PRAMANA
— journal of
physics

© Indian Academy of Sciences

Vol. 81, No. 6
December 2013
pp. 911–924

# Exact solutions for nonlinear variants of Kadomtsev–Petviashvili $(n, n)$ equation using functional variable method

M MIRZAZADEH[1,*] and M ESLAMI[2]

[1]Department of Engineering Sciences, Faculty of Technology and Engineering, East of Guilan, University of Guilan, Rudsar, Iran
[2]Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran
*Corresponding author. E-mail: mirzazadehs2@guilan.ac.ir

MS received 22 July 2013; revised 16 September 2013; accepted 3 October 2013
**DOI:** 10.1007/s12043-013-0632-2; ***e*Publication:** 5 December 2013

**Abstract.** Studying compactons, solitons, solitary patterns and periodic solutions is important in nonlinear phenomena. In this paper we study nonlinear variants of the Kadomtsev–Petviashvili (KP) and the Korteweg–de Vries (KdV) equations with positive and negative exponents. The functional variable method is used to establish compactons, solitons, solitary patterns and periodic solutions for these variants. This method is a powerful tool for searching exact travelling solutions in closed form.

**Keywords.** Functional variable method; compacton; solitary pattern;

Kadomtsev–Petviashvili equation.

**PACS Nos 02.30.Jr; 05.45.Yv**

## 1. Introduction

Travelling wave solutions play important roles in mathematical physics and engineering sciences. These solutions may well describe various phenomena in nature, such as vibrations, solitons and propagation with a finite speed, and thus they may give more insight into the physical aspects of problems and may be easily used in other applications. In mathematics, for a nonlinear partial differential equation (PDE), usually the travelling wave solutions are considered first. Recently, many powerful methods to construct the travelling wave solutions of nonlinear PDEs were presented. Some of them are: ansatz method, exp-function method, tanh method, first integral method, adomian decomposition method, simplest equation method, sine–cosine method, Hirota's bilinear method, functional variable method and so on [1–20].

A large number of equations in many areas of applied mathematics, physics and engineering appear as nonlinear wave equations. One of the most important one-dimensional nonlinear wave equation is the KdV equation which describes the evolution of weakly nonlinear and weakly dispersive wave used in various fields such as solid-state physics, plasma physics, fluid physics and quantum field theory. A decade ago this equation was generalized to the well-known K$(m, n)$ equation. There are several nonlinear variants of the KdV equation which have compacton solutions. Another variant is the CSS equation which is discussed in [21, 22]. The nonlinear variants of the KdV equations are called the K$(m, n)$ equations. Rosenau and Hyman [23] studied the role of nonlinear dispersion in the formation of patterns in liquid drops and introduced the nonlinearly dispersive K$(m, n)$ equation:

$$\frac{\partial u}{\partial t} + a\frac{\partial}{\partial x}(u^m) + \frac{\partial^3}{\partial x^3}(u^n) = 0, \quad m > 0,\ n > 1, \tag{1}$$

which have the special property for certain $m$ and $n$. Here, the first term is the linear term, while the second term represents the nonlinear term and the third term is the dispersion term. This equation is the generalized form of the KdV equation. When $a = 1$, eq. (1) is referred to as the focussing $(+)$ branch, the focussing branch exhibits compacton solutions and if $a = -1$, then eq. (1) is referred to as the defocussing $(-)$ branch, the defocussing branch exhibits solitary pattern solutions.

Equation (1) is the major equation for compactons. These are defined as solitons with compact support. Thus they vanish outside a finite core region. Compactons are free from exponential tails. Compactons are generated as a result of delicate interaction between the nonlinear convection term $(u^m)_x$ and the nonlinear dispersion term $(u^n)_{xxx}$. Also compactons do not have the properties of solitons. They do not just suffer a phase shift after scattering. So they only have some properties of particles. A discussion of how even shocks get generated after scattering is given in [24].

The well-known $(1 + 3)$-dimensional Kadomtsev–Petviashvili (4DKP) equation [25]

$$\frac{\partial}{\partial x}\left(u_t + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\right) + 3\kappa\frac{\partial^2 u}{\partial y^2} - 3\frac{\partial^2 u}{\partial z^2} = 0, \tag{2}$$

with $\kappa = \pm 1$, is a universal model for the propagation of weakly nonlinear dispersive long waves which are essentially one directional, with weak transverse effects. The KP equation also arises naturally in many other applications, particularly in plasma physics, gas dynamics, and elsewhere.

The KP equation is the best known two-dimensional generalization of the KdV equation. Wazwaz [15–19] and Ismail *et al* [20] obtained compactons, solitons, solitary patterns and periodic solutions for the following nonlinear variants of KP and KdV equations by using the tanh method, the sine–cosine method and the mathematical transformation:

KP$(n, n)$ equation:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + a\frac{\partial}{\partial x}(u^n) + b\frac{\partial^3}{\partial x^3}(u^n)\right) + \kappa\frac{\partial^2 u}{\partial y^2} = 0. \tag{3}$$

KP-K$(n+1,\ n+1)$ equation:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+a\frac{\partial}{\partial x}(u^{n+1})+b\frac{\partial}{\partial x}\left[u\frac{\partial^2}{\partial x^2}(u^n)\right]\right)+\kappa\frac{\partial^2 u}{\partial y^2}=0. \tag{4}$$

K$(n, n)$ equation:

