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Lag synchronization of chaotic systems with time-delayed linear terms via impulsive control

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Abstract. In this paper, the lag synchronization of chaotic systems with time-delayed linear terms via impulsive control is investigated. Based on the stability theory of impulsive delayed differential equations, some sufficient conditions are obtained guaranteeing the synchronized behaviours between two delayed chaotic systems. Numerical simulations on time-delayed Lorenz and hyperchaotic Chen systems are also carried out to show the effectiveness of the proposed scheme. Note that under the scheme the chaotic system is controlled only at discrete time instants, and so it reduces the control cost in real applications.

Keywords. Chaotic systems; time delays; impulsive control; lag synchronization.

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1. Introduction

Inspired by the pioneering work of Pecora and Carroll [1–3], the subject of how to synchronize chaotic systems has attracted increasing attention due to their potential applications in secure communication. Various methods have been put forward to synchronize chaotic systems, which include nonlinear observer approach [4,5], self-adaptive control method [6,7], parameter control [8], impulsive control [9–11], etc. Recently, impulsive control has been widely used to stabilize and synchronize chaotic systems, because it allows the stabilization and synchronization of chaotic systems using only impulsive control at discrete time instants, even though the chaotic behaviour may follow unpredictable patterns. More recently, Yang [12], Yang and Leon [13], Sun *et al* [14] and Li and Liao [15] have obtained many impulsive complete and lag synchronization criteria for some well-known chaotic systems.

Lag synchronization, which is different from complete synchronization, has been proposed as the coincidence of the states of two coupled systems in which one of the systems is delayed by a finite time δ, i.e., $x(t) \to y(t+\delta)$ as $t \to \infty$ where x, y denote the states of the interacting systems. Thus, knowledge of the lag synchronization is of considerable practical importance. Shahverdive *et al* [16] carried out the analytical analysis about lag synchronization in unidirectionally coupled time-delayed systems. Note that impulsive control scheme can reduce the control cost significantly, and so it is of great use in practical applications. Now, in this paper, lag synchronization of chaotic systems with time-delayed linear terms will be investigated. The scheme is showed effective through numerical simulations on chaotic systems.

The rest of the paper is organized as follows. In §2, some results of impulsive control are presented. In §3, impulsive lag synchronization of chaotic systems with time delay is obtained. In §4, two numerical examples are given to verify the obtained theoretical results. Finally, some conclusions are drawn in §5.

2. Preliminaries and problem formation

Consider the general nonlinear functional differential system

$$\dot{x} = f(t, x_t), \quad t \geq t_0, \tag{1}$$

where $x \in R^n$ is the state variable, $f\colon [t_0, \infty) \times PC \to R^n$ and $PC = \{\varphi\colon [-\tau, 0] \to R^n, \varphi(t)$ is continuous everywhere except at the finite number of points $\bar{x}$ at which $\varphi(\bar{x}^+)$ and $\varphi(\bar{x}^-)$ exist and $\varphi(\bar{x}^+) = \varphi(\bar{x})\}$. For any $t \geq t_0, x_t \in PC$ is defined as $x_t(s) = x(t+s), -\tau \leq s \leq 0$. For $\varphi \in PC$, the norm of ϕ is defined by $\|\varphi\| = \sup_{-\tau \leq s \leq 0} |\varphi(s)|$, where $|\cdot|$ denotes the norm of the vector in R^n.

Take a discrete set $\{t_k\}$ of time instants, where $0 \leq t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \ldots, t_k \to \infty$ as $k \to \infty$. Let

$$U(k, x) = \Delta x|_{t=t_k} \equiv x(t_k^+) - x(t_k^-) \tag{2}$$

be the 'jump' in the state variable at the time instant t_k. Then this impulsive functional differential system is described by

$$\begin{cases} \dot{x} = f(t, x_t), & t \neq t_k, \\ \Delta x = U(k, x), & t = t_k, k = 1, 2, \ldots, \\ x(t_0) = x_0, & t_0 \geq 0. \end{cases} \tag{3}$$

Let $S(\rho) = \{x \in R^n | |x| < \rho\}$, where $|\cdot|$ denotes the norm of the vector in R^n, $K = \{\omega \in C(R^+, R^+), \omega(s)$ is strictly increasing and $\omega(0) = 0\}$, $K^* = \{\psi \in K, \psi(s) < s, \forall s > 0\}$, $\Omega = \{H \in C(R^+, R^+), H(0) = 0, H(s) > 0, \forall s > 0\}$. To study the stability of impulsive functional differential system, the following definitions and lemma will be used.

