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Lie algebra symmetries and quantum phase transitions in nuclei

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Abstract. In this paper, an overview of some aspects of quantum phase transitions (QPT) in nuclei is given and they are: (i) QPT in interacting boson model (sdIBM), (ii) QPT in two-level models, (iii) critical point $E(5)$ and $X(5)$ symmetries, (iv) QPT in a simple solvable model with three-body forces. In addition, some open problems are also given.

Keywords. Symmetries; quantum phase transitions; nuclear structure; two-level models; critical point symmetries; $SU(3)$; three-body forces.

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1. Introduction

Starting with Wigner's spin–isospin $SU(4)$ symmetry, Elliott's rotational $SU(3)$, Racah's pairing $SU(2)$ and its extension to proton–neutron pairing with j–j coupling giving $SO(5)$, Hecht and Arima's pseudospin, Rowe's $Sp(6, R)$, Ginocchio and Feng's $(k-i)-SO(8)$ symmetries in shell model are well studied and applied during 1950–1990. Similarly, within sd interacting boson model (IBM) of Arima and Iachello for even–even nuclei with pairing and quadrupole deformation, the vibrational $U(5)$, rotational $SU(3)$ and γ-soft $SO(6)$, their analogues in proton–neutron sdIBM with F-spin, $U(15)$ sdgIBM with hexadecupole degree of freedom, $U(16)$ $sdpf$IBM with dipole and octupole degrees of freedom, for odd-A nuclei $Spin(5)$, $Spin(6)$, $U(5) \otimes SU(2)$, $SU(3) \otimes U(2)$, $SO(6) \otimes SU(2)$, $U(6) \otimes SU(2)$, $U(6) \otimes U(20)$ etc., symmetries are well studied and applied during 1975–1998. Beyond these, in the last 15 years many new directions are opened for symmetries defined by Lie algebras and the closely related topic of solvable models in nuclear structure. Some of these are: (i) $SO(8)$ proton–neutron pairing symmetries in L–S coupling, (ii) quantum group extensions of $SO(5)$ pairing in shell model, (iii) introduction of $U(n) \supset SO(n)$ class of symmetries in sdIBM with internal degrees of freedom, (iv) Richardson–Gaudin (RG) and other related methods for generalized pairing Hamiltonians in IBM and shell model, (v) partial dynamical symmetries, (vi) supersymmetry (SUSY) defined by graded Lie algebras within IBM (SUSY describes simultaneously the structure of even–even, odd-A and odd–odd nuclei) (see [1–12] and references therein).

One of the most significant outcome of the studies, in the last decade, using symmetries in nuclear structure, is the discovery by Iachello, Jolie and Casten [13–15] that the change from one type of symmetry to another, as we change neutron or proton number is indeed a quantum phase transition (QPT). As Iachello and Zamfir state [14]: "Quantum phase transitions – that is phase transitions that occur at zero temperature as a function of a coupling constant – have become very important . . .. The concept of quantum phase transition can also be used in mesoscopic systems, that is, systems with finite number of particles, . . . nuclei, molecules, atomic clusters, and finite polymers. The transitions in these systems are between shapes or geometric configurations." Phase transitions within sdIBM were studied in 1980s using mean-field methods [2] and it is known that $U(5)$ to $SU(3)$ is first order, $U(5)$ to $SO(6)$ is second order and for $SO(6)$ to $SU(3)$ there is no phase transition. However, only recently it is understood [15,16] that these are indeed QPT. As Cejnar *et al* state in their latest review article [13]: "It was argued that the models of nuclear collective motion, apart from their empirical content, represent a useful laboratory for testing and even inventing new theoretical descriptions of various types of critical phenomena in quantum many-body systems." Finally, as stated in [17]: "the algebraic IBM and the geometric collective model (GCM), - - - -the coexistence of simple and complex features disclosed in the IBM and GCM studies make the collective models an excellent theoretical laboratory which in many respects surpasses the quantum billiards commonly used in to study the interplay between regular and chaotic motion in finite quantum systems." The purpose of this paper is to give an overview of some topics in QPT in nuclei. The topics selected are closely related to some of the studies made by the author and his collaborators on Lie algebraic symmetry schemes in nuclei. Now, a preview will be given.

Section 2 gives a brief discussion of QPT in sdIBM. Section 3 gives results for QPT in two-level models. Section 4 introduces critical point symmetries. Section 5 gives results for QPT in a simple solvable model with three-body forces in sdIBM. Finally, §6 gives a list of some open problems.

2. Quantum phase transitions in even–even nuclei: Coherent states and sdIBM symmetry limits

A Hamiltonian that covers the three sdIBM symmetry limits is

$$H(\zeta,\chi)=\epsilon_0\left[(1-\zeta)\,n_d-\frac{\zeta}{4N}\,Q^\chi\cdot Q^\chi\right], \tag{1}$$

where $Q^\chi=(d^\dagger\tilde{s}+s^\dagger\tilde{d})^2+\chi(d^\dagger\tilde{d})^2$ and ϵ_0 is a scale factor. Note that $\zeta=0$ gives the $U(5)$ limit, $\zeta=1$, $\chi=0$ gives the $SO(6)$ limit and $\zeta=1$, $\chi=\pm(\sqrt{7}/2)$ gives the

$SU(3)$ limit (in fact the + sign gives $\overline{SU(3)}$ limit). Using the (β_2, γ) parameters for a quadrupole surface, most general intrinsic state or coherent state (CS) for sdIBM is

$$|N;\beta_2,\gamma\rangle = \left[N!\left(1+\beta_2^2\right)^N\right]^{-1/2} \times \left\{s_0^\dagger + \beta_2\left[\cos\gamma\, d_0^\dagger + 2^{-1/2}\sin\gamma\left(d_2^\dagger + d_{-2}^\dagger\right)\right]\right\}^N |0\rangle\ , \qquad (2)$$

where $\beta_2 \geq 0$ and $0° \leq \gamma \leq 60°$. Now the equilibrium shape parameters $\left(\beta_2^0,\ \gamma_0\right)$ are obtained by using the expectation value of the H operator in the CS state