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}(au+bu^n)+\kappa\frac{\partial^3}{\partial x^3}(u^n)=0. \tag{5}$$

$(1+k)$-dimensional KP$(3n, 3n)$ equation:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+a\frac{\partial}{\partial x}(u^{3n})+b\frac{\partial}{\partial x}\left[u^n\frac{\partial^2}{\partial x^2}(u^{2n})\right]\right)+\frac{\partial^2 u}{\partial y^2}+(k-2)\frac{\partial^2 u}{\partial z^2}=0,$$
$$k=2,3. \tag{6}$$

In this paper, we shall obtain new compactons, solitons, solitary patterns and periodic solutions for the KP$(n, n)$, the KP$(n+1, n+1)$, the K$(n, n)$ and the $(1+k)$-dimensional-KP$(3n, 3n)$ equations with positive and negative exponents by using the functional variable method.

## 2. The functional variable method

The functional variable method, which is a direct and effective algebraic method for the computation of compactons, solitons, solitary patterns and periodic solutions, was first proposed by Zerarka *et al* [11]. This method was further developed by many authors [12–14]. We now summarize the functional variable method, established by Zerarka *et al* [11], the details of which can be found in [11–14] among many others.

Consider a general nonlinear PDE in the form

$$P\left(u,\frac{\partial u}{\partial t},\frac{\partial u}{\partial y},\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial t^2},\frac{\partial^2 u}{\partial y^2},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial t\partial x},\ldots\right)=0, \tag{7}$$

where $P$ is a polynomial in $u$ and its partial derivatives. Use a wave variable $\xi=\alpha_0 t+\alpha_1 x+\alpha_2 y+\delta$ so that

$$u(x,y,t)=U(\xi). \tag{8}$$

Equation (7) can be converted to an ordinary differential equation (ODE) as

$$Q(U,U',U'',U''',\ldots)=0, \tag{9}$$

where $Q$ is a polynomial in $U=U(\xi)$ and prime denotes derivative with respect to $\xi$. If all terms contain derivatives, then eq. (9) is integrated where integration constants are considered zeros.

Let us make a transformation in which the unknown function $U(\xi)$ is considered as a functional variable of the form

$$U_\xi=F(U) \tag{10}$$

and some successive derivatives of $U$ are

$$U_{\xi\xi} = \frac{1}{2}(F^2)',$$
$$U_{\xi\xi\xi} = \frac{1}{2}(F^2)''\sqrt{F^2},$$
$$U_{\xi\xi\xi\xi} = \frac{1}{2}[(F^2)'''F^2 + (F^2)''(F^2)'], \tag{11}$$

where $' = \mathrm{d}/\mathrm{d}U$.

The ODE (9) can be reduced in terms of $U$, $F$ and its derivatives upon using the expressions of eq. (11) into eq. (9) gives

$$R(U, F, F', F'', F''', ...) = 0. \tag{12}$$

The key idea of this particular form (eq. (12)) is of special interest because it admits analytical solutions for a large class of nonlinear wave-type equations. After integration, eq. (12) provides the expression of $F$, and this in turn together with eq. (10) give the relevant solutions to the original problem.

*Remark.* The functional variable method definitely can be applied to nonlinear PDEs which can be converted to a second-order ordinary differential equations (ODE) through the travelling wave transformation.

## 3. Applications

In this section, we present four examples to illustrate the applicability of the functional variable method to establish compactons, solitons, solitary patterns and periodic solutions of nonlinear PDEs.

### 3.1 *KP(n, n) equation*

3.1.1 *The positive exponents.* Let us first consider the KP($n, n$) equation with positive exponents

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + a\frac{\partial}{\partial x}\left(u^n\right) + b\frac{\partial^3}{\partial x^3}(u^n)\right) + \kappa\frac{\partial^2 u}{\partial y^2} = 0. \tag{13}$$

Under the travelling wave transformation

$$u(x, y, t) = U(\xi), \quad \xi = \alpha_0 t + \alpha_1 x + \alpha_2 y + \delta \tag{14}$$

we have

$$\alpha_1(\alpha_0 U' + a\alpha_1(U^n)' + b\alpha_1^3(U^n)''')' + \kappa\alpha_2^2 U'' = 0, \tag{15}$$

where $U = U(\xi)$ and prime denotes derivative with respect to $\xi$.

Integrating eq. (15) twice with respect to $\xi$ and neglecting constants of integration, we get

$$(\alpha_0\alpha_1 + \kappa\alpha_2^2)U + a\alpha_1^2(U^n) + b\alpha_1^4(U^n)'' = 0. \tag{16}$$

We use the transformation

$$U(\xi) = V^{1/n}(\xi), \tag{17}$$

that will reduce eq. (16) into the ODE

$$\left(\alpha_0\alpha_1 + \kappa\alpha_2^2\right) V^{1/n} + a\alpha_1^2 V + b\alpha_1^4 V'' = 0. \tag{18}$$

Then we use the transformation

$$V_\xi = F(V), \tag{19}$$

that will convert eq. (18) to

$$\left(\alpha_0\alpha_1 + \kappa\alpha_2^2\right) V^{1/n} + a\alpha_1^2 V + \frac{b\alpha_1^4 (F^2(V))'}{2} = 0. \tag{20}$$

Thus, we get from eq. (20) the expression of the function $F(V)$ as

$$F(V) = \sqrt{-\frac{a}{b\alpha_1^2}} V \sqrt{1 + \frac{2n(\alpha_0\alpha_1 + \kappa\alpha_2^2)}{a(n+1)\alpha_1^2} V^{(1-n)/n}}. \tag{21}$$

