PRAMANA — journal of physics | © Indian Academy of Sciences | Vol. 80, No. 3 March 2013 pp. 413–427

Solutions and conservation laws of Benjamin–Bona–Mahony–Peregrine equation with power-law and dual power-law nonlinearities

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MS received 3 January 2012; accepted 8 October 2012

Abstract. In this paper, exact solutions of Benjamin–Bona–Mahony–Peregrine equation are obtained with power-law and dual power-law nonlinearities. The Lie group analysis as well as the simplest equation method are used to carry out the integration of these equations. The solutions obtained are cnoidal waves, periodic solutions and soliton solutions. Subsequently, the conservation laws are derived for the underlying equations.

Keywords. Lie symmetries; integrability; solitons; conservation laws; simplest equation method.

PACS Nos 02.30.Jr; 02.30.Ik; 02.70.Wz; 47.10.ab

1. Introduction

Finding the exact solutions of nonlinear evolution equations (NLEEs) plays an important role in the study of many physical phenomena in various fields such as fluid mechanics, solid-state physics, plasma physics, chemical physics, optical fibre and geochemistry. Thus, it is important to investigate the exact explicit solutions of NLEEs [1–20]. Finding solutions of such an equation is a difficult task and only in certain special cases one can write down the solutions explicitly.

One such NLEE is the celebrated Korteweg–de Vries (KdV) equation which governs the dynamics of solitary waves. Originally it was derived to describe shallow water waves of long wavelength and small amplitude. Recently, another equation that models long waves in shallow water in ocean beaches was proposed. This is called the Benjamin–Bona–Mahony equation (BBM) [21] and has the form

$$u_t + u_x + \alpha u u_x - u_{xxt} = 0.$$

DOI: 10.1007/s12043-012-0489-9; *e*Publication: 18 February 2013

The BBM equation is also known as the regularized long-wave equation and was also derived by Peregrine [22]. Hence the name Benjamin–Bona–Mahony–Peregrine (BBMP).

The notion of conservation laws plays an important role in the solution process of differential equations. Finding the conservation laws of a system of differential equations (DEs) is often the first step towards finding the solution. In fact, the existence of a large number of conservation laws of a system of partial differential equations (PDEs) is a strong indication of its integrability [23]. In [24], the invariance of a conservation law was used to obtain solutions for a problem in thin films. In jet problems, the conserved quantity plays an essential role in the derivation of the solution. Recently, in [25] the conserved quantity was used to determine the unknown exponent in the similarity solution which cannot be obtained from the homogeneous boundary conditions.

In this paper we study the BBMP equation with power-law and dual power-law nonlinearities, namely

$$q_t + aq_x + bq^n q_x + cq_{xxt} = 0 \tag{1}$$

and

$$q_t + aq_x + (bq^n + cq^{2n})q_x + kq_{xxt} = 0, \tag{2}$$

where a, b, c, k and n are real constants. Here, in (1) and (2) the first term represents the evolution term, while the last term in both equations represents the dispersion term. The third term in (1) and the third and fourth terms in (2) are the nonlinear terms.

The purpose of this paper is two-fold. Firstly, we use Lie symmetry method along with the simplest equation method to obtain exact solutions of (1) and (2). Secondly, we derive conservation laws for both the equations.

The outline of the paper is as follows. In §2, we obtain exact solutions of (1) and (2) using the Lie group and simplest equation methods. Then in §3, we construct conservation laws for both the equations. Finally, in §4 concluding remarks are made.

2. Exact solutions

In this section we present some exact solutions of (1) and (2) using Lie symmetry approach and simplest equation method.

2.1 *Group-invariant solutions using Lie symmetry approach*

In this section we first calculate the Lie point symmetries of (1) and (2) and later use them to construct group-invariant solutions. For the theory and applications of Lie group analysis the reader is referred to the well-known books such as [23,26–28].

Recall that a Lie point symmetry of a partial differential equation (PDE) is an invertible transformation of the dependent and independent variables that leaves the equation unchanged. In general, determining all the symmetries of a partial differential equation is a formidable task. However, Sophus Lie (1842–1899) observed that if we restrict ourself to symmetries that depend continuously on a small parameter and that form a group (continuous one-parameter group of transformations), one can linearize the symmetry conditions and end up with an algorithm for calculating continuous symmetries.

2.1.1 *Group-invariant solutions of* (1). The symmetry group of BBMP equation with power-law nonlinearity (1) will be generated by the vector field of the form

$$\Gamma = \xi^1(t,x,q)\frac{\partial}{\partial t} + \xi^2(t,x,q)\frac{\partial}{\partial x} + \eta(t,x,q)\frac{\partial}{\partial q}.$$

Applying the third prolongation $\mathrm{pr}^{(3)}\Gamma$ [26] to (1) we obtain the following overdetermined system of ten linear partial differential equations:

$$\begin{aligned}
&\xi^1_q = 0,\\
&\xi^2_q = 0,\\
&\eta_{qq} = 0,\\
&\xi^1_x = 0,\\
&\xi^2_t = 0,\\
&\eta_{tq} = 0,\\
&b\eta_x q^n + \eta_t + a\eta_x + c\eta_{txx} = 0,\\
&bn\eta q^n + b\xi^1_t q^{n+1} - b\xi^2_x q^{n+1} - bc\eta_{xxq} q^{n+1} + a\xi^1_t q - a\xi^2_x q\\
&\quad + 2c\eta_{txq} q - ac\eta_{x,x,q} q = 0,\\
&2\xi^2_x + c\eta_{xxq} = 0,\\
&2\eta_{xq} - \xi^2_{xx} = 0.
\end{aligned}$$