DEFINITION 1

Let $V\colon R_+ \times R^n \to R_+$, then V is said to belong to class V_0 if

(1) V is continuous in $(t_{k-1}, t_k] \times S(\rho)$ and for each $x \in S(\rho)$,

$$\lim_{(t,y)\to(t_k^-,x)} V(t, y) = V\left(t_k^-, x\right)$$

exists, $k = 1, 2, \ldots$.

(2) V is locally Lipschitzian in $x \in S(\rho)$ and for all $t \geq t_0$, $V(t, 0) = 0$.

DEFINITION 2

For $(t, x) \in (t_{k-1}, t_k] \times S(\rho)$, the right and upper Dini derivative D^+V of $V(t, x) \in V_0$ along the solution $x(t)$ of system (1) is defined as

$$D^+V(t, x(t)) = \lim \sup_{\delta \to 0^+} \frac{1}{\delta}[V(t+\delta, \delta f(t, x)) - V(t, x(t))]. \tag{4}$$

Lemma 1 (Lyapunov-like stability theorem [17]). *Consider the impulsive function differential system*

$$\begin{cases} \dot{x}(t) = f(t, x_t), & t \geq t_0, \\ x\left(t_k^+\right) = J_k(x(t_k^-)), & k \in N, \end{cases} \tag{5}$$

where f: $[t_0, \infty) \times PC \to R^n$ *and* $J_k(x)$: $S(\rho) \to R^n$ *for each* $k \in N$. *Assume that* $f(t, 0) = 0$, $J_k(0) = 0$, *so* $x(t) = 0$ *is the solution of eq.* (5).

The zero solution of eq. (5) *is uniformly asymptotically stable if there exist* $V \in V_0, \omega_1, \omega_2 \in K, \psi \in K^*$ *and* $H \in \Omega$ *such that*

(i) $\omega_1(|x|) \leq V(t, x) \leq \omega_2(|x|)$, *for any* $(t, x) \in [t_0, \infty) \times S(\rho)$;
(ii) *for all* $x \in S(\rho_1), 0 < \rho_1 \leq \rho$ *and* $k \in N$, $V(t_k, J_k(x)) \leq \psi(V(t_k^-, x))$;
(iii) *for any solution* $x(t)$ *of eq.* (5), $V(t+s, x(t+s)) \leq \psi^{-1}(V(t, x)), -\tau \leq s \leq 0$ *implies that* $D^+V(t, x(t)) \leq g(t)H(V(t, x(t)))$, *where* g: $[t_0, \infty) \to R^+$ *is locally integrable*, ψ^{-1} *is the inverse function of* ψ;
(iv) H *is nondecreasing and there exist constants* $\lambda_2 \geq \lambda_1 > 0$ *and* $A > 0$ *such that for all* $k \in N$ *and* $\mu > 0, \lambda_1 \leq t_k - t_{k-1} \leq \lambda_2$ *and*

$$\int_{\psi(\mu)}^{\mu} \frac{\mathrm{d}u}{H(u)} - \int_{t_{k-1}}^{t_k} g(s)\mathrm{d}s \geq A.$$