After making the change of variables

$$-\frac{2n(\alpha_0\alpha_1 + \kappa\alpha_2^2)}{a(n+1)\alpha_1^2} V^{(1-n)/n} = Z, \tag{22}$$

and using the relation (19), the two possible solutions of eq. (21) are in the following forms:

$$V_1(\xi) = \left\{ -\frac{a(n+1)\alpha_1^2}{2n(\alpha_0\alpha_1 + \kappa\alpha_2^2)} \operatorname{sech}^2 \left[ \frac{1-n}{2n\alpha_1} \sqrt{-\frac{a}{b}} \xi \right] \right\}^{n/(1-n)} \tag{23}$$

and

$$V_2(\xi) = \left\{ \frac{a(n+1)\alpha_1^2}{2n(\alpha_0\alpha_1 + \kappa\alpha_2^2)} \operatorname{csch}^2 \left[ \frac{1-n}{2n\alpha_1} \sqrt{-\frac{a}{b}} \xi \right] \right\}^{n/(1-n)}. \tag{24}$$

Now, from eqs (23) and (24) we deduce the solutions of the original problem

$$U_1(\xi) = \left\{ -\frac{a(n+1)\alpha_1^2}{2n(\alpha_0\alpha_1 + \kappa\alpha_2^2)} \operatorname{sech}^2 \left[ \frac{1-n}{2n\alpha_1} \sqrt{-\frac{a}{b}} \xi \right] \right\}^{1/(1-n)} \tag{25}$$

and

$$U_2(\xi) = \left\{ \frac{a(n+1)\alpha_1^2}{2n(\alpha_0\alpha_1 + \kappa\alpha_2^2)} \operatorname{csch}^2 \left[ \frac{1-n}{2n\alpha_1} \sqrt{-\frac{a}{b}} \xi \right] \right\}^{1/(1-n)}. \tag{26}$$

We can easily obtain the following solitary pattern solutions:

$$\begin{aligned} u_1(x, y, t) = & \left\{ -\frac{2n(\alpha_0\alpha_1 + \kappa\alpha_2^2)}{a(n+1)\alpha_1^2} \right. \\ & \left. \times \cosh^2 \left[ \frac{n-1}{2n\alpha_1} \sqrt{-\frac{a}{b}} (\alpha_0 t + \alpha_1 x + \alpha_2 y + \delta) \right] \right\}^{1/(n-1)} \end{aligned} \tag{27}$$

and

$$u_2(x,y,t)=\left\{\frac{2n(\alpha_0\alpha_1+\kappa\alpha_2^2)}{a(n+1)\alpha_1^2}\right.$$
$$\left.\times\ \sinh^2\left[\frac{n-1}{2n\alpha_1}\sqrt{-\frac{a}{b}}\,(\alpha_0 t+\alpha_1 x+\alpha_2 y+\delta)\right]\right\}^{1/(n-1)}, \qquad (28)$$

for $(a/b)<0$, it is easy to see that solutions (27) and (28) can be reduced to compacton solutions as follows:

$$u_3(x,y,t)=\left\{-\frac{2n(\alpha_0\alpha_1+\kappa\alpha_2^2)}{a(n+1)\alpha_1^2}\right.$$
$$\left.\times\ \cos^2\left[\frac{n-1}{2n\alpha_1}\sqrt{\frac{a}{b}}\,(\alpha_0 t+\alpha_1 x+\alpha_2 y+\delta)\right]\right\}^{1/(n-1)} \qquad (29)$$

and

$$u_4(x,y,t)=\left\{-\frac{2n(\alpha_0\alpha_1+\kappa\alpha_2^2)}{a(n+1)\alpha_1^2}\right.$$
$$\left.\times\ \sin^2\left[\frac{n-1}{2n\alpha_1}\sqrt{\frac{a}{b}}\,(\alpha_0 t+\alpha_1 x+\alpha_2 y+\delta)\right]\right\}^{1/(n-1)}, \qquad (30)$$

for $(a/b)>0$.

3.1.2 *The negative exponents.* In this case, the KP$(n,n)$ equation becomes

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+a\frac{\partial}{\partial x}(u^{-n})+b\frac{\partial^3}{\partial x^3}(u^{-n})\right)+\kappa\frac{\partial^2 u}{\partial y^2}=0. \qquad (31)$$

In view of the results (27) and (28) for $(a/b)<0$, and $n$ is replaced by $-n$, we deduce the following exact soliton-like solutions for eq. (31):

$$u_1(x,y,t)=\left\{\frac{a(1-n)\alpha_1^2}{2n(\alpha_0\alpha_1+\kappa\alpha_2^2)}\right.$$
$$\left.\times\ \operatorname{sech}^2\left[\frac{n+1}{2n\alpha_1}\sqrt{-\frac{a}{b}}\,(\alpha_0 t+\alpha_1 x+\alpha_2 y+\delta)\right]\right\}^{1/(n+1)} \qquad (32)$$

and

$$u_2(x,y,t)=\left\{\frac{a(n-1)\alpha_1^2}{2n(\alpha_0\alpha_1+\kappa\alpha_2^2)}\right.$$
$$\left.\times\ \operatorname{csch}^2\left[\frac{n+1}{2n\alpha_1}\sqrt{-\frac{a}{b}}\,(\alpha_0 t+\alpha_1 x+\alpha_2 y+\delta)\right]\right\}^{1/(n+1)}. \qquad (33)$$

