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Infinitely-many conservation laws for two (2+1)-dimensional nonlinear evolution equations in fluids

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Abstract. In this paper, a method that can be used to construct the infinitely-many conservation laws with the Lax pair is generalized from the (1+1)-dimensional nonlinear evolution equations (NLEEs) to the (2+1)-dimensional ones. Besides, we apply that method to the Kadomtsev–Petviashvili (KP) and Davey–Stewartson equations in fluids, and respectively obtain their infinitely-many conservation laws with symbolic computation. Based on that method, we can also construct the infinitely-many conservation laws for other multidimensional NLEEs possessing the Lax pairs, including the cylindrical KP, modified KP and (2+1)-dimensional Gardner equations, in fluids, plasmas, optical fibres and Bose–Einstein condensates.

Keywords. Infinitely-many conservation laws; (2+1)-dimensional nonlinear evolution equations; Kadomtsev–Petviashvili equation; Davey–Stewartson equation; Lax pair; symbolic computation.

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1. Introduction

Nonlinear evolution equations (NLEEs) and solitons have their applications in, e.g., fluid dynamics, plasma physics, optical fibres and Bose–Einstein condensates [1–4]. For example, the Kadomtsev–Petviashvili (KP) equation can be used to describe, in fluids and plasmas, nonlinear long waves of small amplitude with slow dependence on the transverse coordinate [5,6], and the Davey–Stewartson (DS) equations can model the evolution of weakly nonlinear packets of water waves of the finite depth that travel in one direction but whose amplitudes are modulated in two spatial directions [7]. Consequently, some NLEEs have been investigated in such aspects as the Painlevé property, Lax pairs, Bäcklund transformations (BTs), Darboux transformations, multiple exponential function method and conservation laws [8–14]. Conservation laws mean that certain physical quantities such as the mass, momentum, energy and electric charge will not change with the time during the physical processes [15]. In the context of mathematics, conservation laws are the scalar partial differential equations that describe the quantity affected by the associated flux in a close region [16]. Besides, conservation laws have some applications in the study of the qualitative properties such as the bi- or tri-Hamiltonian structures, Liouville integrability and recursion operators [15–17], and in the theory of non-classical transformations, normal forms and asymptotic integrability [18]. Conservation laws are also helpful to design numerical methods for the NLEEs, which can lead to the stable numerical schemes and be free of nonlinear instabilities and blow-up [15,19]. Solitonic investigation on the conservation laws may promote techniques to solve the NLEEs [15–21]. For example, Miura transformation, Lax pair and inverse scattering technique are discovered as a result of the investigation on the conservation laws for the Korteweg–de Vries (KdV) equation [16,18,20,21].

The $(n+1)$-dimensional NLEEs can be expressed as

$$G[\mathbf{u}^{(p)}(\mathbf{x},t)] = 0, \tag{1}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and t respectively denote the space and time variables, x_i's are the spatial coordinates $(i = 1, 2, \ldots, n)$, $\mathbf{u}^{(p)}$ stands for the dependent variable $\mathbf{u} = (u_1, u_2, \ldots, u_m)$ and its partial derivatives with respect to $\mathbf{x}$ and t, u_i's are the components of the vector function $\mathbf{u}$ $(i = 1, 2, \ldots, m)$, p denotes the order of the derivation, G is a smooth operator which does not explicitly depend on $\mathbf{x}$ and t, and has no restrictions on the number of components, order, and degree of nonlinearity [15,16,20]. Correspondingly, conservation laws for eq. (1) have the following form [12–24]:

$$D_t\rho + \nabla \cdot \mathbf{J} = 0, \tag{2}$$

where the scalar function $\rho = \rho\big[\mathbf{x}, t, \mathbf{u}^{(l)}(\mathbf{x},t)\big]$ is the conserved density and vector function $\mathbf{J} = \mathbf{J}\big[\mathbf{x}, t, \mathbf{u}^{(k)}(\mathbf{x},t)\big] = (J_1, J_2, \ldots, J_n)$ is the associated flux in a close region, the superscript l and k denote the orders of the derivation, D_t is the total derivative operator with respect to the time t as follows:

$$D_t\rho = \frac{\partial\rho}{\partial t} + \sum_{i=1}^{n}\sum_{s=0}^{\tau_i} u_{i,(s+1)t}\frac{\partial\rho}{\partial u_{i,st}}, \tag{3}$$

where τ_i's are the orders of the derivation on the components u_i's and ∇ is the total divergence operator as below:

$$\nabla \cdot \mathbf{J} = D_{x_1}J_1 + D_{x_2}J_2 + \cdots + D_{x_n}J_n, \tag{4}$$

where D_{x_i}'s are the total derivative operators.

Nowadays, several methods have been used to obtain the conservation laws [12–15,18,24,25]. Infinitely-many conservation laws have been constructed [12] for the (1+1)-dimensional NLEEs such as the KdV, sine-Gordon and modified KdV equations through the Lax pairs and BTs. An algorithmic method has been presented [13] for finding the conservation laws. Different methods have been introduced to construct the conservation laws, and applied to some NLEEs [14]. A method that construct the conservation laws with the calculus, variational calculus and linear algebra for the multi-dimensional NLEEs has been given [15]. Ways of obtaining the conservation laws with respect to the complexity and other features have been compared [18]. Neutral-action method for deriving the conservation laws by virtue of the concept of symmetry has been introduced [24]. Noether's theorem has also been applied to obtain the conservation laws [25].

Different from the aforementioned methods for constructing the infinitely-many conservation laws for the (2+1)-dimensional NLEEs, in this paper, we shall generalize a way, which has been used to construct the infinitely-many conservation laws with the Lax pairs for the (1+1)-dimensional NLEEs [12], to the (2+1)-dimensional ones. In §2, we shall respectively obtain the infinitely-many conservation laws for the KP and DS equations with symbolic computation [26]. Conclusions are presented in §3.

2. Infinitely-many conservation laws

Generally, the existence of the infinitely-many conservation laws for the NLEEs denotes the complete integrability, i.e., the NLEEs can be solved through the inverse scattering technique and possess multisoliton solutions [14–16]. In this section, we shall generalize a way, which has been used to construct the infinitely-many conservation laws with the Lax pairs for the (1+1)-dimensional NLEEs [12], to the (2+1)-dimensional ones, and respectively construct the infinitely-many conservation laws for the KP and DS equations.

2.1 *Infinitely-many conservation laws for the KP equation*

The KP equation in fluids and plasmas, which can describe the nonlinear, long waves of small amplitude with slow dependence on the transverse coordinate, has the following normalized form [5,6]:

$$(u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0, \tag{5}$$

where the elevation u is a function of the longitudinal coordinate x, transverse coordinates y and time t, the subscripts denote partial derivatives, and $\sigma^2 = \pm 1$, depending on the relevant magnitude of gravity and surface tension. When $\sigma^2 = -1$, eq. (5) is known as the KPI equation, whereas when $\sigma^2 = 1$, eq. (5) is known as the KPII equation [5,6].

The Lax pair for eq. (5) is as follows [5,6]:

$$\sigma\phi_y + \phi_{xx} + u\phi = 0, \tag{6a}$$

$$\phi_t + 4\phi_{xxx} + 6u\phi_x + 3u_x\phi - 3\sigma v\phi = 0, \tag{6b}$$

where $v_x = u_y$. It is worth noting that although an infinite set of conserved quantities have been presented for eq. (5) through the Lax pair [27], its procedure is completely different from ours. Next, we shall construct the infinitely-many conservation laws for eq. (5) on the basis of eqs (6). Relevant issues can be seen, e.g., in refs [28,29].