3. Synchronization between delayed chaotic systems via impulsive control

Consider a class of time-delayed chaotic systems, which are described by the following DDE:

$$\dot{x} = Ax + Bx(t-\tau) + f(x), \tag{6}$$

where $x \in R^n$ is the state variable, $A, B \in R^{n \times n}$ and f: $R^n \to R^n$ is a nonlinear function satisfying the Lipschiztian condition, that is, there exists a scalar $L > 0$ such that for any $x, y \in R^n$,

$$\|f(x) - f(y)\| \leq L\|x - y\|. \tag{7}$$

To lag-synchronize system (6), the response system is given by

$$\begin{cases} \dot{y} = Ay + By(t-\tau) + f(y), & t \neq t_k \\ \Delta y = y(t_k^+) - y(t_k^-) = B_k[y(t_k - \sigma) - x(t_k)], & t = t_k, \end{cases} \tag{8}$$

where $\{t_k\}$: $0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots$ with $\lim_{t\to\infty} t_k = \infty$, $y(t_k^+) = y(t_k + 0)$, $y(t_k^-) = y(t_k - 0)$, B_k is the control gain. Throughout this paper, $\dot{y}(t)$ denotes the right-hand derivative of $y(t)$ in impulsive DDE.

DEFINITION 3

Systems (6) and (8) are called lag-synchronized if there exists a constant $\sigma > 0$ such that the states of two systems are nearly identical, but one system lags in time to the other, i.e., $\lim_{t\to+\infty} \|y(t-\sigma) \to x(t)\| = 0$.

Define the error

$$e(t) = y(t-\sigma) - x(t), \quad \text{for } t \geq \sigma, \tag{9}$$

then we have the error system

$$\begin{cases} \dot{e} = Ae + Be(t-\tau) + \varphi(x(t), y(t-\sigma)), & t \neq t_k, \\ \Delta e = B_k e, & t = t_k, \end{cases} \tag{10}$$

where $\varphi(x(t), y(t-\sigma)) = f(y(t-\sigma)) - f(x(t))$.

Here, the problem of lag synchronization between (6) and (8) is shifted to that of the stability of controlled system (10). Then from Lemma 1, we have the following result.

Theorem 1. *Suppose that there exists a symmetric and positive definite matrix $Q \in R^{n\times n}$ and constants $\varepsilon > 0$, $\gamma > 0$, $0 < d < 1$ such that the following conditions are satisfied*:

(i) $\Omega_1 \equiv (I + B_k)^T Q(I + B_k) - dQ \leq 0$;
(ii) $r = (\lambda_M(\Omega_2)/\lambda_m(Q)) + (\varepsilon/d) + (\gamma L^2/\lambda_m(Q)) + (\lambda_M(Q)/\gamma) > 0$;
(iii) *there exist constant $\lambda_2 \geq \lambda_1 > 0$ such that for all $k \in N$,*

$$\lambda_1 \leq t_k - t_{k-1} \leq \lambda_2 < -\frac{\ln(d)}{r},$$

where

$$\Omega_2 \equiv A^T Q + QA + \frac{1}{\varepsilon} QBQ^{-1}B^T Q.$$

Then the origin of error system (10) is uniformly asymptotically stable.

Proof. Construct Lyapunov function as follows:

$$V(t, e) = e^T Qe, \tag{11}$$

where Q is a symmetric and positive definite matrix. Let $\omega_1(\|e\|) = \lambda_m(Q)\|e\|^2$ and $\omega_2(\|e\|) = \lambda_M(Q)\|e\|^2$, where λ_m and λ_M denote the smallest and largest eigenvalues of a square matrix, respectively. Then eq. (11) implies that $\omega_1(\|e\|) \leq V(t, e) \leq \omega_2(\|e\|)$. Suppose $\psi(x) = dx$, $H(x) = x$, then

$$\begin{aligned} V(J_k(e)) &= V((I + B_k)e) = e^T((I + B_k)^T Q(I + B_k)e \\ &= e^T[(I + B_k)^T Q(I + B_k) - dQ + dQ]e \\ &\leq de^T Qe = \psi(V(e)). \end{aligned}$$