$$\begin{aligned}
E^{(\zeta,\chi)}(N;\beta_2,\gamma) &= \langle N;\beta_2,\gamma|H(\zeta,\chi)|N;\beta_2,\gamma\rangle\ ,\\
E^{(\zeta,\chi)}(N;\beta_2,\gamma)/(\epsilon_0 N) &= V^{(\zeta,\chi)}(N,\beta_2,\gamma)\\
&= \frac{\beta_2^2}{1+\beta_2^2}\left[(1-\zeta)-\left(\frac{\zeta}{4N}\right)(1+\chi^2)\right]+\frac{5}{1+\beta_2^2}\left(-\frac{\zeta}{4N}\right)\\
&\quad +\frac{(N-1)}{(1+\beta_2^2)^2}\left(\frac{-\zeta}{4N}\right)\left[4\beta_2^2-4\sqrt{\frac{2}{7}}\,\chi\,\beta_2^3\cos 3\gamma+\frac{2}{7}\,\chi^2\beta_2^4\right]
\end{aligned} \qquad (3)$$

and minimizing it with respect to β_2 and γ. Then, $(\partial E/\partial\beta_2) = 0$ and $(\partial E/\partial\gamma) = 0$ will give (β_2^0, γ_0). To confirm the minimum, we have to check whether the second derivative is positive, e.g., $(\partial^2 E/\partial(\beta_2)^2) > 0$. It is seen that, using absolute minimum, the $U(5)$ limit gives spherical ($\beta_2^0 = 0$, vibrational, E is independent of γ) shape, $SU(3)$ limit gives axially deformed ($\beta_2^0 \neq 0$, $\gamma_0 = 0°$, rotational) shape and the $SO(6)$ limit is γ-unstable ($\beta_2^0 \neq 0$ and E is independent of γ).

In the potential $V^{(\zeta,\chi)}(N, \beta_2, \gamma)$, one can take $\gamma = 0°$, a fixed value of χ (to represent $SU(3)$ or $SO(6)$ quadrupole operator) and study $V_{\text{min}}(\beta_2)$ by varying the control parameter ζ. This will establish if there is a phase transition. If the first derivative of $V_{\text{min}}^{\zeta}(\beta_2)$ with respect to ζ is discontinuous, we have first-order phase transition and the second derivative is discontinuous, then second-order phase transition. It is established that $U(5)$ to $SU(3)$ is first order, $U(5)$ to $SO(6)$ is second order and for $SO(6)$ to $SU(3)$ there is no phase transition. More importantly, it is shown that $SO(6)$ is not only a dynamical symmetry but also a critical point of prolate–oblate (i.e., $SU(3)$ to $\overline{SU(3)}$) first-order phase transition. Thus: Landau's theory of continuous phase transitions applies to IBM and the phase diagram for sdIBM is now completely determined [13]. Note that Landau theory is for $V \to \infty$ as $\beta_2 \to \infty$ but in sdIBM, $V(\beta_2)$ is finite for $\beta_2 \to \infty$. Therefore, the phase transition will be smoothed out for finite boson number (N), as it is the situation with real nuclei, but the signatures remain. Iachello and Zamfir [14] suggested that quantities such as isomer shifts (an order parameter) will distinguish first-order from second-order transitions. Let us add that Arias *et al* [16] argued that $U(5)$ to $SO(6)$ transition is second order QPT due to integrability, $U(5)$ to $SU(3)$ is first order due to level repulsion and $SU(3)$ to $\overline{SU(3)}$ is due to level crossing.

Let us add that the sdIBM CS given by eq. (2) was extended to sdgIBM that includes hexadecupole degree of freedom and/or pairs coupled angular momentum $J^\pi = 4^+$ and shapes in the symmetry limits of this model were determined in the past. Details are given in the first three references of [8]. Applications of this CS to QPT in sdgIBM model will be briefly discussed in §6.

3. Second-order QPT in two-level models

A general class of interacting boson models (IBMs) that are simple to study QPT are two-level models with degeneracies n_1 and n_2 for the two levels respectively [18,19]. Then the spectrum generating algebra (SGA) is $U(n_1+n_2)$ and it allows for two general group structures. These are: (i) $U(n_1+n_2) \supset U(n_1) \oplus U(n_2) \supset SO(n_1) \oplus SO(n_2) \supset K$ and (ii) $U(n_1+n_2) \supset SO(n_1+n_2) \supset SO(n_1) \oplus SO(n_2) \supset K$. As we shall see, the transition from (i) to (ii) is a second-order QPT.

Let us consider N bosons in two levels with degeneracies n_1 and n_2, respectively. We can think of n_1 degrees of freedom of a single boson arising with the bosons carrying angular momenta $\ell_1, \ell_2, \ldots, \ell_p$ such that $n_1 = \sum_{i=1}^{p}(2\ell_i + 1)$. Similarly, n_2 from $\ell'_1, \ell'_2, \ldots, \ell'_q$ such that $n_2 = \sum_{j=1}^{q}(2\ell'_j + 1)$. With this we can introduce boson creation operators $y_0^\dagger$ and $z_0^\dagger$ [19],

$$y_0^\dagger = \frac{1}{\sqrt{p}} \sum_{i=1}^{p} b^\dagger_{\ell_i,0}\,, \qquad z_0^\dagger = \frac{1}{\sqrt{q}} \sum_{j=1}^{q} b^\dagger_{\ell'_j,0} \tag{4}$$

and the pair creation operators $S_+(i)$, $i = 1, 2$ for the two levels are $S_+(1) = \sum_{i=1}^{p} b^\dagger_{\ell_i} \cdot b^\dagger_{\ell_i}$ and $S_+(2) = \sum_{j=1}^{q} b^\dagger_{\ell'_j} \cdot b^\dagger_{\ell'_j}$. For the combined system, the pair creation operator $S_+ = S_+(1) - S_+(2)$ and annihilation operator is $S_- = (S_+)^\dagger$. Note that S_+S_- is related to the quadratic Casimir invariant of $SO(n_1+n_2)$ in a simple manner. Now, N-boson coherent state can be written as [19]

$$|N, \alpha\rangle = \frac{1}{\sqrt{N!}}(\cos\alpha\; y_0^\dagger + \sin\alpha\; y_0^\dagger)^N \;|0\rangle\,. \tag{5}$$

