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Chaotic behaviour of nonlinear coupled reaction–diffusion system in four-dimensional space

LI ZHANG[1,*], SHUTANG LIU[2] and CHENGLONG YU[3]

[1]Business School, Shandong University of Political Science and Law, Jinan 250014, China
[2]College of Control Science and Engineering, Shandong University, Jinan 250061, China
[3]Department of Mathematics, Statistics and Computer Science, University of Illinois, Chicago, IL 60607-7045, USA
*Corresponding author. E-mail: zhanglisdu2008@yahoo.com

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Abstract. In recent years, nonlinear coupled reaction–diffusion (CRD) system has been widely investigated by coupled map lattice method. Previously, nonlinear behaviour was observed dynamically when one or two of the three variables in the discrete system change. In this paper, we consider the chaotic behaviour when three variables change, which is called as four-dimensional chaos. When two parameters in the discrete system are unknown, we first give the existing condition of the chaos in four-dimensional space by the generalized definitions of spatial periodic orbits and spatial chaos. In addition, the chaotic behaviour will vary with the parameters. Then we propose a generalized Lyapunov exponent in four-dimensional space to characterize the different effects of parameters on the chaotic behaviour, which has not been studied in detail. In order to verify the chaotic behaviour of the system and the different effects clearly, we simulate the dynamical behaviour in two- and three-dimensional spaces.

Keywords. Chaos; coupled map lattice; partial differential equation; higher dimension.

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1. Introduction

Coupled reaction–diffusion system with different initial and boundary conditions as

$$\begin{cases} \dfrac{\partial \phi(x)}{\partial t} = D_1 \Delta \phi(x) + f(\phi(x), \psi(x)), \\ \dfrac{\partial \psi(x)}{\partial t} = D_2 \Delta \psi(x) + g(\phi(x), \psi(x)), \end{cases} \tag{1}$$

has been naturally related with various phenomena in many different fields, such as in material science [1,2], engineering applications [3,4], physics [5–13] and so on. In system

(1), $x \in \Omega \subset R^n$, $t \in (0, +\infty)$, $D_i > 0$ stands for the diffusion constant where $i = 1, 2$, $f(\phi(x), \psi(x))$ and $g(\phi(x), \psi(x))$ mean all kinds of noises in reality.

As the behaviour of coupled reaction–diffusion phenomena is very complicated, according to nonlinear noise in the real physical world, $f(\phi(x), \psi(x))$ and $g(\phi(x), \psi(x))$ are often taken as nonlinear functions in system (1). However, it is difficult to solve system (1) as coupled partial differential equations analytically according to the nonlinearity. Therefore, the discrete solution of system (1) is usually studied to approximate its precise solution.

Coupled map lattice (CML) method is an effective discrete method, which has been intensively studied for many years (see, for example, [14–17] and the references cited therein). The CML model for system (1), known as coupled reaction diffusion (CRD) system, is obtained as follows:

$$X_{m,n}(s+1) = F(X_{m,n}(s)) + \varepsilon G(\Delta X_{m,n}(s)), \tag{2}$$

where

$$X_{m,n}(s) = (\phi_{m,n}(s), \psi_{m,n}(s))^T,$$

$$F(X_{m,n}(s)) = (\phi_{m,n}(s) + f(\phi_{m,n}(s), \psi_{m,n}(s)), \psi_{m,n}(s) + g(\phi_{m,n}(s), \psi_{m,n}(s)))^T,$$

$$\Delta X_{m,n}(s) = (\Delta\phi_{m,n}(s), \Delta\psi_{m,n}(s))^T,$$

$$\triangle\phi_{m,n}(s) = \phi_{m+1,n}(s) + \phi_{m-1,n}(s) + \phi_{m,n+1}(s) + \phi_{m,n-1}(s) - 4\phi_{m,n}(s),$$

$$\triangle\psi_{m,n}(s) = \psi_{m+1,n}(s) + \psi_{m-1,n}(s) + \psi_{m,n+1}(s) + \psi_{m,n-1}(s) - 4\psi_{m,n}(s),$$

$\varepsilon = \begin{pmatrix} D_1, & 0 \\ 0, & D_2 \end{pmatrix}$, $m, n \in N_r = \{r, r+1, r+2, \ldots\}$, $r \geq 0$, s stands for the discrete time and ε means the coupled parameter.

System (2) has three variables m, n and s, and so we call it 3D discrete system. Some interesting results have been established for this 3D discrete system (2) quantitatively, such as when the nonlinear function is in polynomial [18–23] or exponential form [24–27]. These results focus on the numerical or analytical characterizations of system (2) according to the change of one or two of the three variables, m, n and s. In fact, these three variables independently play important roles in the CRD phenomena. Hence, it is important to study the role of the three variables m, n and s for system (2), which will be helpful to further understand the dynamical behaviour of system (2). However, there are few results on this. So we consider the nonlinear behaviour dynamically for 3D discrete system (2) in four-dimensional space.