For any solution $e(t)$ of eq. (10), if $V(e(t+s)) \leq \psi^{-1}(V(e))$, for $-\tau \leq s \leq 0$, then $e^T(t-\tau)Qe(t-\tau) \leq d^{-1}e^T(t)Qe(t)$. Therefore,

$$\begin{aligned} D^+V(e(t)) &= e^T(A^TQ + QA)e + e(t-\tau)^T B^T Qe \\ &\quad + e^T QBe(t-\tau) + (\varphi^T Qe + e^T Q\varphi) \\ &\leq e^T(A^TQ + QA)e + \varepsilon e(t-\tau)^T Qe(t-\tau) \\ &\quad + \frac{1}{\varepsilon} e^T(t)QBQ^{-1}B^TQe(t) + \gamma\varphi^T\varphi + \frac{1}{\gamma}e^TQ^TQe \\ &\leq e^T\Omega_2 e + \left[\frac{\varepsilon}{d} + \frac{\gamma}{\lambda_m(Q)}L^2 + \frac{\lambda_M(Q)}{\gamma}\right]e^TQe \\ &\leq \left[\frac{\lambda_M(\Omega_2)}{\lambda_m(Q)} + \frac{\varepsilon}{d} + \frac{\gamma L^2}{\lambda_m(Q)} + \frac{\lambda_M(Q)}{\gamma}\right]V(e(t)) \equiv rV(e(t)). \end{aligned}$$

Let $g(t) = r$, $A = -\ln d - r\lambda_2$, then $A > 0$ and for any $\mu > 0$ and $k \in N$,

$$\begin{aligned} \int_{\psi(\mu)}^{\mu} \frac{\mathrm{d}u}{H(u)} &- \int_{t_{k-1}}^{t_k} g(s)\mathrm{d}s \\ &= \int_{\mathrm{d}\mu}^{\mu} \frac{\mathrm{d}u}{u} - \int_{t_{k-1}}^{t_k} r\mathrm{d}s \\ &= -\ln d - r(t_k - t_{k-1}) \\ &\geq -\ln d - r\lambda_2 = A. \end{aligned}$$

Thus, from Lemma 1 we may conclude that the origin of (10) is uniformly asymptotically stable.

COROLLARY 1

Let $Q = I$, then the origin of error system (10) is uniformly asymptotically stable if there exist constants $\varepsilon > 0, \gamma > 0$, $0 < d < 1$ such that the following conditions are satisfied:

(i) $(I + B_k)^T(I + B_k) - \mathrm{d}I \leq 0$;
(ii) $r = \lambda_M(\Omega_3) + (\varepsilon/d) + \gamma L^2 + (1/\gamma) > 0$;
(iii) *there exist constant $\lambda_2 \geq \lambda_1 > 0$ such that for all $k \in N$,*

$$\lambda_1 \leq t_k - t_{k-1} \leq \lambda_2 < -\frac{\ln(d)}{r},$$

where

$$\Omega_3 \equiv A^T + A + (1/\varepsilon)BB^T.$$

4. Numerical examples

In this section, numerical simulations are presented on the lag synchronization of chaotic systems via impulsive control. For the notional and illustrative convenience, we always assume the time intervals of impulses to be equidistant and the control gain to be constant, i.e., $t_k - t_{k-1} \equiv \delta$ for any $k \in N$.

Example 1. Consider a time-delayed Lorenz system in the following form:

$$\begin{cases} \dot{x_1} = a(x_2(t-\tau) - x_1), \\ \dot{x_2} = bx_1 - x_2 - x_1x_3, \\ \dot{x_3} = x_1x_2 - cx_3(t-\tau), \end{cases} \tag{12}$$

where $a = 10, b = 28, c = 8/3, f(x) = [0, -x_1x_3, x_1x_2]^T, \tau > 0$ is the time delay.

System (12) may be in the chaotic state through the suitable selection of time delay τ (see figure 1). Now take (12) as the drive system, and the response system with impulsive control is given by

$$\begin{cases} \dot{y_1} = a(y_2(t-\tau) - y_1), \\ \dot{y_2} = by_1 - y_2 - y_1y_3, \\ \dot{y_3} = y_1y_2 - cy_3(t-\tau), & t \neq t_k \\ \Delta y = y(t_k^+) - y(t_k^-) = B_k[y(t_k - \sigma) - x(t_k)], & t = t_k, \end{cases} \tag{13}$$

in which $k = 1, 2, \ldots$. The error dynamical system is characterized by

$$\begin{cases} \dot{e} = Ae + Be(t-\tau) + \varphi(x(t), y(t-\sigma)), & t \neq t_k, \\ \Delta e = B_k e, & t = t_k, \end{cases} \tag{14}$$

where $e = y(t-\sigma) - x(t), \varphi(x(t), y(t-\sigma)) = f(y(t-\sigma)) - f(x(t))$

$$A = \begin{bmatrix} -a & 0 & 0 \\ b & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -c \end{bmatrix}.$$