A simple one-parameter Hamiltonian that interpolates the $U(n_1) \oplus U(n_2)$ and $SO(n_1+n_2)$ limits, without changing the $SO(n_1)$ and $SO(n_2)$ quantum numbers ω_1 and ω_2, respectively, is

$$H = \frac{1}{N}\hat{n}_2 + \frac{1}{N(N-1)}\eta S_+ S_-\,. \tag{6}$$

Then, the CS expectation value of H is $E(\alpha)$ where

$$E(\alpha) = \langle H\rangle^{N,\alpha} = \sin^2\alpha + \frac{\eta}{4}\cos^2 2\alpha\,. \tag{7}$$

Now, the minimum value of E, i.e., $E_{\min}(\alpha)$, is obtained using $\partial E/\partial\alpha = \sin 2\alpha(1 - \eta\cos 2\alpha) = 0$ and $\partial^2 E/\partial\alpha^2 = 2\cos 2\alpha - 2\eta\cos 4\alpha > 0$. Therefore, $\alpha = 0$ and $\cos 2\alpha = 1/\eta$ will give $E_{\min}$ with

$$\begin{aligned} E_{\min}(\alpha = 0 : \eta) &= \frac{\eta}{4} \quad \text{for} \quad \eta \le 1\,, \\ E_{\min}(\cos 2\alpha = 1/\eta : \eta) &= \frac{1}{4\eta}(2\eta - 1) \quad \text{for} \quad \eta \ge 1\,. \end{aligned} \tag{8}$$

For QPT we shall examine $\partial E_{\min}/\partial\eta$ and $\partial^2 E_{\min}/\partial\eta^2$. We have $\partial E_{\min}/\partial\eta = 1/4$ for $\eta \le 1$ and $1/4\eta^2$ for $\eta \ge 1$. This shows no discontinuity and hence, there is no first-order QPT. However, the second derivative is

$$\begin{aligned}\partial^2 E_{\min}/\partial\eta^2 &= 0 \quad \text{for} \quad \eta \le 1, \\ &= -\frac{1}{2\eta^3} \quad \text{for} \quad \eta \ge 1 \,. \end{aligned} \tag{9}$$

This gives $\eta_c = 1$ and at this value the second derivative changes from 0 to $-1/2$. Thus, the second derivative shows discontinuity and hence, the system exhibits second-order phase transition for all (n_1, n_2). Numerical results for some values of N and (n_1, n_2) with $(\omega_1, \omega_2) = (0, 0)$ are shown in figure 1. Here, the Hamiltonian used is

$$H = \frac{1-\xi}{N}\,\hat{n}_2 + \frac{\xi}{N^2}[4S_+S_- - \hat{N}(\hat{N} + n_1 + n_2 - 2)] \,. \tag{10}$$

The first part is the number operator giving the number of bosons in the n_2 orbit and this will preserve $U(n_1) \oplus U(n_2)$ symmetry. Similarly, the second part is the repulsive pairing interaction with eigenvalues $-\omega(\omega + n_1 + n_2 - 2)$ in the $SO(n_1 + n_2)$ limit. The $SO(n_1 + n_2)$ quantum number ω takes values $\omega = N, N-2, \ldots, 0$ or 1. Results in the figure and the CS description confirm that for $N \gg n_1 + n_2$ there is a QPT. Note that $\xi/(1-\xi) = \eta/4$ and therefore, $\eta_c = 1$ gives $\xi_c = 0.2$ and this is seen in figure 1. More discussions and examples are available in [18,19].

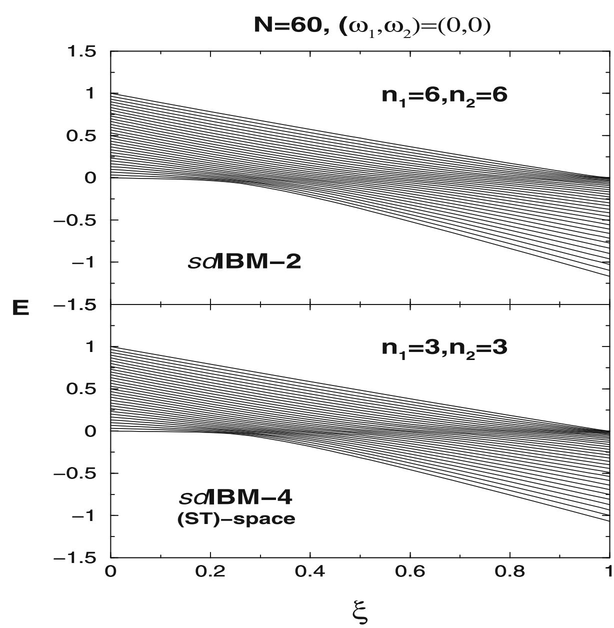

Figure 1. Spectra as a function of the mixing parameter ξ for $(n_1, n_2) = (6, 6)$ and $(3, 3)$ with the boson number $N = 60$. Results are shown for $(\omega_1, \omega_2) = (0, 0)$. All the results are obtained using the mathematical formalism given in [20].

4. Critical point symmetries

4.1 *Critical point symmetries for $U(5) \rightarrow SO(6)$ transition*

With QPT, from the point of view of experiments, an important question is: is it possible to obtain analytical predictions for observables at the phase transition point, i.e., are there solvable models or symmetries that describe the structure at the phase transition point? A better starting point now is [21] to consider the five-dimensional Schrödinger equation for the Bohr's Hamiltonian in the (β, γ) coordinates and the three Euler's angles $(\theta_1, \theta_2, \theta_3)$,

$$H = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2 \sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} - \frac{1}{4\beta^2}\sum_k \frac{L_k^2}{\sin^2\left(\gamma - \frac{2}{3}\pi k\right)}\right] + V(\beta, \gamma) \tag{11}$$

and solve for $H\Psi = E\Psi$. For the $U(5) \rightarrow SO(6)$ transition, first, $V(\beta, \gamma)$ is independent of γ, i.e., $V(\beta, \gamma) = U(\beta)$ and secondly, at the phase transition point, the potential $U(\beta)$ has flat behaviour. Therefore, as a first approximation one can choose an infinite square well with width β_w,

$$U(\beta) = 0 \;\text{ for }\; \beta \leq \beta_w\,, \qquad U(\beta) = \infty \qquad \text{for }\; \beta > \beta_w\,. \tag{12}$$