Moreover, there has been a growing interest in the chaotic behaviour of CML model such as proof of the existence of chaos [28], measure and distribution of chaos with Belusov–Zhabotinskii reaction [29], synchronous chaos [30–32], control of chaos in CML [33–36] and growth patterns for global weather models [37]. But there exist few results for the chaotic behaviour of 3D discrete system (2). So we study the chaotic behaviour in four-dimensional space for system (2). In order to verify our method, we take the following nonlinear functions for system (2):

$$f(\phi_{m,n}(s), \psi_{m,n}(s)) = (\mu - 1)\phi_{m,n}(s) - \mu^2\phi_{m,n}(s)\exp[-\beta\psi_{m,n}(s)],$$

$$g(\phi_{m,n}(s), \psi_{m,n}(s)) = -\phi_{m,n}(s)\exp[-\beta\psi_{m,n}(s)].$$

Then we obtain the famous model as an example:

$$\begin{cases} \phi_{m,n}(s+1) = D_1\Delta\phi_{m,n}(s) + \mu\phi_{m,n}(s)[1-\phi_{m,n}(s)]\exp[-\beta\psi_{m,n}(s)], \\ \psi_{m,n}(s+1) = D_2\Delta\psi_{m,n}(s) + \phi_{m,n}(s)(1-\exp[-\beta\psi_{m,n}(s)]), \end{cases} \tag{3}$$

which was proposed by Solé and Valls [38] to study the process of host–parasite interaction in ecological system, where $\phi_{m,n}(s)$ is the host density in generations s and $s+1$, $\psi_{m,n}(s)$ represents the parasite density in generations s and $s+1$, μ is the increasing rate of the host population and β means the average area that the parasite searched in its lifetime effectively. The identification [39–41] and the effects of the initial condition for chaotic signals within the CML system (3) as lower-dimensional (dimension $d \leq 2$) graph [42] were considered mostly.

In this paper, we obtain the existing condition of the chaotic behaviour in four-dimensional space as two parameters are unknown, which is generalized by the definitions of spatial periodic orbits and spatial chaos of Li–Yorke in §2. The variation of the two parameters has different effects on the chaotic behaviour in four-dimensional space for system (2). So we propose a generalized Lyapunov exponent to characterize the different effects in §3. The chaotic variations, the spatial variations and cross-section of Lyapunov exponent of system (2) in two- and three-dimensional spaces are simulated illustratively in §4. Finally, conclusions and discussions are given in §5.

2. Chaotic existence using generalized definitions by spatial orbits

For convenience, we assume

$$H(X_{m,n}(s)) = F(X_{m,n}(s)) + \varepsilon G(\Delta X_{m,n}(s)).$$

Then system (2) is replaced as follows:

$$X_{m,n}(s+1) = H(X_{m,n}(s)). \tag{4}$$

First, we give some definitions generalized by spatial orbits [43].

DEFINITION 2.1

Let non-empty set $V \subseteq R^4$ and $I \subset R$ are any non-empty subsets of V. $X_{m,n}(1)$ is the k periodic point in four-dimensional space, if $H(X_{m,n}(s))$ is continuous and self-mapping, which means that $H(X_{m,n}(s)) \in C^0(I^2, I^2)$, $H(I^2) \subset I^2$, and

$$\begin{cases} H^k(X_{m,n}(1)) = X_{m,n}(1), \\ H^s(X_{m,n}(1)) \neq X_{m,n}(1), \quad 1 \leq s < k, \end{cases}$$

where $k = 1, 2, \ldots$.

Based on Definition 2.1, we shall first prove the existence of a k periodic point in four-dimensional space for system (3).

Theorem 2.1. *For system* (3), *there exists a k-periodic point in four-dimensional space, if*

$$X_{m,n}(1), X_{m,n}(2), \ldots, X_{m,n}(k)$$

is a real-function sequence in which $X_{m,n}(i) \neq 0$, $X_{m,n}(i) \neq X_{m,n}(j)$, $i, j = 1, 2, \ldots, k$, $i \neq j$ *and*

$$\mu = \frac{\phi_{m,n}(s-1)\exp[\beta\psi_{m,n}(s-2)]}{\phi_{m,n}(s-2)(1-\phi_{m,n}(s-2))},$$

where $s = 3, 4, \ldots, k+2$.

Proof. For system (3), we have

$$\begin{aligned} X_{m,n}(s+1) &= H(X_{m,n}(s)) \\ &= \begin{pmatrix} D_1\Delta\phi_{m,n}(s) + \mu\phi_{m,n}(s)[1-\phi_{m,n}(s)]\exp[-\beta\psi_{m,n}(s)] \\ D_2\Delta\phi_{m,n}(s) + \phi_{m,n}(s)(1-\exp[-\beta\psi_{m,n}(s)]) \end{pmatrix}, \end{aligned}$$

satisfying

$$\begin{cases} H(X_{m,n}(1)) = \begin{pmatrix} D_1\Delta\phi_{m,n}(1) + \mu\phi_{m,n}(1)[1-\phi_{m,n}(1)]\exp[-\beta\psi_{m,n}(1)] \\ D_2\Delta\phi_{m,n}(1) + \phi_{m,n}(1)(1-\exp[-\beta\psi_{m,n}(1)]) \end{pmatrix}, \\ H(X_{m,n}(2)) = \begin{pmatrix} D_1\Delta\phi_{m,n}(2) + \mu\phi_{m,n}(2)[1-\phi_{m,n}(2)]\exp[-\beta\psi_{m,n}(2)] \\ D_2\Delta\phi_{m,n}(2) + \phi_{m,n}(2)(1-\exp[-\beta\psi_{m,n}(2)]) \end{pmatrix}, \\ \cdots \\ H(X_{m,n}(k)) = \begin{pmatrix} D_1\Delta\phi_{m,n}(k) + \mu\phi_{m,n}(k)[1-\phi_{m,n}(k)]\exp[-\beta\psi_{m,n}(k)] \\ D_2\Delta\phi_{m,n}(k) + \phi_{m,n}(k)(1-\exp[-\beta\psi_{m,n}(k)]) \end{pmatrix}, \end{cases}$$