Similarly, we now use the results (29) and (30) for $(a/b)>0$, and $n$ is replaced by $-n$, we have the following analytical periodic solutions for eq. (31):

$$u_3(x,y,t)=\left\{\frac{a(1-n)\alpha_1^2}{2n(\alpha_0\alpha_1+\kappa\alpha_2^2)}\right.$$
$$\left.\times\ \sec^2\left[\frac{n+1}{2n\alpha_1}\sqrt{-\frac{a}{b}}\,(\alpha_0 t+\alpha_1 x+\alpha_2 y+\delta)\right]\right\}^{1/(n+1)} \qquad (34)$$

and

$$u_4(x, y, t) = \left\{ \frac{a(1-n)\alpha_1^2}{2n(\alpha_0\alpha_1 + \kappa\alpha_2^2)} \right.$$
$$\left. \times \csc^2\left[\frac{n+1}{2n\alpha_1}\sqrt{-\frac{a}{b}}\,(\alpha_0 t + \alpha_1 x + \alpha_2 y + \delta)\right]\right\}^{1/(n+1)}. \qquad (35)$$

### 3.2 $(1+k)$-*dimensional KP*$(3n,3n)$ *equation*

3.2.1 *The positive exponents.* Let us demonstrate the application of functional variable method for finding the exact travelling wave solutions of the $(1+k)$-dimensional KP$(3n, 3n)$ equation

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + a\frac{\partial}{\partial x}(u^{3n}) + b\frac{\partial}{\partial x}\left[u^n\frac{\partial^2}{\partial x^2}(u^{2n})\right]\right) + \frac{\partial^2 u}{\partial y^2} + (k-2)\frac{\partial^2 u}{\partial z^2} = 0, \quad (36)$$

where $k = 2,\ 3$, indicates the dimensions of the two- and three-dimensional spaces.

To look for the exact solutions of eq. (36), we make transformation

$$u(x, y, t) = U(\xi), \quad \xi = \alpha_0 t + \alpha_1 x + \alpha_2 y + (k-2)\alpha_3 z + \delta, \qquad (37)$$

and generate the reduced nonlinear ODE in the form

$$\alpha_1(\alpha_0 U' + a\alpha_1(U^{3n})' + b\alpha_1^3[U^n(U^{2n})'']')' + ((k-2)^3\alpha_3^2 + \alpha_2^2)U'' = 0. \quad (38)$$

Integrating (38) twice with respect to $\xi$ and setting the constants of integration to be zero we find

$$\left(\alpha_0\alpha_1 + (k-2)^3\alpha_3^2 + \alpha_2^2\right)U + a\alpha_1^2 U^{3n} + b\alpha_1^4 U^n (U^{2n})'' = 0. \qquad (39)$$

We use the transformation

$$U(\xi) = V^{1/2n}(\xi), \qquad (40)$$

that will reduce eq. (39) into the ODE

$$\left(\alpha_0\alpha_1 + (k-2)^3\alpha_3^2 + \alpha_2^2\right)V^{1/2n} + a\alpha_1^2 V^{3/2} + b\alpha_1^4 V^{1/2} V'' = 0. \qquad (41)$$

Following eq. (11), it is easy to deduce from (41) the expression of the function $F(V)$ which reads as

$$F(V) = \sqrt{-\frac{a}{b\alpha_1^2}}\,V\sqrt{1 + \frac{4n\left(\alpha_0\alpha_1 + (k-2)^3\alpha_3^2 + \alpha_2^2\right)}{a(n+1)\alpha_1^2}V^{(1-3n)/2n}}. \qquad (42)$$

Using the change of variables

$$-\frac{4n\left(\alpha_0\alpha_1 + (k-2)^3\alpha_3^2 + \alpha_2^2\right)}{a(n+1)\alpha_1^2}V^{(1-3n)/2n} = Z, \qquad (43)$$

and using the relation (19), we can obtain the two possible solutions of eq. (41) in the following forms:

$$V_1(\xi) = \left\{-\frac{a(n+1)\alpha_1^2}{4n\left(\alpha_0\alpha_1 + (k-2)^3\alpha_3^2 + \alpha_2^2\right)}\operatorname{sech}^2\left[\frac{1-3n}{4n\alpha_1}\sqrt{-\frac{a}{b}}\,\xi\right]\right\}^{2n/(1-3n)} \qquad (44)$$

and

$$V_2(\xi) = \left\{ \frac{a(n+1)\alpha_1^2}{4n\left(\alpha_0\alpha_1 + (k-2)^3\alpha_3^2 + \alpha_2^2\right)} \mathrm{csch}^2 \left[ \frac{1-3n}{4n\alpha_1} \sqrt{-\frac{a}{b}}\xi \right] \right\}^{2n/(1-3n)}. \tag{45}$$

Using eq. (40), we get the following exact solutions for eq. (38):

$$U_1(\xi) = \left\{ -\frac{a(n+1)\alpha_1^2}{4n\left(\alpha_0\alpha_1 + (k-2)^3\alpha_3^2 + \alpha_2^2\right)} \mathrm{sech}^2 \left[ \frac{1-3n}{4n\alpha_1} \sqrt{-\frac{a}{b}}\xi \right] \right\}^{1/(1-3n)} \tag{46}$$

and

$$U_2(\xi) = \left\{ \frac{a(n+1)\alpha_1^2}{4n\left(\alpha_0\alpha_1 + (k-2)^3\alpha_3^2 + \alpha_2^2\right)} \mathrm{csch}^2 \left[ \frac{1-3n}{4n\alpha_1} \sqrt{-\frac{a}{b}}\xi \right] \right\}^{1/(1-3n)}. \tag{47}$$