Via eq. (6a), and setting

$$\omega_1 = \frac{\phi_x}{\phi} - \lambda \quad \text{and} \quad \omega_2 = \frac{\phi_y}{\phi}, \tag{7}$$

where λ is a formal parameter, we obtain

$$\sigma\omega_2 + \omega_{1,x} + (\omega_1 + \lambda)^2 + u = 0. \tag{8}$$

By virtue of the compatibility condition

$$\omega_{1,y} = \omega_{2,x}, \tag{9}$$

we have

$$\sigma\omega_{1,y} + \omega_{1,xx} + 2\,\omega_{1,x}(\omega_1 + \lambda) + u_x = 0, \tag{10}$$

and then expand ω_1 in the following form:

$$\omega_1 = \sum_{j=1}^{\infty} f_j \lambda^{-j} . \tag{11}$$

Substituting eq. (11) into eq. (10), and collecting the coefficients of each order of λ, we obtain a recursion formula for f_j's as follows:

$$\sigma f_{j-1,y} + f_{j-1,xx} + 2\sum_{k=1}^{j-2} f_{k,x} f_{j-1-k} + 2 f_{j,x} + u_x \delta_{j,0} = 0, \tag{12}$$

where $f_0 = 0$, $\delta_{j,0} = 1$ $(j = 1)$ and $\delta_{j,0} = 0$ $(j \neq 1)$. From eq. (12), we can obtain

$$f_1 = -\frac{u}{2}, \tag{13a}$$

$$f_2 = \frac{\sigma\partial_x^{-1} u_y + u_x}{4}, \tag{13b}$$

$$f_3 = -\frac{\sigma^2\partial_x^{-2} u_{yy} + 2\sigma u_y + u_{xx} + u^2}{8}, \tag{13c}$$

$$\vdots$$

via eqs (6)–(8), and setting that

$$\theta = \frac{\phi_t}{\phi}, \tag{14}$$

we derive

$$\theta - 4\sigma\omega_{1,y} + (4\omega_{1,x} + 6u)(\omega_1 + \lambda) + 4(\omega_1 + \lambda)^3 - u_x - 3\sigma v = 0. \tag{15}$$

Through the compatibility condition

$$\omega_{1,t} = \theta_x, \tag{16}$$

substituting eqs (11) and (13) into eq. (16), and collecting the coefficients of each order of λ, we have

$$\lambda^1: \quad \text{LHS} = 0, \quad \text{RHS} = (-6u - 12 f_1)_x = 0, \tag{17a}$$

$$\lambda^0: \quad \text{LHS} = 0, \quad \text{RHS} = (-4 f_{1,x} - 12 f_2 + u_x + 3\sigma v)_x = 0, \tag{17b}$$

$$\lambda^{-1}: \quad \text{LHS} = f_{1,t} = -\frac{u_t}{2},$$

$$\text{RHS} = (4\sigma f_{1,y} - 6uf_1 - 12f_1^2 - 4f_{2,x} - 12f_3)_x$$

$$= \frac{3\sigma^2\partial_x^{-1}u_{yy} + 6uu_x + u_{xx}}{2}, \tag{17c}$$

$$\vdots$$

Here, the LHS and RHS respectively mean the left-hand side and right-hand side of an equation. We find that eqs (17a) and (17b) can be automatically established. From eq. (17c), we have the first conservation law for eq. (5) as follows:

$$D_t(u_x) + D_x(6uu_x + u_{xx}) + D_y(3\sigma^2 u_y) = 0. \tag{18}$$

Similarly, the infinitely-many conservation laws for eq. (5) can be derived through eqs (17).

2.2 *Infinitely-many conservation laws for the DS equations*

In this part, we shall construct the infinitely-many conservation laws for a coupled (2+1)-dimensional NLEEs, i.e., the DS equations [6,7]:

$$iu_t + u_{xx} + \frac{u_{yy}}{\alpha^2} + \frac{u(s_1 - s_2)}{\alpha} = 0, \tag{19a}$$

$$s_{1,y} - \alpha s_{1,x} - \varepsilon\left(|u|^2\right)_x - \frac{\varepsilon\left(|u|^2\right)_y}{\alpha} = 0, \tag{19b}$$

$$s_{2,y} + \alpha s_{1,x} - \varepsilon\left(|u|^2\right)_x + \frac{\varepsilon\left(|u|^2\right)_y}{\alpha} = 0, \tag{19c}$$