where $m, n \in N_r$, $s = 1, 2, \ldots, k$. According to Systems (3) and (4), we get

$$\begin{cases} X_{m,n}(2) = \begin{pmatrix} D_1\Delta\phi_{m,n}(1) + \mu\phi_{m,n}(1)[1-\phi_{m,n}(1)]\exp[-\beta\psi_{m,n}(1)], \\ D_2\Delta\phi_{m,n}(1) + \phi_{m,n}(1)(1-\exp[-\beta\psi_{m,n}(1)]) \end{pmatrix}, \\ X_{m,n}(3) = \begin{pmatrix} D_1\Delta\phi_{m,n}(2) + \mu\phi_{m,n}(s)[1-\phi_{m,n}(2)]\exp[-\beta\psi_{m,n}(2)] \\ D_2\Delta\phi_{m,n}(2) + \phi_{m,n}(2)(1-\exp[-\beta\psi_{m,n}(2)]) \end{pmatrix}, \\ \cdots \\ X_{m,n}(1) = \begin{pmatrix} D_1\Delta\phi_{m,n}(k) + \mu\phi_{m,n}(k)[1-\phi_{m,n}(s)]\exp[-\beta\psi_{m,n}(k)] \\ D_2\Delta\phi_{m,n}(k) + \phi_{m,n}(k)(1-\exp[-\beta\psi_{m,n}(k)]) \end{pmatrix}, \end{cases}$$

and we also obtain the following condition:

$$\mu = \frac{\phi_{m,n}(s-1)\exp[\beta\psi_{m,n}(s-2)]}{\phi_{m,n}(s-2)(1-\phi_{m,n}(s-2))},$$

where $s = 3, 4, \ldots, k+2$. This completes the proof. □

Next we give the definition of chaos in four-dimensional space generalized by the spatial chaos in ref. [43].

DEFINITION 2.2

Let $V \subseteq R^4$, V_0 is the non-empty subset of V, $I \subseteq V_0$ and $I \subseteq R$. System (3) is chaotic in Li–Yorke in four-dimensional space, if there exists a point $c \in I$ and a continuous function $H: I \to I$ satisfying $H^3(c) \leq c < H(c) < H^2(c)$ or $H^3(c) \geq c > H(c) > H^2(c)$.

Using Theorem 2.1 and Definition 2.2, we obtain the following theorem.

Theorem 2.2. *Let $V \subseteq R^4$, V_0 is the non-empty subset of V, $I^2 \subseteq V_0$ and $I^2 \subseteq R^2$. System (3) is chaotic in Li–Yorke in four-dimensional space, if the following conditions are satisfied*:

(1) *For any $m, n \in N_0, i, j, s = 1, 2, 3$, $X_{m,n}(s) \in I^2$, $X_{m,n}(s) \neq 0$, and $X_{m,n}(i) \neq X_{m,n}(j)$, for $i \neq j$.*

(2) *For any $m, n \in N_0$,*

$$X_{m,n}(1) > X_{m,n}(2) > X_{m,n}(3)$$

or

$$X_{m,n}(1) < X_{m,n}(2) < X_{m,n}(3).$$

(3) *Let*

$$H(X_{m,n}(s)) = \begin{pmatrix} D_1\Delta\phi_{m,n}(s) + \mu\phi_{m,n}(s)[1-\phi_{m,n}(s)]\exp[-\beta\psi_{m,n}(s)] \\ D_2\Delta\phi_{m,n}(s) + \phi_{m,n}(s)(1-\exp[-\beta\psi_{m,n}(s)]) \end{pmatrix},$$

where

$$\mu = \frac{\phi_{m,n}(2)\exp[\beta\psi_{m,n}(1)]}{\phi_{m,n}(1)(1-\phi_{m,n}(1))} = \frac{\phi_{m,n}(3)\exp[\beta\psi_{m,n}(2)]}{\phi_{m,n}(2)(1-\phi_{m,n}(2))} = \frac{\phi_{m,n}(1)\exp[\beta\psi_{m,n}(3)]}{\phi_{m,n}(3)(1-\phi_{m,n}(3))}.$$

(4) $H(I^2) \subset I^2$.