Choose $Q = I, B_k = \mathrm{diag}\{-0.97, -0.97, -0.97\}, \sigma = 0.4, \tau = 0.2, \varepsilon = 1, \gamma = 1, L = 20$, then $d = 0.001, r = 499, \delta = 0.023$. From figure 2, the error states of the drive system (12) and the response system (13) converge to zero, which shows the effectiveness of the scheme.

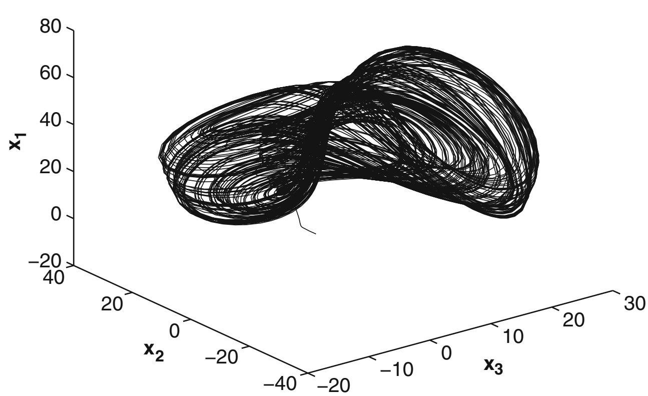

Figure 1. Chaotic attractor of system (12) with $\tau = 0.2$.

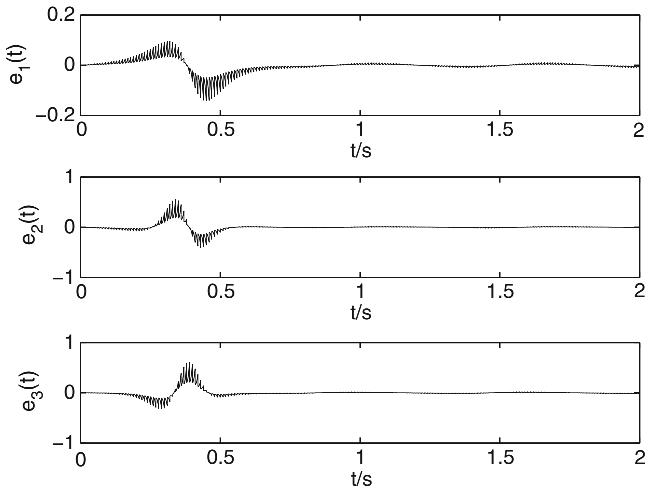

Figure 2. Time evolution of errors with $\tau = 0.2$ and $\sigma = 0.4$.

Example 2. Consider the time-delayed hyperchaotic Chen system in the following form:

$$\begin{cases} \dot{x_1} = a(x_2 - x_1) + x_4, \\ \dot{x_2} = dx_1 - x_1x_3 + cx_2(t-\tau), \\ \dot{x_3} = -bx_3 + x_1x_2, \\ \dot{x_4} = px_4 + x_2x_3, \end{cases} \tag{15}$$

where $a = 35$, $b = 3$, $c = 12$, $d = 7$, $p = 0.5$, $f(x) = [0, -x_1x_3, x_1x_2, x_2x_3]^T$.