Now, one can separate the equation into β and γ parts with $\Psi = f(\beta)\Phi(\gamma, \theta)$. The γ part then gives

$$\left[-\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} + \frac{1}{4}\sum_k \frac{L_k^2}{\sin^2\left(\gamma - \frac{2}{3}\pi k\right)}\right]\Phi(\gamma, \theta) = \Lambda\Phi(\gamma, \theta)\,. \tag{13}$$

This is the equation for $C_2(SO(5))$ and its eigenvalues are $\Lambda = \tau(\tau + 3)$, $\tau = 0, 1, 2, \ldots$ (and $\tau \rightarrow L$ is well known). With $\epsilon = (2B/\hbar^2)E$, $u(\beta) = (2B/\hbar^2)U(\beta)$, the equation for $f(\beta)$ is

$$\left[-\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{\tau(\tau+3)}{\beta^2} + u(\beta)\right] f(\beta) = \epsilon\, f(\beta)\,. \tag{14}$$

The solution, apart from normalization factor, is

$$f_{\xi\tau}(\beta) = \beta^{-3/2} J_{\tau+3/2}\left(\frac{x_{\xi\tau}}{\beta_w}\beta\right), \quad E_{\xi\tau} = \frac{\hbar^2}{2B}\left(\frac{x_{\xi\tau}}{\beta_w}\right)^2, \tag{15}$$

where $x_{\xi\tau}$ is the ξth zero of $J_{\tau+3/2}(z)$. Thus, one has bands with $\xi = 1$, $\tau = 0(L = 0), 1(L = 2), 2(L = 2, 4), 3(L = 0, 3, 4, 6), \ldots$; $\xi = 2$, $\tau = 0(L = 0), 1(L = 2), 2(L = 2, 4), \ldots$ and so on. The $B(E2)$s can be calculated using (with t a parameter)

$$T^{E_2} = (t)\,\beta\left[D^2_{\mu,0}\cos\gamma + \frac{1}{\sqrt{2}}\left(D^2_{\mu,2} + D^2_{\mu,-2}\right)\sin\gamma\right]. \tag{16}$$

Situation with $U(\beta)$ being infinite well is called $E(5)$ limit [21]. Actually, $U(\beta)$ at the critical point is more like β^4. Therefore, studies with $U(\beta) = \beta^4$ and also using sextic-oscillator form which is quasiexactly solvable [22] have been carried out. The nucleus ^{134}Ba is a good example for $E(5)$ limit, so also many other nuclei. Some results are shown in table 1. The $U(5) \to SO(6)$ transition is also solved for very large N both using RG equation and by direct diagonalization. A summary of the various investigations related to $E(5)$ symmetry is given in table 2.

4.2 *Critical point symmetries for $U(5) \to SU(3)$ transition*

Here, the potential $V(\beta,\gamma)$ has a minimum at $\gamma = 0°$. Now, assuming $V(\beta,\gamma) = (2B/\hbar^2)^{-1}[u(\beta) + v(\gamma)]$, noting that around $\gamma = 0°$

$$\sum_k \frac{L_k^2}{\sin^2(\gamma - \frac{2}{3}\pi k)} \simeq \frac{4}{3}L^2 + L_z^2\left(\frac{1}{\gamma^2} - \frac{4}{3}\right) , \tag{17}$$

putting $\Psi = f_L(\beta)\,\eta_K(\gamma)\,D^L_{MK}(\theta)$ and replacing β^2 in the γ part by $\langle\beta^2\rangle$ we get the equations

$$\begin{aligned}
&\left[-\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{L(L+1)}{3\beta^2} + u(\beta)\right] f_L(\beta) = \epsilon_\beta\, f_L(\beta) , \\
&\left[-\frac{1}{\langle\beta^2\rangle\,\gamma}\frac{\partial}{\partial\gamma}\gamma\frac{\partial}{\partial\gamma} + \frac{1}{4\langle\beta^2\rangle}K^2\left(\frac{1}{\gamma^2} - \frac{4}{3}\right) + v(\gamma)\right]\eta_K(\gamma) = \epsilon_\gamma\,\eta_K(\gamma) .
\end{aligned} \tag{18}$$

Examining $V(\beta,\gamma)$ at the critical point, it is seen that $u(\beta)$ is flat and therefore can be replaced by infinite square well and also assume $v(\gamma)$ to be harmonic, i.e., $v(\gamma) \sim \gamma^2$. With these, eq. (18) can be solved and this gives,

$$\begin{aligned}
&E = E_0 + A_1\,(x_{s,L})^2 + A_2\,n_\gamma + A_3\,K^2 , \\
&x_{s,L} \text{ is } s\text{th zero of } J_v(z);\quad v = \left[\frac{L(L+1) - K^2}{3} + \frac{9}{4}\right]^{1/2} , \\
&n_\gamma = 0,\quad K = 0,\quad n_\gamma = 1,\quad K = \pm 2,\quad n_\gamma = 2,\quad K = 0, \pm 4, \ldots , \\
&K = 0 \to L = 0, 2, 4, \ldots , \\
&K \neq 0 \to L = |K|, |K|+1, |K|+2, \ldots .
\end{aligned} \tag{19}$$

Table 1. Some results with different $U(\beta)$.

System	$\frac{E(4_1^+)}{E(2_1^+)}$	$\frac{E(0_2^+)}{E(2_1^+)}$	$\frac{B(E2;4_1^+\to 2_1^+)}{B(E2;2_1^+\to 0_1^+)}$	$\frac{B(E2;0_2^+\to 2_1^+)}{B(E2;2_1^+\to 0_1^+)}$
$U(\beta)$:$E(5)$	2.20	3.03	1.68	0.86
$U(\beta)$:β^4	2.09	2.39	1.82	1.41
$U(\beta)$:Sextic	2.39	3.68	1.70	1.03
^{134}Ba	2.31	3.57	1.56	0.42

Table 2. Summary of $E(5)$ symmetry related investigations.