Proof. Because system (3) is as follows:

$$X_{m,n}(s+1) = H(X_{m,n}(s)),$$

where

$$H(X_{m,n}(s)) = \begin{pmatrix} D_1\Delta\phi_{m,n}(s) + \mu\phi_{m,n}(s)[1-\phi_{m,n}(s)]\exp[-\beta\psi_{m,n}(s)] \\ D_2\Delta\phi_{m,n}(s) + \phi_{m,n}(s)(1-\exp[-\beta\psi_{m,n}(s)]) \end{pmatrix},$$

$s = 1, 2, 3$, and condition (3) holds, we have

$$H^3(X_{m,n}(1)) = X_{m,n}(1)$$

and

$$H^i(X_{m,n}(1)) \neq X_{m,n}(1),$$

$i = 1, 2$, it is obvious that there is a three-periodic point $X_{m,n}(1)$ in four-dimensional space according to Theorem 2.1.

In addition, we easily get that $H(I^2) \subset I^2$ and

$$H^3(X_{m,n}(1)) = Xi_{m,n}(1) < X_{m,n}(2) = H(X_{m,n}(1)) < H^2(X_{m,n}(1)) = X_{m,n}(3).$$

Then based on Definition 2.2, we obtain that system (3) is chaotic of Li–Yorke in four-dimensional space. This completes the proof. □

3. Variation of the chaotic behaviour as μ and β change

In §2, it is clear that we can obtain the chaotic behaviour for system (3) with three-periodic point in four-dimensional space theoretically. Applying the conditions of the three-periodic point in §2, we can calculate the values of μ and β. Moreover, parameters μ and β will have different effects on the chaotic behaviour of system (3). So we propose a generalized Lyapunov exponent in four-dimensional space to consider these different effects by the definition of Lyapunov exponent in ref. [43].

Theorem 3.1. *The Lyapunov exponent λ of system* (3) *in four-dimensional space is defined as follows*:

$$\lambda = \lim_{s\to\infty} \frac{1}{s} \ln \sum_{j=0}^{s} \frac{\partial H^s(X_{m,n}(j))}{\partial X_{m,n}(j)}.$$

Proof. Assume two initial values of system (3) that are very close to each other: $P_0 = X_{m,n}(0)$ and $P_0' = X_{m,n}'(0)$. Then we have

$$X_{m,n}(s) = H^s(X_{m,n}(0))$$

and

$$X_{m,n}'(s) = H^s(X_{m,n}'(0)).$$

If the values of $X_{m,n}(s)$ and $X_{m,n}'(s)$ depart very fast exponentially in the iteration process, we get the following relationship:

$$\|X_{m,n}(s) - X_{m,n}'(s)\| = \|X_{m,n}(0) - X_{m,n}'(0)\| \mathrm{e}^{s\lambda_s},$$

where $\|\cdot\|$ means any matrix norm in ref. [44] and $\lambda_s > 0$. Then we get

$$\begin{aligned}
\lambda &= \lim_{s\to\infty} \frac{1}{s} \lim_{\|P_0-P_0'\|\to 0} \ln \frac{\|X_{m,n}(s) - X_{m,n}'(s)\|}{\|P_0 - P_0'\|} \\
&= \lim_{s\to\infty} \frac{1}{s} \lim_{\|P_0-P_0'\|\to 0} \ln \frac{\|H^s(P_0) - H^s(P_0')\|}{\|P_0 - P_0'\|} \\
&= \lim_{s\to\infty} \frac{1}{s} \ln \sum_{j=0}^{s} \frac{\partial H^s(X_{m,n}(j))}{\partial X_{m,n}(j)}.
\end{aligned}$$

4. Simulations and results

From Theorem 2.2, we obtain the values of μ and β based on the condition that

$$\mu = \frac{\phi_{m,n}(2)\exp[\beta\psi_{m,n}(1)]}{\phi_{m,n}(1)(1-\phi_{m,n}(1))} = \frac{\phi_{m,n}(3)\exp[\beta\psi_{m,n}(2)]}{\phi_{m,n}(2)(1-\phi_{m,n}(2))} = \frac{\phi_{m,n}(1)\exp[\beta\psi_{m,n}(3)]}{\phi_{m,n}(3)(1-\phi_{m,n}(3))}.$$

Then we simulate the chaotic behaviour of system (3) with some three-periodic points in four-dimensional space.

Let $V_0 = V = [-2, 2]^4$, $I = [-1, 1] \subset V_0 \subset R$. The initial and boundary condition is $X_{m,n}(1) = (0.2, 0.1894)^T$. Without loss of generalization, let $X_{m,n}(2) = (0.3, 0.2483)^T$

