effects in both cases.

In the above two string models the other physical and kinematical parameters such as the respective directional Hubble parameters, anisotropic parameter and shear scalar σ in both the string models are infinite at initial epoch and approach zero asymptotically. The models approach isotropic at later stages of the evolution of the Universe for viscous fluid with magnetic field whereas these models are always anisotropic for viscous fluid without magnetic field. The deceleration parameter turns out to be a positive constant in the presence of viscous fluid with magnetic field which shows that the Universe expands with decelerated rate. The string might be dominant over the particles but there is no such observational evidence in the present day Universe. It is zero in the case of viscous fluid only showing marginal inflation during the expansion. This means that the particles dominate over the strings which is in agreement with the results obtained in [77].

In Takabayasi model we have found a quadrature form of average volume which is too difficult to solve, in general. The solution for the proper energy density, string tension density and particle energy density in terms of average scale factor has been represented. These physical parameters approach a constant value asymptotically as $t \to \infty$ due to the bulk viscosity.

We have also discussed the classical potential with respect to time in each string model and have observed that the classical potential changes its behaviour rapidly due to the bulk viscous term. In geometrical string model it remains negative and decreases rapidly in viscous fluid solution without magnetic field. In Takabayasi string model it remains negative in both cases but the effect of viscous fluid with and without magnetic field varies rapidly. Thus, we conclude that the bulk viscous fluid with and without magnetic field plays an important role in the evolution of the Universe in anisotropic models as analysed in this paper.

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References

[1] T W B Kibble, *J. Phys. A* **9**, 1387 (1976)
[2] A E Everett, *Phys. Rev. D* **24**, 858 (1981)
[3] A Vilenkin, *Phys. Rev. D* **24**, 2082 (1981)
[4] P S Letelier, *Phys. Rev. D* **20**, 1294 (1979)
[5] J Stachel, *Phys. Rev. D* **21**, 2171 (1980)
[6] P S Letelier, *Phys. Rev. D* **28**, 2414 (1983)
[7] D R Matraverse, *Gen. Relativ. Grav.* **20**, 279 (1988)
[8] K D Krori, T Chaudhury, C R Mahanta and A Mazumdar, *Gen. Relativ. Grav.* **22**, 123 (1990)
[9] I Yavuz and I Tarhan, *Astrophys. Space Sci.* **240**, 45 (1996)
[10] R Bali and S Dave, *Pramana – J. Phys.* **56**, 513 (2001)
[11] R Bali and S Dave, *Astrophys. Space Sci.* **288**, 503 (2003)
[12] R Bali and R D Upadhaya, *Astrophys. Space Sci.* **283**, 97 (2003)
[13] R Bali and D K Singh, *Astrophys. Space Sci.* **300**, 387 (2005)
[14] R Bali and A Pradhan, *Chin. Phys. Lett.* **24**, 585 (2007)
[15] A Pradhan and D S Chouhan, *Astrophys. Space Sci.* **331**, 697 (2011)
[16] K L Mahanto, S K Biswal, S K Sahoo and M C Adhikary, *Int. J. Theor. Phys.* **51**, 1538 (2012)
[17] E Asseo and H Sol, *Phys. Rep.* **6**, 148 (1987)
[18] M S Madsen, *Mon. Not. R. Astron. Soc.* **237**, 109 (1989)
[19] A M Wolfe, K Lanzetta and A L Oren, *Astrophys. J.* **388**, 17 (1992)
[20] R Kulsrud, R Cen, J P Ostriker and D Ryu, *Astrophys. J.* **380**, 481 (1997)
[21] J D Barrow, *Phys. Rev. D* **55**,7451 (1997)
[22] D R Matravers and C G Tsagas, *Phys. Rev. D* **62**, 103519 (2000)
[23] A Banerjee, A K Sanyal and S Chakraborty, *Pramana – J. Phys.* **34**, 1 (1990)
[24] S Chakraborty, *Ind. J. Pure Appl. Phys.* **29**, 31 (1991)
[25] R Tikekar and L K Patel, *Gen. Relativ. Grav.* **24**, 397 (1992)
[26] R Tikekar and L K Patel, *Pramana – J. Phys.* **42**, 483 (1994)
[27] ShriRam and J K Singh, *Gen. Relativ. Grav.* **27**, 1207 (1995)
[28] L K Patel and S D Maharaj, *Pramana – J. Phys.* **47**, 1 (1996)
[29] G P Singh and T Singh, *Gen. Relativ. Grav.* **31**, 371 (1999)
[30] J K Singh and ShriRam, *Astrophys. Space Sci.* **246**, 65 (1997)
[31] C B Kilinç and I Yavuz, *Astrophys. Space Sci.* **271**, 11 (2000)
[32] A Pradhan, A Rai and S K Singh, *Astrophys. Space Sci.* **312**, 261 (2007)
[33] A Pradhan, *Fizika B* **16**, 205 (2007)
[34] R Bali and S Jain, *Int. J. Mod. Phys. D* **16**, 1769 (2007)
[35] B Saha and M Visinescu, *Astrophys. Space Sci.* **315**, 99 (2008)
[36] A Pradhan, K Jatonia and A Singh, *Braz. J. Phys.* **38**, 167 (2008)
[37] A Pradhan, V Rai and K Jatonia, *Commun. Theor. Phys.* **50**, 279 (2008)
[38] B Saha, V Rikhvitsky and M Visinescu, *Cent. Eur. J. Phys.* **8**, 113 (2010)
[39] A Pradhan, H Amirhashchi and H Zainuddin, *Int. J. Theor. Phys.* **50**, 56 (2011)
[40] H Amirhashchi, H Zainuddin and A Pradhan, *Int. J. Theor. Phys.* **50**, 2531 (2011)
[41] V Rikhvitsky, B Saha and M Visinescu, *Astrophys. Space. Sci.* **339**, 371 (2012)
[42] C Eckart, *Phys. Rev.* **58**, 919 (1940)

