PRAMANA — journal of physics

Vol. 80, No. 6
June 2013
pp. 945–951

The Kepler problem in the Snyder space

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MS received 30 November 2012; accepted 12 February 2013

Abstract. In this paper the Kepler problem in the non-commutative Snyder scenario was studied. The deformations were characterized in the Poisson bracket algebra under a mimic procedure from quantum standard formulations by taking into account a general recipe to build the non-commutative phase space coordinates (in the sense of Poisson brackets). An expression for the deformed potential was obtained, and then the consequences in the precession of the orbit of Mercury were calculated. The result could be used for finding an estimated value for the non-commutative deformation parameter.

Keywords. Snyder space; non-commutativity; Kepler problem.

PACS Nos **04.20.−q; 04.70.Bw; 04.90.+e**

1. Introduction

Non-commutativity has become a serious theory among the physics theories, since minimal fundamental lengths have been introduced by the leading theories of loop quantum gravity and string theory. This minimal fundamental length usually is identified as the Planck length and it is supposed that under that scale physics is totally different, even from the standard quantum physics.

There are many ways to introduce non-commutativity. Usually the Heisenberg algebra is deformed through a matrix that encodes the lack of commutativity between the position operators. This is incompatible with Lorentz symmetry and many difficulties arise due to the many changes that abandoning this fundamental symmetry implies. But there is a safer way. In fact, Snyder, in the 1940s [1], proposed a modification of the Heisenberg algebra that implies discrete spectra of the space-time operators. This modification is included among the κ-deformed space-time modifications.

DOI: 10.1007/s12043-013-0540-5; ***e*Publication: 29 May 2013**

In fact, the non-commutative space-time program was forgotten due to the successful renormalization program in the Standard Model. However, there is a renewed interest due to the development of loop quantum gravity and string theories with their discrete space-times.

One of the problems with the leading theories of quantum gravity today is the lack of experimental confirmations. In that direction, this paper shows a possible way to measure the implications of a non-commutative space-time, using the well-known Kepler celestial mechanics; introducing a deformation parameter in the Kepler potential and forecasting deformations in the orbits of planets. There are some previous efforts dealing with this problem, but they used a non-commutativity that was not compatible with Lorenz symmetry, which was very undesirable [2–5].

The paper is organized as follows: in §3, a short review of non-commutative algebras is given, in §3 the Kepler problem in the Snyder space-time is developed obtaining an advance of perihelion of a planet due to the deformed considerations and, finally conclusions are given in §4.

2. Non-commutative algebras

2.1 *General case*

In a $(n + 1)$-dimensional Minkowski space-time, we introduce the non-commutativity through

$$[\bar{x}_\mu, \bar{x}_\nu] = lM_{\mu\nu}, \tag{1}$$

where $\bar{x}$ is the non-commutative coordinate and l is a parameter measuring the non-commutativity with dimension of squared length, usually identifying $\sqrt{l}$ with l_{p}, the Planck longitude and $M_{\mu\nu}$ the rotations generator.

It is usual to demand that the Poincaré algebra is untouched, then we have the standard commutations relations

$$\begin{aligned}
&[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\mu\nu} M_{\mu\nu} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho} + \eta_{\mu\sigma} M_{\nu\rho}, \\
&[p_\mu, p_\nu] = 0.
\end{aligned} \tag{2}$$

We can obtain a general expression for the new coordinates taking [6]

$$\bar{x}_\mu = x_\mu \phi_1(A) + l(xp)p_\mu \phi_2(A), \tag{3}$$

where ϕ_1 and ϕ_2 are two dependent functions of the quantity $A = sp^2$, and the relation between them is

$$\phi_2 = \frac{1 + 2\phi_1' \phi_1}{\phi_1 - 2A\phi_1'}, \tag{4}$$

where $(')$ denotes derivative with respect to A.

We have the freedom to take any value of ϕ_1 for realizing non-commutativity. The only restriction is that the boundary condition $\phi(0) = 1$, to retrieve the ordinary commutativity.

In general, the commutator between coordinates and momenta is

$$[\bar{x_\mu}, p_\nu] = i(\eta_{\mu\nu}\phi_1 + lp_\mu p_\nu \phi_2). \tag{5}$$

2.2 *Snyder case*

These are infinite possibilities for choosing the value of ϕ_1, and taking the value of ϕ_1 as 1 is a very special case. This choice implies that $\phi_2 = 1$, that leads to the so-called Snyder space, characterized by

$$[x_\mu, x_\nu] = ilM_{\mu\nu}, \tag{6}$$

$$[x_\mu, p_\nu] = i\delta_{\mu\nu} - ilp_\mu p_\nu, \tag{7}$$

$$[p_\mu, p_\nu] = 0. \tag{8}$$

This is a very interesting case and many have investigated about it since the Snyder's paper itself ([1,2,7–10] are some of them).