When $(a/b) < 0$, we have the following solitary pattern solutions:

$$u_1(x,y,z,t) = \left\{ -\frac{4nA}{a(n+1)\alpha_1^2} \times \cosh^2 \left[ \frac{3n-1}{4n\alpha_1} \sqrt{-\frac{a}{b}} \left(\alpha_0 t + \alpha_1 x + \alpha_2 y + (k-2)\alpha_3 z + \delta\right) \right] \right\}^{1/(3n-1)} \tag{48}$$

and

$$u_2(x,y,z,t) = \left\{ \frac{4nA}{a(n+1)\alpha_1^2} \times \sinh^2 \left[ \frac{3n-1}{4n\alpha_1} \sqrt{-\frac{a}{b}} \left(\alpha_0 t + \alpha_1 x + \alpha_2 y + (k-2)\alpha_3 z + \delta\right) \right] \right\}^{1/(3n-1)}, \tag{49}$$

where

$$A = \alpha_0\alpha_1 + (k-2)^3\alpha_3^2 + \alpha_2^2.$$

When $(a/b) > 0$, we obtain the following compacton solutions:

$$u_3(x,y,z,t) = \left\{ -\frac{4nA}{a(n+1)\alpha_1^2} \times \cos^2 \left[ \frac{3n-1}{4n\alpha_1} \sqrt{\frac{a}{b}} \left(\alpha_0 t + \alpha_1 x + \alpha_2 y + (k-2)\alpha_3 z + \delta\right) \right] \right\}^{1/(3n-1)} \tag{50}$$

and

$$u_4(x,y,z,t) = \left\{ -\frac{4nA}{a(n+1)\alpha_1^2} \times \sin^2 \left[ \frac{3n-1}{4n\alpha_1} \sqrt{\frac{a}{b}} \left(\alpha_0 t + \alpha_1 x + \alpha_2 y + (k-2)\alpha_3 z + \delta\right) \right] \right\}^{1/(3n-1)} \tag{51}$$

where

$$A = \alpha_0\alpha_1 + (k-2)^3\alpha_3^2 + \alpha_2^2.$$

3.2.2 *The negative exponents.* The $(1+k)$-dimensional KP$(3n, 3n)$ equation with negative exponents is written as

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+a\frac{\partial}{\partial x}(u^{-3n})+b\frac{\partial}{\partial x}\left[u^{-n}\frac{\partial^2}{\partial x^2}(u^{-2n})\right]\right)+\frac{\partial^2 u}{\partial y^2}+(k-2)\frac{\partial^2 u}{\partial z^2}=0. \tag{52}$$

In view of the results (48) and (49) for $(a/b)<0$, and $n$ is replaced by $-n$, we deduce the following exact soliton-like solutions for eq. (52):

$$u_1(x,y,z,t)=\left\{\frac{a(1-n)\alpha_1^2}{4nA}\operatorname{sech}^2\left[\frac{3n+1}{4n\alpha_1}\sqrt{-\frac{a}{b}}\,(\alpha_0 t+\alpha_1 x+\alpha_2 y +(k-2)\alpha_3 z+\delta)\right]\right\}^{1/(3n+1)} \tag{53}$$

and

$$u_2(x,y,z,t)=\left\{-\frac{a(1-n)\alpha_1^2}{4nA}\operatorname{csch}^2\left[\frac{3n+1}{4n\alpha_1}\sqrt{-\frac{a}{b}}\,(\alpha_0 t+\alpha_1 x+\alpha_2 y +(k-2)\alpha_3 z+\delta)\right]\right\}^{1/(3n+1)}, \tag{54}$$

where

$$A=\alpha_0\alpha_1+(k-2)^3\alpha_3^2+\alpha_2^2.$$

Similarly, we now use the results (50) and (51) for $(a/b)>0$, and $n$ is replaced by $-n$, we have the following analytical periodic solutions to eq. (52):

$$u_3(x,y,z,t)=\left\{\frac{a(1-n)\alpha_1^2}{4nA}\sec^2\left[\frac{3n+1}{4n\alpha_1}\sqrt{\frac{a}{b}}\,(\alpha_0 t+\alpha_1 x+\alpha_2 y +(k-2)\alpha_3 z+\delta)\right]\right\}^{1/(3n+1)} \tag{55}$$

and

$$u_4(x,y,z,t)=\left\{\frac{a(1-n)\alpha_1^2}{4nA}\csc^2\left[\frac{3n+1}{4n\alpha_1}\sqrt{\frac{a}{b}}\,(\alpha_0 t+\alpha_1 x+\alpha_2 y +(k-2)\alpha_3 z+\delta)\right]\right\}^{1/(3n+1)}, \tag{56}$$

where

$$A=\alpha_0\alpha_1+(k-2)^3\alpha_3^2+\alpha_2^2.$$

On comparison, we observe that our solutions (48)–(56) include the solutions of Wazwaz [18].