F Iachello, *Phys. Rev. Lett.* **85**, 3580 (2000) [introduced $E(5)$ limit]
R F Casten *et al*, *Phys. Rev. Lett.* **85**, 3584 (2000) [^{134}Ba as first example]
A Frank *et al*, *Phys. Rev. C* **65**, 014301 (2001) [^{104}Ru example]
N V Zamfir *et al*, *Phys. Rev. C* **65**, 044325 (2002) [^{102}Pd example]
M A Caprio, *Phys. Rev. C* **65**, 031304 (2002) [$U(\beta)$ is a finite depth square well]
J M Arias *et al*, *Phys. Rev. C* **68**, 041302 (R) (2003) [$U(\beta) = \beta^4$, compared with exact results (used Richardson equation) for N up to 1000 for energies and N up to 40 for $B(E2)$s]
L Fortunato and A Vitturi, *J. Phys. G* **29**, 1341 (2003) [Coulomb-like $U(\beta) = -A/\beta$ and Kratzer-like $U(\beta) = -2D\left(\frac{\beta_0}{\beta} - \frac{\beta_0^2}{2\beta^2}\right)$ for γ-unstable nuclei]
L Fortunato and A Vitturi, *J. Phys. G* **30**, 627 (2004) [Coulomb-like or Kratzer-like for $U(\beta)$ and harmonic oscillator for $U(\gamma)$ for β-soft and γ-soft axial rotors with ^{234}U example]
R M Clark *et al*, *Phys. Rev. C* **69**, 064322 (2004) [^{102}Pd, ^{106,108}Cd, ^{124}Te, ^{128}Xe and ^{134}Ba as examples]
G Levai and J M Arias, *Phys. Rev. C* **69**, 014304 (2004) [sextic oscillator for $U(\beta)$]
D Bonatsos *et al*, *Phys. Rev. C* **69**, 044316 (2004) [with $U(\beta) = \beta^2,\ \beta^4,\ \beta^6,\ \beta^8$; predictions for spectra and $B(E2)$s]
D Bonatsos *et al*, *Phys. Rev. C* **74**, 044306 (2006) [^{128,130}Xe between $E(5)$ and $SO(5)$]
J E Garcia Ramos *et al*, *Phys. Rev. C* **72**, 037301 (2005) [$U(\beta) = \beta^4$ is established to be the best – used matrix diagonalization with N up to 10,000 for energies and $B(E2)$s]
M A Caprio and F Iachello, *Nucl. Phys. A* **781**, 26 (2007) [E(5) symmetry details]

Thus, we have ground $K = 0$ band from $s = 1, n_\gamma = 0$, the β-band from $s = 2, n_\gamma = 0$, etc. Situation with $u(\beta)$ being infinite well and $V(\gamma) \sim \gamma^2$ is called $X(5)$ limit [23]. A summary of the various investigations on $X(5)$ symmetry is given in table 3.

Table 3. Summary of $X(5)$ symmetry related investigations.

F Iachello, *Phys. Rev. Lett.* **87**, 052502 (2001) [introduced $X(5)$ limit]
R F Casten *et al*, *Phys. Rev. Lett.* **87**, 052503 (2001) [^{152}Sm as first example]
R Krucken *et al*, *Phys. Rev. Lett.* **88**, 232501 (2002) [^{150}Nd example]
P G Bizzeti *et al*, *Phys. Rev. C* **66**, 031301 (R) (2002) [^{104}Mo with $n_\gamma = 0, 1, 2$]
M A Caprio *et al*, *Phys. Rev. C* **66**, 054310 (2002) [^{156}Dy example]
C Hutter *et al*, *Phys. Rev. C* **67**, 054315 (2003) [^{104,106}Mo examples]
D Tonev *et al*, *Phys. Rev. C* **69**, 034334 (2004) [^{154}Gd example]
E A McCutchan *et al*, *Phys. Rev. C* **69**, 024308 (2004) [^{162}Yb example]
D Bonatsos *et al*, *Phys. Rev. C* **69**, 014302 (2004) [exact solution with $u(\beta) = \beta^2$ and numerical results for $u(\beta) = \beta^4$, β^6 and β^8; ^{148}Nd as example for $X(5) - \beta^2$, ^{160}Yb as an example for $X(5) - \beta^4$ and ^{158}Er as an example for $X(5) - \beta^6$]
D Bonatsos *et al*, *Phys. Rev. C* **70**, 024305 (2004) [application of Davidson potential $u(\beta) = \beta^2 + \frac{\beta_0^4}{\beta^2}$]
A Leviatan, *Phys. Rev. C* **72**, 031305 (R) (2005) [$X(5)$ structure for finite N]
M Sugawara and H Kusakari, *Phys. Rev. C* **75**, 067302 (2007) [X(5) and analytical quadrupole and octupole axially symmetric (AQQA) model applied to ^{148}Nd]

Table 4. Other types of critical point symmetries and QPT in even–even nuclei.

F Iachello, *Phys. Rev. Lett.* **91**, 132502 (2003) [$Y(5)$ symmetry for axial to triaxial angular phase transition]
D Bonatsos *et al*, *Phys. Lett. B* **588**, 172 (2004) [$Z(5)$ symmetry for prolate to oblate shape phase transition]
D Bonatsos *et al*, *Phys. Rev. C* **71**, 064309 (2005) [critical point symmetry for transition from octupole deformation to octupole vibrations, ^{226}Th and ^{226}Ra examples]
R M Clark, A O Machiavelli, L Fortunato and R Krucken, *Phys. Rev. Lett.* **96**, 032501 (2006) [critical point of transition from pair-vibrational to pair-rotational regimes with example from pairing bands in Pb isotopes]
D J Rowe *et al*, *Phys. Rev. Lett.* **93**, 232502 (2004); **93**, 122502 (2004); *Nucl. Phys. A* **745**, 47 (2004); A **753**, 94 (2005); *A* **756**, 333 (2005); *A* **760**, 59 (2005) [phase transitions in GCM and their relation to IBM]
J N Ginocchio, *Phys. Rev. C* **71**, 064325 (2005) [critical point symmetry in the fermionic Ginocchio $SO(8)$ model]
F Pan and J P Draayer, *Nucl. Phys. A* **636**, 156 (1998); *Ann. Phys. (N.Y.)* **271**, 120 (1999); **275**, 224 (1999); *J. Phys. A* **33**, 9095 (2000); *Phys. Lett. A* **339**, 403 (2005) [algebraic Bethe ansatz method and its application to $U(5)$ to $O(6)$ transition]
H Yepez-Martinez, J Cseh and P O Hess, *Phys. Rev. C* **74**, 024310 (2006) [phase transitions in algebraic cluster models]
A Frank, F Iachello and P Van Isacker, *Phys. Rev. C* **73**, 061302(R) (2006) [phase transitions in configuration mixed models]
R Fossion, C E Alonso, J M Arias, L Fortunato and A Vitturi, *Phys. Rev. C* **76**, 014316 (2007) [shape phase transitions and two-nucleon transfer]
Yu Zhang, Z Hou and Y Liu, *Phys. Rev. C* **76**, 011305(R) (2007) [distinguishing first-order QPT from second-order QPT using $B(E2)$ ratios]
Z Hou, Yu Zhang and Y Liu, *Phys. Rev. C* **80**, 054308 (2009) [statistical properties of $E(5)$ and $X(5)$ symmetries]
L R Dai, F Pan, L Liu, L X Wang and J P Draayer, *Phys. Rev. C* **86**, 034316 (2012) [QPT studied using $[QQQ]^0$ where Q is the quadrupole generator of $SO(6)$ of IBM]