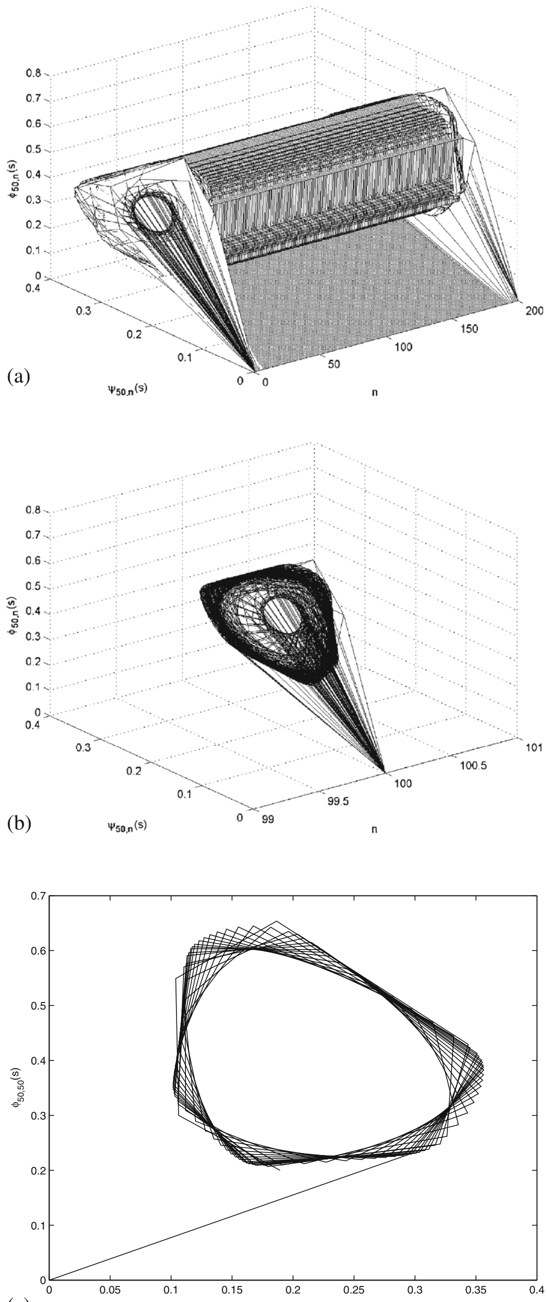

Figure 1. The chaotic behaviour of $X_{50,n}(s) = (\phi_{50,n}(s), \psi_{50,n}(s))^T$ in d-dimensional space, where s and n are taken respectively as $1, 2, \ldots, 200$. (**a**) The spatial chaotic behaviour in three-dimensional space, (**b**) the cross-section of the chaotic behaviour in three-dimensional space and (**c**) the chaotic behaviour in two-dimensional space.

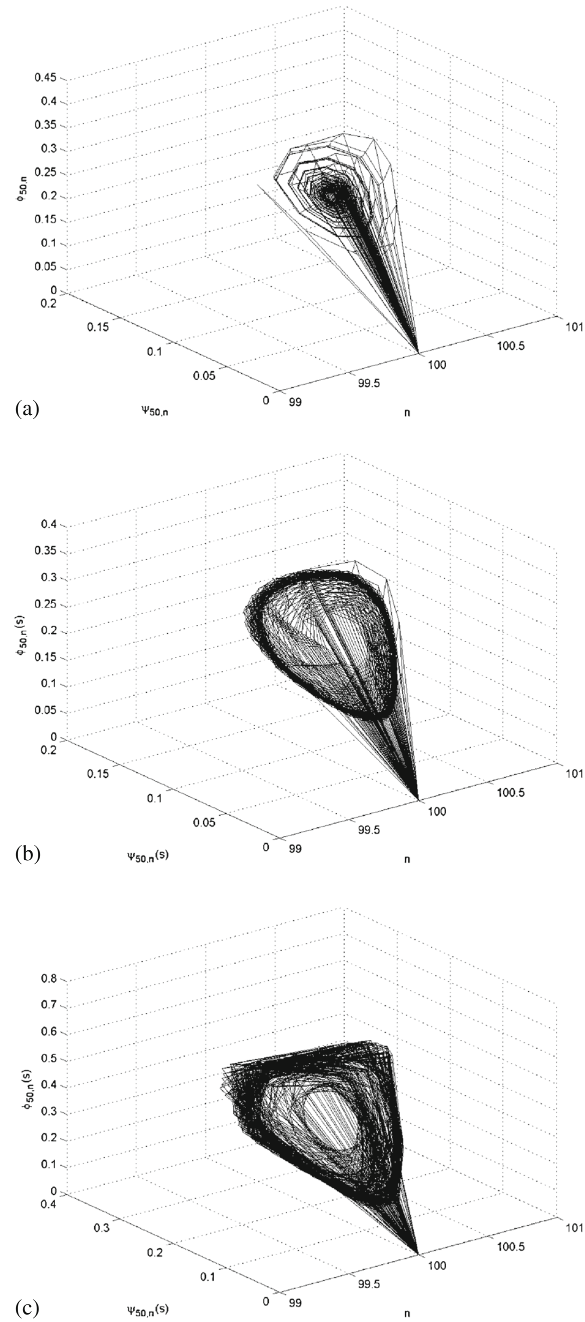

Figure 2. The cross-section of the chaotic behaviour of $X_{50,n}(s) = (\phi_{50,n}(s), \psi_{50,n}(s))^T$ in three-dimensional space, where $s, n = 1, 2, \ldots, 200$ as μ and β vary. (**a**) The cross-section of the chaotic behaviour when $\mu = 2$, $\beta = 5$, (**b**) the cross-section of the chaotic behaviour when $\mu = 2$, $\beta = 6$ and (**c**) the cross-section of the chaotic behaviour when $\mu = 3.5$, $\beta = 5$.