3. The Kepler problem in the Snyder non-commutative Euclidian space

Classical Euclidian n-dimensional Snyder space is characterized by its non-linear commutation relations (in the sense of Poisson brackets), between the variables of the phase space. They can be set following the inverse of Dirac quantization recipe

$$\{x_i, x_j\} = l_\mathrm{p}^2 L_{ij}, \tag{9}$$

$$\{x_i, p_j\} = \delta_{ij} - l_\mathrm{p}^2 p_i p_j, \tag{10}$$

$$\{p_i, p_j\} = 0, \tag{11}$$

where l_p is the Planck longitude which measures the deformation introduced in the canonical Poisson brackets and L_{ij} is defined as a dimensionless matrix proportional to the angular momentum.

The Kepler potential $V = -\kappa/\sqrt{x_i^2}$ is implemented in the general non-commutative case, taking

$$V(\bar{x}) = -\frac{\kappa}{\sqrt{\bar{x}_i \bar{x}_i}}, \tag{12}$$

and considering the recipe from (3), we obtain at the first order in l

$$V(x) = -\frac{\kappa}{\sqrt{x_i^2 \phi_1^2 + 2l_\mathrm{p}^2 (xp)^2 \phi_1 \phi_2}}. \tag{13}$$

For the Snyder realization ($\phi_1 = \phi_2 = 1$), we have

$$V(x) = -\frac{\kappa}{\sqrt{x_i^2 + 2l_\mathrm{p}^2 (xp)^2}}. \tag{14}$$

So, using polar coordinates for a plane motion,

$$x = \rho\hat{\rho}, \tag{15}$$

$$p = m(\dot{\rho}\hat{\rho} + \rho\dot{\theta}\hat{\theta}), \tag{16}$$

the Lagrangian for a particle in the Snyder–Kepler potential can be written as

$$\mathcal{L} = \frac{1}{2}m\left[1 - \frac{2l_{\mathrm{p}}^2 km}{\rho}\right]\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\theta}^2 + \frac{k}{\rho}. \tag{17}$$

We still have the angular momentum $L = m\rho^2\dot{\theta}$ as a constant of motion. So considering a particle with energy E we obtain for the radial equation

$$\dot{\rho}^2 = \frac{2}{mf(\rho)}\left[E - V_{\mathrm{cl}}(\rho)\right], \tag{18}$$

where

$$f(\rho) = \left(1 + \frac{2\kappa l_{\mathrm{p}}^2 m}{\rho}\right) \quad \text{and} \quad V_{\mathrm{cl}}(\rho) = \frac{L^2}{2m\rho^2} - \frac{\kappa}{\rho},$$

is the classical effective potential for the two-bodies problem. In this sense, our interest is to study non-commutative correction to the confined orbit. So the constant of motion E is restricted to the values

$$0 > E > E_{\mathrm{c}} \equiv -\frac{\kappa}{2\rho_{\mathrm{c}}},$$

where $\rho_{\mathrm{c}} = (m\kappa)^{-1}L^2$ is the radius of the circular orbit and E_{c} is the energy at this point. Now, we can write a dimensionless equation of motion in terms of these quantities as

$$(-x')^2 = (2x - x^2 - \mathfrak{E})(1 + 2J^2 x)^{-1}, \tag{19}$$

where $J = (m\kappa l_{\mathrm{p}})/L$, $1 > \mathfrak{E} \equiv E/E_{\mathrm{c}} > 0$, $x_- \geq x \equiv \rho_{\mathrm{c}}/\rho \geq x_+$ (with $x_\pm \equiv \rho_{\mathrm{c}}/\rho_\pm = 1 \mp \sqrt{1-\mathfrak{E}}$), and $x' = \mathrm{d}x/\mathrm{d}\theta$. Performing the substitution $x = A - y$, with $A = (4J^2 - 1)(6J^2)^{-1}$, eq. (19) becomes

$$y'^2 = \frac{1}{8J^2}\frac{4y^3 - g_2 y - g_3}{(h-y)^2}, \tag{20}$$

where $h = (1 + 2J^2)(3J^2)^{-1}$, and the invariants are given by

$$g_2 = \frac{1+4J^2+4J^4(4-3\mathfrak{E})}{3J^4} \quad \text{and} \quad g_3 = \frac{(1+2J^2)\left[1+4J^2-4J^4(8-9\mathfrak{E})\right]}{27J^6}.$$