There are other types of critical point symmetries and also several extensions of QPT to proton–neutron IBM, sdgIBM, excited state QPT (EQPT) and so on. These are summarized in table 4. More interestingly, QPT and critical point symmetries are also studied in odd-A nuclei within the IBFM model [3,5] and experimental examples for these are found. These are summarized in table 5.

5. Example of a simple analytically solvable QPT

Van Isacker *et al* showed that three-body forces will give new results in sdIBM [24]. Going further, recently Draayer *et al* identified a simple situation with three-body forces that gives a solvable QPT [25]. Draayer *et al* considered the $SU(3)$ limit of sdIBM generated by the quadrupole (Q^2_μ) and angular momentum (L^1_μ) operators,

$$Q^2_\mu = [s^\dagger \tilde{d} + d^\dagger \tilde{s}]^2_\mu - \frac{\sqrt{7}}{2}\,(d^\dagger \tilde{d})^2_\mu\,, \qquad L^1_v = \sqrt{10}\,(d^\dagger \tilde{d})^1_v\,. \tag{20}$$

Table 5. Summary of QPT and critical point symmetries in odd-A nuclei.

F Iachello, *Phys. Rev. Lett.* **95**, 052503 (2005); M A Caprio and F Iachello, *Nucl. Phys. A* **781**, 26 (2007) [Transitional odd-A nuclei near the critical point of spherical to deformed γ-unstable transition and E(5/4) SUSY]
M S Fetea *et al*, *Phys. Rev. C* **73**, 051301(R) (2006) [first test of E(5/4) in ^{135}Ba]
C E Alonso, J M Arias and A Vitturi, *Phys. Rev. Lett.* **98**, 052501 (2007); *Phys. Rev. C* **75**, 064316 (2007) [critical point symmetry in odd-A nuclei: $U(5)$ to $SO(6)$ with odd particle in $j = 1/2, 3/2$ and $5/2$ orbits]
J Jolie, S Heinze, P Van Isacker and R F Casten, *Phys. Rev. C* **70**, 011305(R) (2004) [shape phase transitions in odd-mass nuclei using a SUSY approach with $U^B(6) \otimes U^F(12)$ symmetry limits]
C E Alonso, J M Arias, L Fortunato and A Vitturi, *Phys. Rev. C* **72**, 061302(R) (2005) [phase transition between $Spin(5)$ and $Spin(6)$ limits of IBFM]
C E Alonso, J M Arias, L Fortunato and A Vitturi, *Phys. Rev. C* **79**, 014306 (2009) [$U^{BF}(5)$ to SU^{BF} QPT with a fermion in $j = 1/2, 3/2$ and $5/2$ orbits]
D Petrelli, A Leviatan and F Iachello, *Ann. Phys. (N.Y.)* **326**, 926 (2011) [QPT in Bose-Fermi systems with sdIBM plus a particle in a j-orbit]

The $SU(3)$ algebra admits quadratic ($\hat{C}_2$) and cubic Casimir ($\hat{C}_3$) invariants and they are given by

$$\begin{aligned}\hat{C}_2 &= 2Q \cdot Q + \frac{3}{4} L \cdot L,\\ \hat{C}_3 &= -\frac{4}{9}\sqrt{35}\,[Q \times Q \times Q]^0 - \frac{\sqrt{15}}{2}\,[L \times Q \times L]^0 .\end{aligned} \tag{21}$$

With respect to the $SU(3)$ algebra, N boson states are denoted by $|N, (\lambda\mu), K, L, M\rangle$. Note that $(\lambda\mu)$ is a $SU(3)$ irreducible representation (irrep) and K has the geometric meaning as the 'K' in GCM for rotational nuclei. Given an N, the allowed $(\lambda\mu)$, K for a given $(\lambda\mu)$ and L for a given K are given by $(-L \le M \le L)$

$$\begin{aligned}&\lambda = 2f_1 - 2f_2, \quad \mu = 2f_2 - 2f_3;\\ &f_1 \ge f_2 \ge f_3 \ge 0, \quad f_1 + f_2 + f_3 = N,\\ &K = 0, 2, \ldots, \min(\lambda, \mu),\\ &L = 0, 2, \ldots, \max(\lambda, \mu) \qquad \text{for} \ \ K = 0,\\ &L = K, K+1, K+2, \ldots, K + \max(\lambda, \mu) \qquad \text{for} \quad K \ne 0.\end{aligned} \tag{22}$$

Action of the Casimir invariants on any $SU(3)$ state $|N, (\lambda\mu), K, L, M\rangle$ will give the same state multiplied by their eigenvalue that depends only on the $SU(3)$ irrep. The eigenvalues $C_r[(\lambda\mu)]$ of $\hat{C}_r$, $r = 2, 3$ are

$$\begin{aligned}C_2[(\lambda\mu)] &= \lambda^2 + \mu^2 + \lambda\mu + 3(\lambda + \mu),\\ C_3[(\lambda\mu)] &= \frac{1}{9}(\lambda - \mu)(2\lambda + \mu + 3)(\lambda + 2\mu + 3).\end{aligned} \tag{23}$$