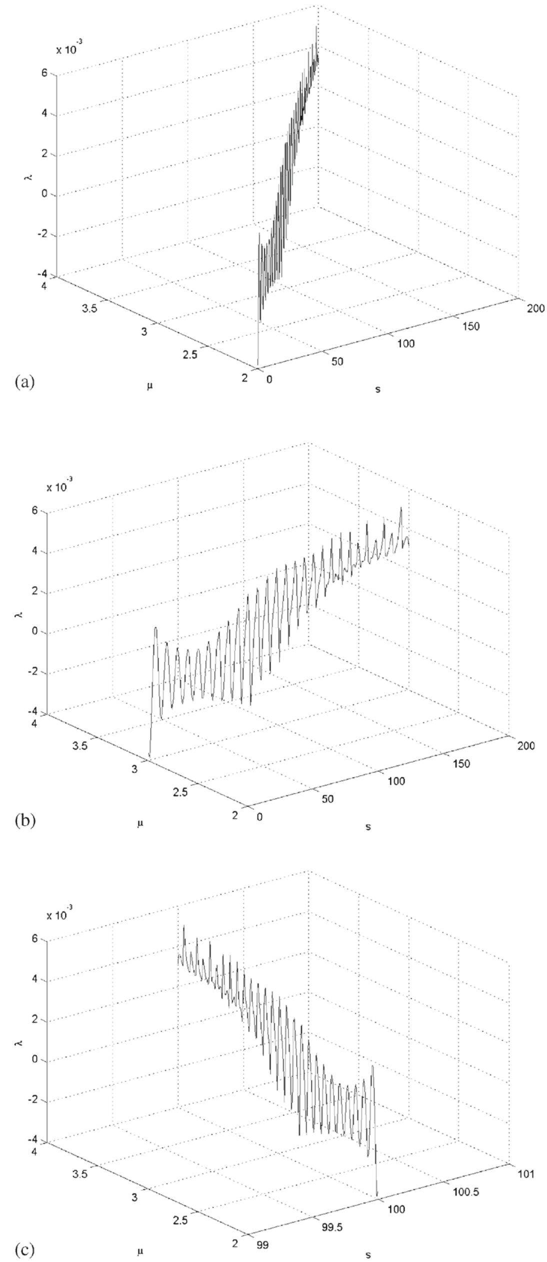

Figure 3. The spatial and cross-section variations of generalized Lyapunov exponent λ in three-dimensional space with $\beta = (3.01, 5)$ and $u = (2.01, 4)$, respectively. (**a**) The spatial variation between λ and μ when $\beta = 5$, (**b**) the cross-section variation between λ and μ when $\beta = 5$, (**c**) the cross-section variation between λ and μ when $\beta = 5$, (**d**) the spatial variation between λ and β when $\mu = 4$, (**e**) the cross-section variation between λ and β when $\mu = 4$ and (**f**) the cross-section variation between λ and β when $\mu = 4$.