### 3.3 $K(n, n)$ equation

3.3.1 *The positive exponents.* We consider a third variant of the KdV equation

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(au+bu^{n}\right)+\kappa\frac{\partial^{3}}{\partial x^{3}}\left(u^{n}\right)=0, \tag{57}$$

which was studied in Wazwaz [19]. It is to be noted that eq. (57) is the linear KdV equation for $b = 0$ and $n = 1$. However, for $a = 0$, eq. (57) will be reduced to the well-known K$(n, n)$ equation.

Next, to find a travelling wave solution of eq. (57), we use

$$u(x,t)=U(\xi),\quad \xi=\alpha_0 t+\alpha_1 x+\delta. \tag{58}$$

Substituting (58) into eq. (57), we get

$$\alpha_0 U'+\alpha_1(aU+bU^{n})'+\kappa\alpha_1^{3}(U^{n})'''=0. \tag{59}$$

Integrating eq. (59) and neglecting constants of integration, we find

$$(\alpha_0+a\alpha_1)U+b\alpha_1(U^{n})+\kappa\alpha_1^{3}(U^{n})''=0. \tag{60}$$

Using the transformation

$$U(\xi)=V^{1/n}(\xi), \tag{61}$$

eq. (60) will be reduced to the following equation:

$$(\alpha_0+a\alpha_1)V^{1/n}+b\alpha_1 V+\kappa\alpha_1^{3}V''=0. \tag{62}$$

Using eq. (11), it is easy to deduce from (62) the expression of the function $F(V)$ as

$$F(V)=\sqrt{-\frac{b}{\kappa\alpha_1^{2}}}V\sqrt{1+\frac{2n(\alpha_0+a\alpha_1)}{b(n+1)\alpha_1}V^{(1-n)/n}}. \tag{63}$$

Using the change of variables

$$-\frac{2n(\alpha_0+a\alpha_1)}{b(n+1)\alpha_1}V^{(1-n)/n}=Z, \tag{64}$$

and proceeding as before, we can obtain the two possible solutions of eq. (62) in the following forms:

$$V_1(\xi)=\left\{-\frac{b(n+1)\alpha_1}{2n(\alpha_0+a\alpha_1)}\operatorname{sech}^{2}\left[\frac{1-n}{2n\alpha_1}\sqrt{-\frac{b}{k}}\xi\right]\right\}^{n/(1-n)} \tag{65}$$

and

$$V_2(\xi)=\left\{\frac{b(n+1)\alpha_1}{2n(\alpha_0+a\alpha_1)}\operatorname{csch}^{2}\left[\frac{1-n}{2n\alpha_1}\sqrt{-\frac{b}{k}}\xi\right]\right\}^{n/(1-n)}. \tag{66}$$

Using eq. (61), we obtain the exact solutions for eq. (59) in the following forms:

$$U_1(\xi)=\left\{-\frac{b(n+1)\alpha_1}{2n(\alpha_0+a\alpha_1)}\operatorname{sech}^{2}\left[\frac{1-n}{2n\alpha_1}\sqrt{-\frac{b}{k}}\xi\right]\right\}^{1/(1-n)} \tag{67}$$

and

$$U_2(\xi) = \left\{ \frac{b(n+1)\alpha_1}{2n(\alpha_0 + a\alpha_1)} \operatorname{csch}^2 \left[ \frac{1-n}{2n\alpha_1} \sqrt{-\frac{b}{k}} \xi \right] \right\}^{1/(1-n)}. \tag{68}$$

When $(a/k) < 0$, we have the following solitary pattern solutions:

$$u_1(x,t) = \left\{ -\frac{2n(\alpha_0 + a\alpha_1)}{b(n+1)\alpha_1} \cosh^2 \left[ \frac{n-1}{2n\alpha_1} \sqrt{-\frac{b}{k}} (\alpha_0 t + \alpha_1 x + \delta) \right] \right\}^{1/(n-1)} \tag{69}$$

and

$$u_2(x,t) = \left\{ \frac{2n(\alpha_0 + a\alpha_1)}{b(n+1)\alpha_1} \sinh^2 \left[ \frac{n-1}{2n\alpha_1} \sqrt{-\frac{b}{k}} (\alpha_0 t + \alpha_1 x + \delta) \right] \right\}^{1/(n-1)}. \tag{70}$$

When $(a/k) > 0$, we get the following compacton solutions:

$$u_3(x,t) = \left\{ -\frac{2n(\alpha_0 + a\alpha_1)}{b(n+1)\alpha_1} \cos^2 \left[ \frac{n-1}{2n\alpha_1} \sqrt{\frac{b}{k}} (\alpha_0 t + \alpha_1 x + \delta) \right] \right\}^{1/(n-1)} \tag{71}$$

and

$$u_4(x,t) = \left\{ -\frac{2n(\alpha_0 + a\alpha_1)}{b(n+1)\alpha_1} \sin^2 \left[ \frac{n-1}{2n\alpha_1} \sqrt{\frac{b}{k}} (\alpha_0 t + \alpha_1 x + \delta) \right] \right\}^{1/(n-1)}. \tag{72}$$

3.3.2 *The negative exponents.* In this subsection, we apply our method to obtain soliton-like and periodic solutions of the K$(n, n)$ equation in the form

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(au + bu^{-n}\right) + \kappa \frac{\partial^3}{\partial x^3}\left(u^{-n}\right) = 0. \tag{73}$$