Let us mention that an easy-to-understand discussion of $SU(3)$ algebra for nuclei is given in [5]. It is easy to see from eq. (23) that $-\hat{C}_2$ will give ground state (gs) with $SU(3)$ irrep having largest value for λ (with $\lambda \gg \mu$) and $\hat{C}_3$ gives gs with $SU(3)$ irrep having largest value for μ (with $\mu \gg \lambda$). It is easy to see from geometric analysis that the former is prolate and the latter is oblate. Therefore, the simple Hamiltonian

$$H(x) = \frac{(x-1)}{N}\,\hat{C}_2 + \frac{x}{N(N-1)}\,\hat{C}_3 \tag{24}$$

will generate prolate–oblate transition in gs as we change the parameter x from 0 to 1. Equations (22) and (23) will give all the eigenvalues of $H(x)$ without any matrix construction. For example, for $N = 8$ the $SU(3)$ irreps are (16, 0), (12, 2), (8, 4), (10, 0), (4, 6), (6, 2), (0, 8), (2, 4), (4, 0) and (0, 2). Then, $H(x)$ with $x = 0$ gives (16, 0) as gs (prolate) and $x = 1$ gives (0, 8) as gs (oblate). Similarly, for $N = 9$ the $SU(3)$ irreps are (18, 0), (14, 2), (10, 4), (12, 0), (6, 6), (8, 2), (2, 8), (4, 4), (6, 0), (0, 6), (2, 2) and (0, 0). Then, $H(x)$ with $x = 0$ gives (18, 0) as gs (prolate) and $x = 1$ gives (2, 8) as gs (oblate). Using the CS defined by eq. (2), the CS expectation value (valid as $N \to \infty$) $\langle N; \beta_2, \gamma|H(x)|N; \beta_2, \gamma\rangle$ of $H(x)$ in eq. (24) can be determined. Then, carrying an analysis similar to the one in §3, it can be shown that $H(x)$ generates prolate–oblate transition and it is a first-order QPT. The transition occurs at $x = x_c = 0.6$ with $\beta = \sqrt{2}$ and $\gamma = 0°$ for $x < 0.6$ and $\beta = 1/\sqrt{2}$ and $\gamma = 60°$ for $x > 0.6$. Using eqs (22) and (23), calculations for finite N are easy to perform and again one sees that as N increases $x_c \to 0.6$ (see [25] for further details). Let us add that the simple model defined by eq. (24) explains the experimental data with prolate–oblate shape phase transition in ^{180}Hf, $^{182-186}$W, ^{188,190}O and $^{192-198}$Pt nuclei.

6. Open problems in QPT

In concluding this article, listed below are some open problems in QPT in nuclei:

(1) Study of phase transitions in sdgIBM using the symmetry limits [8] $SU_{sdg}(3)$, $SO_{sdg}(15)$ and $U(6) \oplus U(9)$ was done [26]. It will be interesting to study QPT by including the other two important symmetry limits $SU_{sdg}(5)$, $SU_{sdg}(6)$ of sdgIBM. Note that $SU_{sdg}(5)$ generates $\Delta L = 4$ staggering.

(2) It is also possible to carry out a simple analysis of QPT using $SU_{sdg}(3)$ with H containing quadratic and cubic Casimir invariants as it is done for $SU_{sd}(3)$ (see §5). In addition, it will be interesting to carry out an analysis of QPT using H as a linear combination of $\hat{C}_2$, $\hat{C}_3$ and $\hat{C}_4$ of $SU_{sdg}(5)$ and similarly also for the $SU_{sdg}(6)$ limit.

(3) QPT and the phase diagram for IBM-2 (proton–neutron IBM) was addressed using $U(12) \supset U(6) \otimes SU(2)$ symmetry limits in [27]. However, IBM-2 also admits $SO(12)$ symmetry limits [12,28] and they appear for systems with F-spin broken but M_F is good. It is important to study QPT in IBM-2 including $SO(12)$ symmetry chains so that the phase diagram for IBM-2 can be completely determined (note that M_F is preserved for all nuclei but F-spin could be broken).

(4) For bosons in n_1 levels and carrying n_2 internal degrees of freedom, the SGA is $U(n_1 n_2)$. Then, we have two subalgebra chains: (i) $U(n_1 n_2) \supset U(n_1) \otimes U(n_2) \supset$

$SO(n_1) \otimes SO(n_2)$ and (ii) $U(n_1 n_2) \supset SO(n_1 n_2) \supset SO(n_1) \otimes SO(n_2)$. They appear in IBM-2, IBM-3 and IBM-4 models [12]. QPT between (i) and (ii) need to be investigated.

(5) QPT in IBFM $SU^{BF}(3)$ for odd-A nuclei [29] and $SU^{BFF}(3)$ of IBFFM for odd–odd nuclei [30], as it was done for even–even nuclei (see §5), will give new insights into QPT in odd-A and odd–odd nuclei. In general, it will be interesting and useful to study QPT in heavy $N = Z$ odd–odd nuclei as they are of astrophysical interest and also they exhibit proton–neutron pairing correlations. These nuclei are being studied using IBM-4, shell model $SO(8)$ pairing algebra and the so-called deformed shell model [6,31,32].

(6) Phase transitions between the three limits of the shell model $SO(8)$ pairing symmetries need to be studied. It should be noted that the three limits of this fermion model generate vibrations, rotations and γ-soft spectra in isospin space [6].

(7) In sdIBM, for $L \leq 4$ it is possible to write analytical formulas for Hamiltonian matrix elements (V K B Kota, unpublished) when H is one plus two-body. Therefore, QPT related issues can be analysed in sdIBM for very large values of boson number N with $L \leq 4$. Implementing this will be useful in understanding the results obtained so far using sdIBM and also it can be used for EQPT studies.

(8) EQPT [33,34] is related to changes in level densities and other statistical quantities. Further studies on EQPT will give new insights into the relationship between QPT, quantum chaos and random matrix theory.