which describe the evolution of weakly nonlinear packets of water waves of the finite depth that travel in one direction but whose amplitudes are modulated in two spatial directions, where u is the complex wave envelope, s_1 and s_2 are related to the nonlocal flow generated by the wave packet, $(s_2 - s_1)^* = \alpha^2(s_2 - s_1)$, $*$ denotes the complex conjugate, x, y and t respectively denote the longitudinal coordinate, transverse coordinate and time, α is a complex constant and ε is a real one. Through

$$v = -\varepsilon|u|^2 + \frac{\alpha(s_1 - s_2)}{2}, \tag{20}$$

eqs (19) become

$$iu_t + u_{xx} + \frac{u_{yy}}{\alpha^2} + \frac{2\varepsilon|u|^2 u}{\alpha^2} + \frac{2uv}{\alpha^2} = 0, \tag{21a}$$

$$v_{yy} - \alpha^2 v_{xx} - 2\alpha^2\varepsilon\left(|u|^2\right)_{xx} = 0, \tag{21b}$$

where v describes the nonlocal flow generated by the wave packet [6,7]. When $\alpha^2 = 1$, eqs (21) are the DSI equations, while when $\alpha^2 = -1$, eqs (21) are the DSII equations [7].

The Lax pair for eqs (19) is as follows [6,7]:

$$\Phi_y = \mathcal{M}_1 \Phi_x + \mathcal{M}_2 \Phi, \tag{22a}$$

$$\Phi_t = \mathcal{N}_1 \Phi_{xx} + \mathcal{N}_2 \Phi_x + \mathcal{N}_3 \Phi, \tag{22b}$$

where

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad \mathcal{M}_1 = \alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathcal{M}_2 = \begin{pmatrix} 0 & u \\ -\varepsilon u^* & 0 \end{pmatrix}, \tag{23}$$

$$\mathcal{N}_1 = 2i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathcal{N}_2 = \frac{2i}{\alpha} \begin{pmatrix} 0 & u \\ -\varepsilon u^* & 0 \end{pmatrix}, \tag{24}$$

$$\mathcal{N}_3 = \frac{i}{\alpha^2} \begin{pmatrix} \alpha s_1 & \alpha u_x + u_y \\ -\varepsilon \alpha u_x^* + \varepsilon u_y^* & \alpha s_2 \end{pmatrix}. \tag{25}$$

Through eqs (22), we shall construct the infinitely-many conservation laws for eqs (19).

Setting

$$m_1 = \frac{\phi_{1,x}}{\phi_1} - \lambda, \qquad m_2 = \frac{\phi_{1,y}}{\phi_1}, \qquad n = \frac{\phi_2}{\phi_1}, \tag{26}$$

through eq. (22a) and the compatibility condition

$$m_{1,y} = m_{2,x}, \tag{27}$$

we obtain the following equations:

$$n_y + \alpha n_x + 2\alpha n(m_1 + \lambda) + \varepsilon u^* + u n^2 = 0, \tag{28a}$$

$$m_{1,y} - (\alpha m_1 + u n)_x = 0. \tag{28b}$$

We then respectively expand m_1 and n in the following form:

$$m_1 = \sum_{j=1}^{\infty} h_j \lambda^{-j}, \qquad n = \sum_{j=1}^{\infty} \chi_j \lambda^{-j}. \tag{29}$$

Substituting eq. (29) into eqs (28), and collecting the coefficients of each order of λ, we obtain recursion formulae for h_j's and l_j's as follows:

$$\chi_{j-1,y} + \alpha \chi_{j-1,x} + 2\alpha \sum_{k=1}^{j-2} \chi_k h_{j-1-k} + 2\alpha \chi_j + \varepsilon u^* \delta_{j,0} + u \sum_{k=1}^{j-2} \chi_k \chi_{j-1-k} = 0, \tag{30a}$$