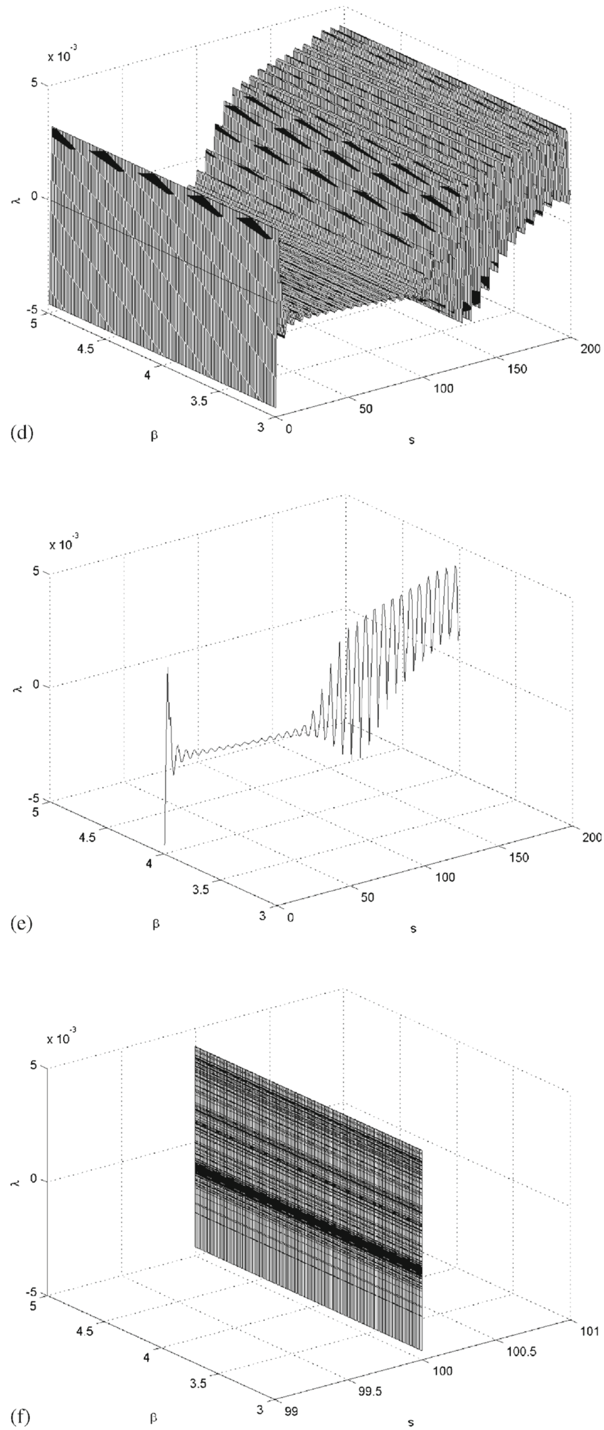

Figure 3. *Continued*

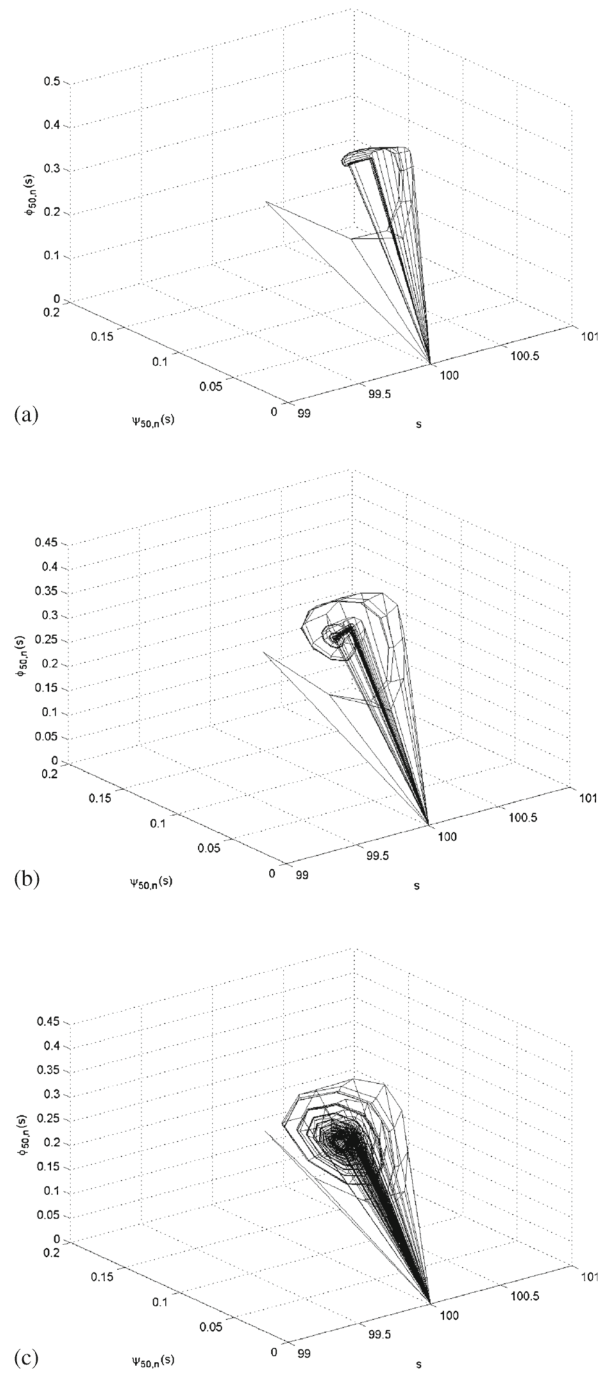

Figure 4. The chaotic behaviour of $X_{50,n}(s) = (\phi_{50,n}(s), \psi_{50,n}(s))^T$ in three-dimensional space, where $s, n = 1, 2, \ldots, 200$, $\mu = 2$ as β changes. (**a**) $\beta = 3$, (**b**) $\beta = 4$ and (**c**) $\beta = 5$.