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References

[1] P Van Isacker, *Rep. Prog. Phys.* **62**, 1661 (1999)

[2] F Iachello and A Arima, *The interacting boson model* (Cambridge University Press, Cambridge, 1987)

[3] F Iachello and P Van Isacker, *The interacting boson–fermion model* (Cambridge University Press, Cambridge, 1991)

[4] D J Rowe and J L Wood, *Fundamentals of nuclear models: Foundational models* (World Scientific, Singapore, 2010)

[5] V K B Kota and Y D Devi, *Nuclear shell model and the interacting boson model: Lecture notes for practitioners*, published by IUC-DAEF, Calcutta Centre (Kolkata, India, 1996)

[6] V K B Kota and J A Castilho Alcarás, *Nucl. Phys. A* **764**, 181 (2006)

[7] S Lerma H *et al*, *Phys. Rev. Lett.* **99**, 032501 (2007)
S Lerma H, *Phys. Rev. C* **82**, 014302 (2010)

[8] Y D Devi and V K B Kota, *Z. Phys. A* **337**, 15 (1990); in *Symmetries in Science VII* edited by B Gruber and T Otsuka (Plenum, New York, 1994), p. 307; *Pramana – J. Phys.* **39**, 413 (1992)
Y Liu *et al*, *Phys. Rev. C* **59**, 2511 (1999)

[9] J Jolie, *Scientific American* **287**, 54 (2002)
A Frank, P Van Isacker and J Jolie, *Symmetries in atomic nuclei: From isospin to supersymmetry* (Springer, Heidelberg, 2010)
G Levai, J Cseh and P Van Isacker, *J. Phys. G* **34**, 1729 (2007)
[10] Y Alhassid and A Leviatan, *J. Phys. A* **25**, L1265 (1992)
A Leviatan, *Phys. Rev. Lett.* **77**, 818 (1996)
J Escher and A Leviatan, *Phys. Rev. Lett.* **84**, 1866 (2000)
A Leviatan and P Van Isacker, *Phys. Rev. Lett.* **89**, 222501 (2002)
A Leviatan, *Phys. Rev. Lett.* **98**, 242502 (2007)
A Shirokov, N A Smirnova and P Van Isacker, *Phys. Lett. B* **434**, 237 (1998)
D Kusnezov, *Phys. Rev. Lett.* **79**, 537 (1997)
[11] V K B Kota, *Pramana – J. Phys.* **48**, 1035 (1997)
[12] V K B Kota, in *Focus on boson research* edited by A V Ling (Nova Science Publishers Inc., New York, 2006) p. 57
[13] P Cejnar, J Jolie and R F Casten, *Rev. Mod. Phys.* **82**, 2155 (2010)
[14] F Iachello and N V Zamfir, *Phys. Rev. Lett.* **92**, 212501 (2004)
[15] F Iachello, N V Zamfir and R F Casten, *Phys. Rev. Lett.* **81**, 1191 (1998)
P Cejnar and J Jolie, *Phys. Rev. E* **61**, 6237 (2000)
J Jolie, R F Casten, P von Brentano and V Werner, *Phys. Rev. Lett.* **87**, 62501 (2001)
J Jolie, P Cejnar, R F Casten, S Heinze, A Linnemann and V Werner, *Phys. Rev. Lett.* **89**, 182502 (2002)
[16] J M Arias, J Dukelsky and J E García-Ramos, *Phys. Rev. Lett.* **91**, 162502 (2003)
[17] P Cejnar, P Stransky and M Macek, *Nucl. Phys. News* **21(4)**, 22 (2011)
[18] P Cejnar and F Iachello, *J. Math. Phys.* **40**, 581 (2007)
M A Caprio, J H Skrabacz and F Iachello, *J. Phys. A: Mat. Theor.* **44**, 075303 (2011)
[19] V K B Kota, *AIP Conf. Proc.* **1524**, 52 (2013)
[20] V K B Kota, *J. Math. Phys.* **38**, 6639 (1997)
[21] F Iachello, *Phys. Rev. Lett.* **85**, 3580 (2000)
[22] A G Ushveridze, *Quasi-exactly solvable models in quantum mechanics* (Institute of Physics, Bristol, 1994)
[23] F Iachello, *Phys. Rev. Lett.* **87**, 052502 (2001)
[24] P Van Isacker, *Phys. Rev. Lett*. **83**, 4269 (1999)
Y F Smirnov, N A Smirnova and P Van Isacker, *Phys. Rev. C* **61**, 041302(R) (2000)
J E Garcia-Ramos, A Leviatan and P Van Isacker, *Phys. Rev. Lett.* **102**, 112502 (2009)
[25] Yu Zhang, Feng Pan, Yu-Xin Liu, Yan-An Luo and J P Draayer, *Phys. Rev. C* **85**, 064312 (2012)
[26] P Van Isacker, A Bouldjedri and S Zerguine, *Nucl. Phys. A* **836**, 225 (2010)
[27] M A Caprio and F Iachello, *Phys. Rev. Lett.* **93**, 242502 (2004); *Ann. Phys. (N.Y.)* **318**, 454 (2005)
J M Arias, J E Garcia-Ramos and J Dukelsky, *Phys. Rev. Lett.* **93**, 212501 (2004)
A Giannatiempo, L Fortunato and A Vitturi, *Phys. Rev. C* **86**, 034311 (2012)
[28] V K B Kota, *Pramana – J. Phys.* **60**, 59 (2003)
[29] R Bijker and V K B Kota, *Ann. Phys. (N.Y.)* **187**, 148 (1988)
[30] V K B Kota and U Datta Pramanik, *Euro. Phys. J. A* **3**, 243 (1998)
[31] V K B Kota, *Ann. Phys. (N.Y.)* **280**, 1 (2000)
[32] V K B Kota and R Sahu, in: *Proceedings of the Workshop on Frontiers in Gamma Ray Spectroscopy (FIG09)* edited by R Palit and R Pillay (TIFR, Mumbai, India, 2011) p. 83
[33] P Cejnar, M Macek, S Heinze, J Jolie and J Dobes, *J. Phys. A. Math. Gen.* **39**, L515 (2006)
[34] M A Caprio, P Cejnar and F Iachello, *Ann. Phys. (N.Y.)* **323**, 1106 (2008)