In view of the results (69) and (70) for $(b/k) < 0$, and $n$ is replaced by $-n$, we deduce the following exact soliton-like solutions for eq. (73):

$$u_1(x,t) = \left\{ \frac{b(1-n)\alpha_1}{2n(\alpha_0 + a\alpha_1)} \operatorname{sech}^2 \left[ \frac{n+1}{2n\alpha_1} \sqrt{-\frac{b}{k}} (\alpha_0 t + \alpha_1 x + \delta) \right] \right\}^{1/(n+1)} \tag{74}$$

and

$$u_2(x,t) = \left\{ \frac{b(n-1)\alpha_1}{2n(\alpha_0 + a\alpha_1)} \operatorname{csch}^2 \left[ \frac{n+1}{2n\alpha_1} \sqrt{-\frac{b}{k}} (\alpha_0 t + \alpha_1 x + \delta) \right] \right\}^{1/(n+1)}. \tag{75}$$

Similarly, we now use the results (71) and (72) for $(b/k) > 0$, and $n$ is replaced by $-n$, we have the following analytical periodic solutions to eq. (73):

$$u_3(x,t) = \left\{ \frac{b(1-n)\alpha_1}{2n(\alpha_0 + a\alpha_1)} \sec^2 \left[ \frac{n+1}{2n\alpha_1} \sqrt{\frac{b}{k}} (\alpha_0 t + \alpha_1 x + \delta) \right] \right\}^{1/(n+1)} \tag{76}$$

and

$$u_4(x,t) = \left\{ \frac{b(1-n)\alpha_1}{2n(\alpha_0 + a\alpha_1)} \csc^2 \left[ \frac{n+1}{2n\alpha_1} \sqrt{\frac{b}{k}} (\alpha_0 t + \alpha_1 x + \delta) \right] \right\}^{1/(n+1)}. \tag{77}$$

### 3.4 *KP-K*$(n+1,\ n+1)$ *equation*

We finally consider KP-K$(n+1,\ n+1)$ equation

$$\frac{\partial}{\partial x}\left( \frac{\partial u}{\partial t} + a\frac{\partial}{\partial x}(u^{n+1}) + b\frac{\partial}{\partial x}\left[ u\frac{\partial^2}{\partial x^2}(u^n) \right] \right) + \kappa\frac{\partial^2 u}{\partial y^2} = 0. \tag{78}$$

Using the wave variable $\xi = \alpha_0 t + \alpha_1 x + \alpha_2 y + \delta$ and proceeding as before we find

$$\alpha_1(\alpha_0 U' + a\alpha_1 (U^{n+1})' + b\alpha_1^3 [U(U^n)'']')' + \kappa\alpha_2^2 U'' = 0. \tag{79}$$

Integrating eq. (79) and neglecting constants of integration, we find

$$(\alpha_0\alpha_1 + \kappa\alpha_2^2)U + a\alpha_1^2 U^{n+1} + b\alpha_1^4 U(U^n)'' = 0. \tag{80}$$

Using the transformation

$$U(\xi) = V^{1/n}(\xi), \tag{81}$$

eq. (80) will be reduced to the following equation:

$$\left(\alpha_0\alpha_1 + \kappa\alpha_2^2\right)V^{1/n} + a\alpha_1^2 V^{(1/n)+1} + b\alpha_1^4 V^{(1/n)} V'' = 0. \tag{82}$$

Following eq. (11), it is easy to deduce from (82) the expression of the function $F(V)$ as

$$F(V) = \sqrt{-\frac{a}{b\alpha_1^2}}\, V \sqrt{1 + \frac{2(\alpha_0\alpha_1 + \kappa\alpha_2^2)}{Va\alpha_1^2}}. \tag{83}$$

Using the change of variables

$$-\frac{2(\alpha_0\alpha_1 + \kappa\alpha_2^2)}{Va\alpha_1^2} = Z, \tag{84}$$

and proceeding as before, we can obtain the two possible solutions of eq. (82) in the following forms:

$$V_1(\xi) = -\frac{2\left(\alpha_0\alpha_1 + \kappa\alpha_2^2\right)}{a\alpha_1^2} \cosh^2 \left[ \frac{1}{2\alpha_1}\sqrt{-\frac{a}{b}}\xi \right] \tag{85}$$

and

$$V_2(\xi) = \frac{2\left(\alpha_0\alpha_1 + \kappa\alpha_2^2\right)}{a\alpha_1^2} \sinh^2 \left[ \frac{1}{2\alpha_1}\sqrt{-\frac{a}{b}}\xi \right]. \tag{86}$$

Using eq. (81), we obtain the exact solutions for eq. (80) in the following forms:

$$U_1(\xi) = \left\{-\frac{2\left(\alpha_0\alpha_1 + \kappa\alpha_2^2\right)}{a\alpha_1^2}\cosh^2\left[\frac{1}{2\alpha_1}\sqrt{-\frac{a}{b}}\xi\right]\right\}^{1/n} \tag{87}$$

and

$$U_2(\xi) = \left\{\frac{2\left(\alpha_0\alpha_1 + \kappa\alpha_2^2\right)}{a\alpha_1^2}\sinh^2\left[\frac{1}{2\alpha_1}\sqrt{-\frac{a}{b}}\xi\right]\right\}^{1/n}. \tag{88}$$