$$h_{j,y} - \alpha h_{j,x} - (u \chi_j)_x = 0, \tag{30b}$$

where $\chi_0 = 0$. From eqs (30) we obtain

$$2\alpha\chi_1 + \varepsilon u^* = 0, \tag{31a}$$

$$h_{1,y} - \alpha h_{1,x} - (u\chi_1)_x = 0, \tag{31b}$$

$$\chi_{1,y} + \alpha\chi_{1,x} + 2\alpha\chi_2 = 0, \tag{31c}$$

$$h_{2,y} - \alpha h_{2,x} - (u\chi_2)_x = 0, \tag{31d}$$

$$\vdots$$

Setting

$$\Theta = \frac{\phi_{1,t}}{\phi_1}, \tag{32}$$

and through eqs (22) and (26), we have

$$\Theta = 2i\left[m_{1,x} + (m_1+\lambda)^2\right] + \frac{2iu}{\alpha}\left[n_x + n(m_1+\lambda)\right] + \frac{is_1}{\alpha} + \frac{in(\alpha u_x + u_y)}{\alpha^2}. \tag{33}$$

By virtue of eqs (29) and (31) and the compatibility condition

$$\Theta_y - \alpha\Theta_x = (un)_t\,, \tag{34}$$

we collect the coefficients of each order of λ and obtain

$$\begin{aligned}
\lambda^0:\quad \text{LHS} &= \left(4ih_1 + \frac{2iu\chi_1}{\alpha} + \frac{2is_1}{\alpha}\right)_y - \alpha\left(4ih_1 + \frac{2iu\chi_1}{\alpha} + \frac{2is_1}{\alpha}\right)_x \\
&= \frac{1}{\alpha}\left[s_{1,y} - \alpha s_{1,x} - \varepsilon\left(|u|^2\right)_x - \frac{\varepsilon\left(|u|^2\right)_y}{\alpha}\right] = 0, \\
\text{RHS} &= 0,
\end{aligned} \tag{35a}$$

$$\begin{aligned}
\lambda^{-1}:\quad \text{LHS} &= \frac{i\varepsilon}{2\alpha}\left(uu^*_{xx} - u_{xx}u^* + \frac{uu^*_{yy} - u_{yy}u^*}{\alpha^2}\right) \\
&= \frac{i\varepsilon}{2\alpha}(uu^*_x - u_x u^*)_x + \frac{i\varepsilon}{2\alpha^3}(uu^*_y - u_y u^*)_y, \\
\text{RHS} &= \left(-\frac{\varepsilon uu^*}{2\alpha}\right)_t,
\end{aligned} \tag{35b}$$

$$\vdots$$

We find that eq. (35a) can be automatically established since it corresponds to eq. (19b). Through eq. (35b), we have the first conservation law for eqs (19) as follows:

$$D_t(-iuu^*) + D_x(uu^*_x - u_x u^*) + D_y\left(\frac{uu^*_y - u_y u^*}{\alpha^2}\right) = 0. \tag{36}$$

Moreover, the infinitely-many conservation laws for eqs (19) can be similarly constructed via eqs (35). Besides, setting

$$m_1' = \frac{\phi_{2,x}}{\phi_2} - \lambda, \qquad m_2' = \frac{\phi_{2,y}}{\phi_2}, \qquad n = \frac{\phi_1}{\phi_2}, \tag{37}$$

we can also achieve the infinitely-many conservation laws for eqs (19) through similar procedures.

3. Conclusions

In this paper, we have generalized a method, which has been used to construct the infinitely-many conservation laws with the Lax pairs for the (1+1)-dimensional NLEEs, to the (2+1)-dimensional ones. Furthermore, we have applied the method to a single (2+1)-dimensional NLEE (i.e., eq. (5)) and a coupled (2+1)-dimensional ones (i.e., eqs (19)), which can describe the nonlinear waves in fluids and plasmas, and have respectively derived their infinitely-many conservation laws (i.e., eqs (17) and (35)). Based on that method, we can also construct the infinitely-many conservation laws for other multidimensional NLEEs possessing the Lax pair in such applied sciences as fluids, plasmas, optical fibres and Bose–Einstein condensates.

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