[43] G L Murphy, *Phys. Rev. D* **8**, 4231 (1973)
[44] V A Belinskii and I M Khalatrikov, *Soviet Phys. JETP* **42**, 205 (1976)
[45] W H Huang, *Phys. Lett. A* **129**, 429 (1988)
[46] W Zimdahl, *Phys. Rev. D* **53**, 5483 (1996)
[47] W Zimdahl, *Phys. Rev. D* **61**, 083511 (2000)
[48] L P Chimento, A S Jakubi, V Méndez and R Maartens, *Class. Quantum Grav.* **14**, 3363 (1997)
[49] M D Mak and T Harko, *J. Math. Phys.* **39**, 5458 (1998)
[50] R Maartens, *Class. Quantum Grav.* **12**, 1455 (1995)
[51] C P Singh, *Pramana – J. Phys.* **71**, 33 (2008)
[52] M K Yadav, A Rai and A Pradhan, *Int. J. Theor. Phys.* **46**, 2677 (2007)
[53] G Mohanty and C S Gauranga, *Tur. J. Phys.* **32**, 251 (2008)
[54] G F R Ellis, *Gen. Relativ. Grav.* **38**, 1003 (2006)
[55] A A Coley, *Gen. Relativ. Grav.* **22**, 3 (1990)
[56] C P Singh, M Zeyauddin and ShriRam, *Astrophys. Space Sci.* **315**, 181 (2008)
[57] C P Singh, M Zeyauddin and ShriRam, *Int. J. Mod. Phys. A* **23**, 2719 (2008)
[58] C P Singh, M Zeyauddin and ShriRam, *Int. J. Theor. Phys.* **47**, 3162 (2008)
[59] C P Singh and A Beesham, *Int. J. Mod. Phys. A* **25**, 3825 (2010)
[60] C P Singh, *Braz. J. Phys.* **41**, 323 (2011)
[61] N C Chakraborty and S Chakraborty, *Int. J. Mod. Phys. D* **10**, 723 (2001)
[62] R Bali, *Eur. J. Theor. Phys.* **5**, 105 (2008)
[63] R Bali, U K Parrek and A Pradhan, *Chin. Phys. Lett.* **24**, 2455 (2007)
[64] R Bali, R Banerjee and S K Banerjee, *Astrophys. Space Sci.* **317**, 21 (2008)
[65] M Sharif and S Waheed, *Int. J. Mod. Phys. D* **55**, 21500 (2012)
[66] C P Singh, *Astrophys. Space Sci.* **343**, 773 (2013)
[67] L D Landuam and E M Lifshitz, *Fluid mechanics* (Pergamon Press, Oxford, 1936) p. 505
[68] A Linchnerowicz, *Relativistic hydrodynamics and magnetohydrodynamics* (Benjamin, New York, 1967) p. 13
[69] R Maartens, *Pramana – J. Phys.* **55**, 575 (2000)
[70] D R K Reddy, *Astrophys. Space Sci.* **286**, 365 (2003)
[71] D R K Reddy, *Astrophys. Space Sci.* **286**, 397 (2003)
[72] D R K Reddy and M V S Rao, *Astrophys. Space Sci.* **305**, 139 (2006)
[73] B Saha, *Astrophys. Space Sci.* **299**, 149 (2005)
[74] B Saha and M Visinescu, *Int. J. Theor. Phys.* **49**, 1411 (2010)
[75] B Saha, *Mod. Phys. Lett. A* **20**, 2127 (2005)
[76] B Saha and T Boyadjiev, *Phys. Rev. D* **69**, 124010 (2004)
[77] J A Belinchon, *Astrophys. Space Sci.* **323**, 307 (2009)