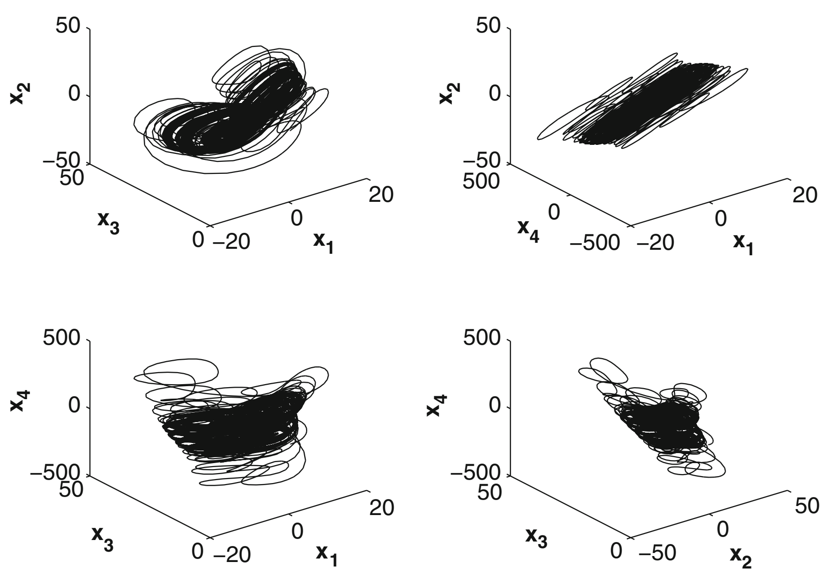

Figure 3. Chaotic attractor of hyperchaotic Chen system with $\tau = 0.2$.

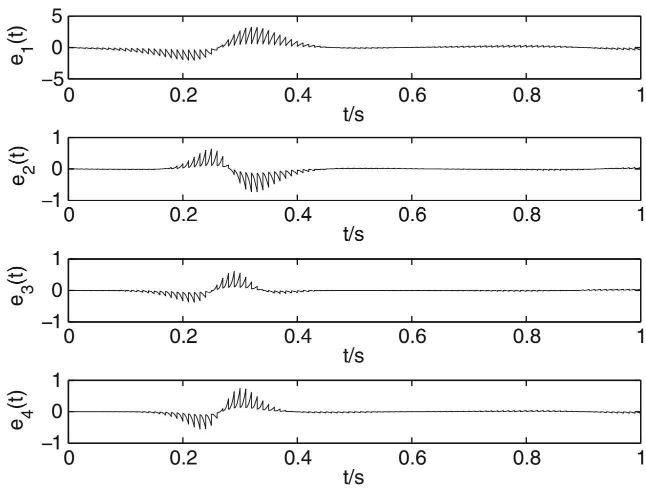

Figure 4. Time evolution of errors with $\tau = 0.2$ and $\sigma = 0.3$.

Take (15) as the drive system and it shows chaotic behaviour when $\tau = 0.4$ (see figure 3). The response system with impulsive control is described by

$$\begin{cases} \dot{y}_1 = a(y_2 - y_1) + y_4, & \\ \dot{y}_2 = dy_1 - y_1 y_3 + cy_2(t-\tau), & \\ \dot{y}_3 = -by_3 + y_1 y_2, & \\ \dot{y}_4 = py_4 + y_2 y_3, & t \neq t_k, \\ \Delta y = y\left(t_k^+\right) - y\left(t_k^-\right) = B_k[y(t_k - \sigma) - x(t_k)], & t = t_k, \end{cases} \tag{16}$$

where $k = 1, 2, \ldots$. Choose $Q = I$, $B_k = \text{diag}\{-0.98, -0.98, -0.98\}$, $\sigma = 0.4$, $\tau = 0.3$, $\varepsilon = 1$, $\gamma = 1$, $L = 20$, then $d = 0.0005$, $r = 578$, $\delta = 0.0132$. From figure 4, the error states of the drive system (15) and the response system (16) converge to zero, and so the obtained result is also effective for hyperchaotic systems with time delays.

5. Conclusion

In this paper, based on the stability theory for impulsive functional equations, the impulsive lag synchronization scheme for a class of delayed chaotic systems has been proposed, which is applied to two timed-delayed systems, i.e. time-delayed chaotic Lorenz system and hyperchaotic Chen system. Numerical simulations have shown the effectiveness of the synchronization method.

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