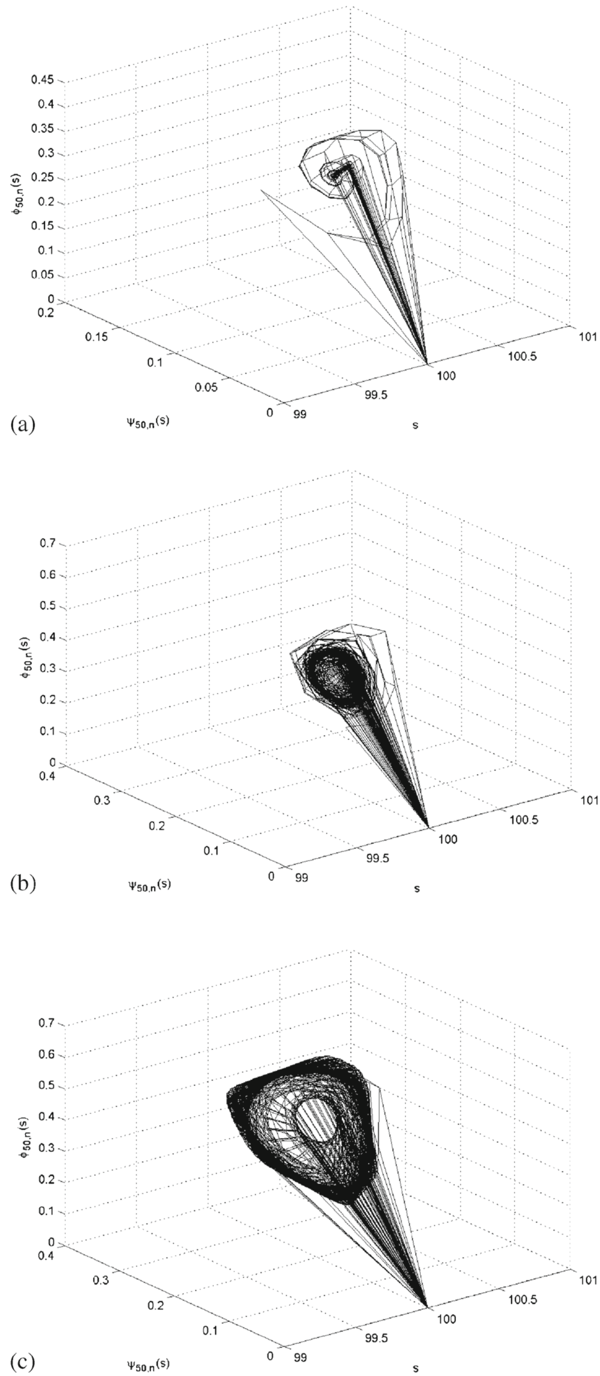

Figure 5. The chaotic behaviour of $X_{50,n}(s) = (\phi_{50,n}(s), \psi_{50,n}(s))^T$ in three-dimensional space, where $s, n = 1, 2, \ldots, 200$, $\beta = 4$ as μ changes. (**a**) $\mu = 2$, (**b**) $\mu = 3$ and (**c**) $\mu = 4$.

and $X_{m,n}(3) = (0.4, 0.3922)^T$. In order to calculate the values easily, we take $D_1 = 0.001$, $D_2 = 0.2$, which is the same as in ref. [38]. From Theorem 2.2, using

$$\mu = \frac{\phi_{m,n}(2)\exp(\beta\psi_{m,n}(1))}{\phi_{m,n}(1)(1-\phi_{m,n}(1))} = \frac{\phi_{m,n}(3)\exp(\beta\psi_{m,n}(2))}{\phi_{m,n}(2)(1-\phi_{m,n}(2))} = \frac{\phi_{m,n}(1)\exp(\beta\psi_{m,n}(3))}{\phi_{m,n}(3)(1-\phi_{m,n}(3))},$$

we get

$$\begin{cases} 0.3 = \mu 0.2(1-0.2)\exp(-0.1894\beta), \\ \\ 0.2483 = 0.2[1-\exp(-0.1894\beta)]. \end{cases}$$

Then we have $\mu = 3.9987$, $\beta = 3.9987$. In addition, $X_{m,n}(1) = (0.2, 0.1894)^T$ is a three-periodic point in four-dimensional space, satisfying

$$\begin{aligned} H^3(X_{m,n}(1)) = (0.2, 0.1894)^T &< (0.3, 0.2483)^T = H(X_{m,n}(1)) \\ &< (0.4, 0.3922)^T = H^2(X_{m,n}(1)). \end{aligned}$$

Hence, system (3) is chaotic of Li–Yorke in four-dimensional space according to Theorem 2.2.

Because of the difficulty in visualizing the dynamical behaviour in four-dimensional space for system (3), we simulate the chaotic behaviour of $X_{50,n}(s) = (\phi_{50,n}(s), \psi_{50,n}(s))^T$ shown in figure 1 in two- and three-dimensional spaces, respectively.

Moreover, the chaotic behavior of system (3) will be different according to the variation of μ and β, which is illustrated in figure 2. To find the effects of μ and β on the chaotic variation, we also simulate the quantitative effects of parameters μ and β on the Lyapunov exponent λ defined in §3 (see figure 3), which has not been studied in detail. It is clear that the chaotic behaviour of system (3) can be obtained by choosing different values for μ and β from the simulation results shown in figure 3, where the initial value is $X_{m,n}(1) = (0.2, 0.1515)^T$. Figure 3 also shows that Lyapunov exponent λ increases as μ or β increases in general. Based on this result, we predict that the chaotic behaviour will be more complex when μ and β increase, respectively. Figures 4 and 5 illustrate the two increasing effects on the chaotic behaviour of system (3) in three-dimensional space, respectively.

5. Discussions and conclusions

Coupled reaction–diffusion (CRD) phenomena can be observed in a variety of spatially extended systems in fields as diverse as biology, chemistry and engineering. Because of the difficulty for solving coupled partial differential equations with nonlinearity analytically, the discrete form of the CRD with three variables that are obtained by the CML method has been in focus in recent years. Most researchers studied the properties of the discrete CRD by changing one or two variables. In addition, chaotic behaviour [45–47] is an important nonlinear behaviour. So we considered the chaotic variation of the discrete CRD dynamically by changing these three variables.

In this paper, we investigated the chaotic behaviour of the nonlinear CRD system with three variables which was named as four-dimensional chaos. At first, we presented a chaotic definition in four-dimensional space and then proved the existing condition of the chaos with two unknown parameters. Based on the existing condition, we simulated two- and three-dimensional chaos illustratively. Because of the variation of chaos with two parameters in four-dimensional space, we also proposed a generalized Lyapunov exponent to characterize different variations. Simulations illustrated that Lyapunov exponent increased as the two parameters increase, respectively. Using these relationships, we can predict the complexity of the chaotic behaviour when the two parameters change. It can be seen that the complexity increases as two parameters increase in simulations. Our results in this paper are very useful to further understand and control the nonlinear behaviour for complex reaction–diffusion phenomena in reality.

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