When $(a/b) < 0$, we have the following solitary pattern solutions:

$$u_1(x, y, t) = \left\{-\frac{2\left(\alpha_0\alpha_1 + \kappa\alpha_2^2\right)}{a\alpha_1^2}\cosh^2\left[\frac{1}{2\alpha_1}\sqrt{-\frac{a}{b}}\,(\alpha_0 t + \alpha_1 x + \alpha_2 y + \delta)\right]\right\}^{1/n} \tag{89}$$

and

$$u_2(x, y, t) = \left\{\frac{2\left(\alpha_0\alpha_1 + \kappa\alpha_2^2\right)}{a\alpha_1^2}\sinh^2\left[\frac{1}{2\alpha_1}\sqrt{-\frac{a}{b}}\,(\alpha_0 t + \alpha_1 x + \alpha_2 y + \delta)\right]\right\}^{1/n}. \tag{90}$$

When $(a/b) > 0$, we get the following compacton solutions:

$$u_3(x, y, t) = \left\{-\frac{2\left(\alpha_0\alpha_1 + \kappa\alpha_2^2\right)}{a\alpha_1^2}\cos^2\left[\frac{1}{2\alpha_1}\sqrt{\frac{a}{b}}\,(\alpha_0 t + \alpha_1 x + \alpha_2 y + \delta)\right]\right\}^{1/n} \tag{91}$$

and

$$u_4(x, y, t) = \left\{-\frac{2\left(\alpha_0\alpha_1 + \kappa\alpha_2^2\right)}{a\alpha_1^2}\sin^2\left[\frac{1}{2\alpha_1}\sqrt{\frac{a}{b}}\,(\alpha_0 t + \alpha_1 x + \alpha_2 y + \delta)\right]\right\}^{1/n}. \tag{92}$$

*Remark.* Comparing our results with earlier results [15–20], it can be seen that the current results are new.

## 4. Conclusion

Nonlinear phenomena play crucial roles in applied mathematics and physics. Exact solutions for nonlinear PDEs play important roles in many phenomena such as fluid mechanics, hydrodynamics, optics, plasma physics and so on. For the past several decades, many powerful methods [15–20] were used in solitary wave theory to investigate compactons, solitons, solitary patterns and periodic solutions for the nonlinear variants of KP and KdV equations. In this paper, we obtained compactons, solitons, solitary patterns and periodic solutions of the nonlinear variants of KP and KdV equations by using the functional variable method. It shows that the method is powerful and straightforward for nonlinear differential equations. It is said that this method can be applied to other kinds of nonlinear problems. Also, we predict that the obtained solutions in this paper will be important for analysing the nonlinear phenomena arising in applied physical sciences. The work reveals the power of this method in handling nonlinear PDEs which can be converted to a second-order ODE through the travelling wave transformation.

**Acknowledgement**

The authors are grateful to the anonymous referee for careful checking of the details and for helpful comments that improved this paper.

## References

[1] A M Wazwaz, *Comput. Math. Appl.* **49**, 565 (2005)
[2] A Biswas, A H Kara and E Zerrad, *Open Nuclear and Particle Phys. J.* **4**, 21 (2011)
[3] A Biswas, *Int. J. Theor. Phys.* **49**, 79 (2010)
[4] A M Wazwaz, *Appl. Math. Comput.* **154**, 713 (2004)
[5] A M Wazwaz, *Bull. Inst. Math. Acad. Sinica* **29**, 125 (2001)
[6] I Aslan, *Appl. Math. Comput.* **217**, 8134 (2011)
[7] I Aslan, *Math. Meth. Appl. Sci.* **35**, 716 (2012)
[8] W X Ma, A Abdeljabbar and M G Asaad, *Appl. Math. Comput.* **217**, 10016 (2011)
[9] N A Kudryashov and N B Loguinova, *Commun. Nonlinear Sci. Numer. Simul.* **14**, 1881 (2009)
[10] I Aslan, *Appl. Math. Comput.* **219**, 2825 (2012)
[11] A Zerarka, S Ouamane and A Attaf, *Appl. Math. Comput.* **217**, 2897 (2010)
[12] A Zerarka and S Ouamane, *World J. Model. Simul.* **6**, 150 (2010)
[13] A C Cevikel, A Bekir, M Akar and S San, *Pramana – J. Phys.* **79**, 337 (2012)
[14] A Nazarzadeh, M Eslami and M Mirzazadeh, *Pramana – J. Phys.* **81**, 225 (2013)
[15] A M Wazwaz, *Appl. Math. Comput.* **170**, 361 (2005)
[16] A M Wazwaz, *Appl. Math. Comput.* **190**, 633 (2007)
[17] A M Wazwaz, *Appl. Math. Comput.* **204**, 227 (2008)
[18] A M Wazwaz, *Appl. Math. Comput.* **161**, 561 (2005)
[19] A M Wazwaz, *Appl. Math. Comput.* **132**, 29 (2002)
[20] M S Ismail, M D Petkovic and A Biswas, *Appl. Math. Comput.* **216**, 2220 (2010)
[21] F Cooper, H Shepard and P Sodano, *Phys. Rev. E* **48**, 4027 (1993)
[22] A Khare and F Cooper, *Phys. Rev. E* **48**, 4843 (1993)
[23] P Rosenau and J M Hyman, *Phys. Rev. Lett.* **70**, 564 (1993)
[24] A Cardenas, B Mihaila, F Cooper and A Saxena, *Phys. Rev. E* **83**, 066705 (2011)
[25] B B Kadomtsev and V I Petviashvili, *Sov. Phys. Dokl.* **15**, 539 (1970)