Therefore, choosing $\theta = 0$ at $y = y_+$ and integrating eq. (20), we find

$$\frac{\theta}{\sqrt{8J^2}} = W(y_+) - W(y), \tag{21}$$

where

$$W(y) = (C - y)\wp(y; g_2, g_3) - \zeta(y; g_2, g_3), \tag{22}$$

where $\wp$ is the Weierstrass-p function and ζ is the Weierstrass-z function. Equation (21) represents the formal solution for the Kepler's problem when the non-commutativity is taken into account. But we still can say something more about the deformation parameter, l_{p}. To do this, we study the advance of perihelion starting from (19), expanding to order J^2, and neglecting x^3 terms. Thus, we obtain

$$\begin{aligned}\left(-\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 &\approx -\mathfrak{E} + 2(1 + J^2\mathfrak{E})x - (1 + 4J^2)x^2 \\ &= \frac{(1 + J^2\mathfrak{E})^2}{(1 + 4J^2)} - \mathfrak{E} - (1 + 4J^2)\left(x - \frac{(1 + J^2\mathfrak{E})}{(1 + 4J^2)}\right)^2.\end{aligned} \tag{23}$$

So, it yields

$$x \equiv \frac{\rho_{\mathrm{c}}}{\rho} = C_1 + C_2 \cos(k\theta + \theta_0), \tag{24}$$

where

$$C_1 = \frac{1 + J^2\mathfrak{E}}{1 + 4J^2}, \quad C_2 = k^{-1}\left(\frac{(1 + J^2\mathfrak{E})^2}{1 + 4J^2} - \mathfrak{E}\right)^{1/2}, \quad k = \sqrt{1 + 4J^2}.$$

Therefore, the correction for the advance of perihelion is given by

$$\Delta\theta = \frac{2\pi}{k} = 2\pi(1 + 4J^2)^{-1/2}, \tag{25}$$

which can be approximated as a deviation of the Newtonian orbit

$$\Delta\theta \simeq 2\pi(1 - 2J^2) = 2\pi + \delta\theta_{\mathrm{nc}}, \tag{26}$$

where $\delta\theta_{\mathrm{nc}} = -4\pi J^2$ is the non-commutative correction. To obtain the value of the deformation parameter, we can consider that the discrepancy of the observational data and the theoretical value in the specific case of Mercury (see table 1), could be due to the

Table 1. Sources of the precession of perihelion for Mercury.

Amount (arcsec/Julian century)	Cause
5028.83 ± 0.04 [11]	Coordinate (due to the precession of the equinoxes)
530 [12]	Gravitational tugs of the other planets
0.0254	Oblateness of the Sun (quadrupole moment)
42.98 ± 0.04 [13]	General relativity
5603.24	Total
5599.7	Observed
−3.54	Discrepancy

non-commutativity scenario. Obviously we choose Mercury because it is a natural laboratory to check deformations as it is expected that any little effect can be observable in its orbit as Mercury is the nearest planet to the Sun. Therefore, we obtain $l_{\mathrm{p}} = 1.68 \times 10^{-32}$.

4. Final remarks

In this article we have described the effects of Snyder space non-commutativity on Kepler problem and have studied its effect on a planetary orbit. We have introduced non-commutativity by performing the deformations in Poisson bracket algebra under a mimic procedure from quantum standard formulations and then, using a general recipe built the non-commutative phase-space coordinates (in the sense of Poisson brackets). We found that the deformation in the central potential allows us to write a Lagrangian for a particle in the Snyder–Kepler potential and to obtain formal solution for the Kepler's problem when non-commutativity is taken into account. Our solution is given in terms of Weierstrass-p ($\wp$) and Weierstrass-z (ζ) functions. Then, we used our analytical results to compute the advance of perihelion of a planetary orbit which is given by

$$\Delta\theta = \frac{2\pi}{k} = 2\pi(1 + 4J^2)^{-1/2}.$$

This can be approximated as a deviation of the Newtonian orbit as $\Delta\theta \simeq 2\pi(1 - 2J^2) = 2\pi + \delta\theta_{\mathrm{nc}}$ where $\delta\theta_{\mathrm{nc}} = -4\pi J^2$ is the non-commutative correction.

Finally, we applied this formula to fix the discrepancy between observational data and the theoretical value obtained from different classical sources and, under the hypothesis that the discrepancy is due to the non-commutativity of the space, we obtained an estimated value for the non-commutative deformation parameter given by $l_{\mathrm{p}} = 1.68 \times 10^{-32}$. In future we would like to see the value of deformation parameter in more general setting as the advance of perihelion in the neighborhood of a black hole.

Acknowledgements

C L was supported by Universidad de Tarapacá Grant 4723-11. J R V was supported by Universidad de Tarapacá Grant 4720-11. J S was supported by COMISION NACIONAL DE CIENCIAS Y TECNOLOGIA through FONDECYT Grant 1110076, 1090613 and 1110230. This work was also partially supported by PUCV DI-123.713/2011.

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