After some straightforward, albeit tedious and lengthy calculations, the above system gives the two translation symmetries

$$\Gamma_1 = \frac{\partial}{\partial t}, \quad \Gamma_2 = \frac{\partial}{\partial x}.$$

One of the main reasons for finding symmetries of a differential equation is to use them for finding exact solutions. We now utilize these symmetries to deduce exact solutions of (1). The symmetry $V = \Gamma_1 + \nu\Gamma_2$, where ν is a constant, reduces the BMBP equation (1) to a nonlinear ordinary differential equation (ODE). It can be seen that the symmetry V yields the following two invariants:

$$z = x - \nu t, \qquad F = q. \tag{3}$$

Treating F as the new dependent variable and z as new independent variable, the BMBP equation (1) transforms to the third-order nonlinear ODE

$$c\nu F'''(z) - bF^n(z)F'(z) - (a-\nu)F'(z) = 0. \tag{4}$$

It should be noted here that one can assume $q = F(z)$, $z = x - \nu t$ and arrive at (4). However, when using the Lie symmetry approach no such form of the solution is assumed. It is the beauty of this approach that one automatically gets the form of the solution as $q = F(x - \nu t)$. Integrating equation (4) twice with respect to z and taking the constants to be zero we obtain

$$\left(\frac{\mathrm{d}F}{\mathrm{d}z}\right)^2 = \frac{1}{c\nu}\left[(a-\nu)F^2 + \frac{2b}{(n+1)(n+2)}F^{n+2}\right]. \tag{5}$$

This is a first-order variables separable equation. Integrating the above equation and reverting back to the original variables, we obtain the following solution of the BMBP equation (1) for arbitrary values of n in the form

$$q(x,t) = \left(\frac{(n+1)(n+2)(\nu-a)}{2b}\right)^{1/n} \operatorname{sech}^{2/n}\left[\frac{n}{2}\sqrt{\frac{a-\nu}{c\nu}}(x-\nu t)\right]. \quad (6)$$

By taking $a = 3$, $b = 1$, $c = 1$, $n = 1$ and $\nu = 2$ in (6) we have the following profile of the solution of (6) (see figure 1).

Likewise, one can obtain more group-invariant solutions of the BMBP equation (1). However, we list here a few exact solutions.

$$q(t,x) = \frac{1}{b}\{\nu - a + 8\nu c + 12\nu c \tan^2(x-\nu t)\}, \quad (7)$$

$$q(t,x) = \frac{1}{b}\{\nu - a - 4\nu c + 8\nu c m^2 - 12\nu c\, m^2 \operatorname{cn}^2(x - \nu t|m)\}. \quad (8)$$

Here $\operatorname{cn}(Z|m)$ is the Jacobian elliptic functions [29], which is defined as follows: If

$$Z = \int_0^{\phi} \frac{\mathrm{d}\theta}{\sqrt{1 - m\sin^2\theta}},$$

where the angle ϕ is called the amplitude, then the function $\operatorname{cn}(Z|m)$ is defined as $\operatorname{cn}(Z|m) = \cos\phi$, where m is the modulus of the elliptic function and $0 \le m \le 1$.

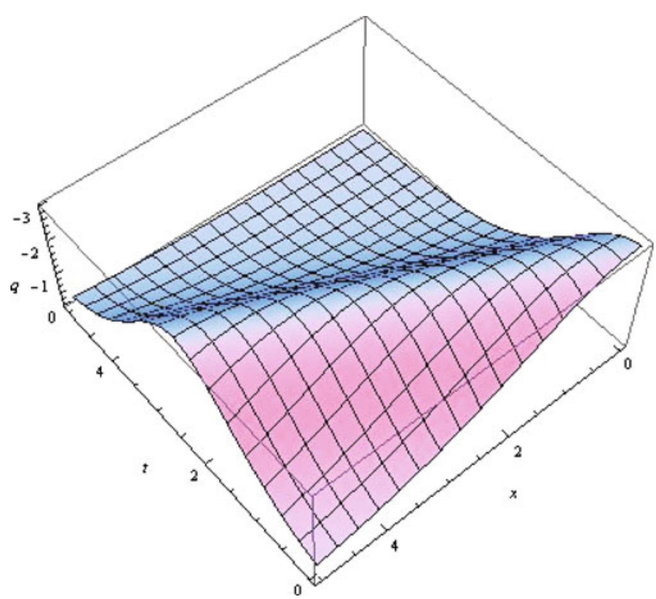

Figure 1. Profile of solution (6).

By taking $a = 1, b = 1, c = 3, m = 0.5$ and $\nu = 1$ in (8) we have the following profile of the solution of (8) (see figure 2).

2.1.2 *Group-invariant solutions of* (2). The BBMP equation with dual power-law nonlinearity (2) has the same two symmetries as the BBMP equation with power-law nonlinearity. Using the same procedure as in the previous section we obtain the following implicit solution of (2) for arbitrary values of n in the form

$$\frac{1}{n\sqrt{a-\nu}}\ln\left[\frac{q^n}{(n+1)(n+2)(a+P)+bq^n}\right] = \pm\sqrt{\frac{1}{k\nu}}(x-\nu t)+C_1, \quad (9)$$

where

$$P = \sqrt{(a-\nu)\left(a+\frac{2bq^n}{(n+1)(n+2)}+\frac{cq^{2n}}{(n+1)(2n+1)}-\nu\right)}-\nu$$

and C_1 is an arbitrary constant of integration.

2.2 *Solutions using the simplest equation method*

We now use the simplest equation method which was introduced by Kudryashov [30,31] and modified by Vitanov [32] (see also [33]) to solve (1) and (2) for $n = 1$. The simplest equations that will be used are the Bernoulli and Riccati equations.

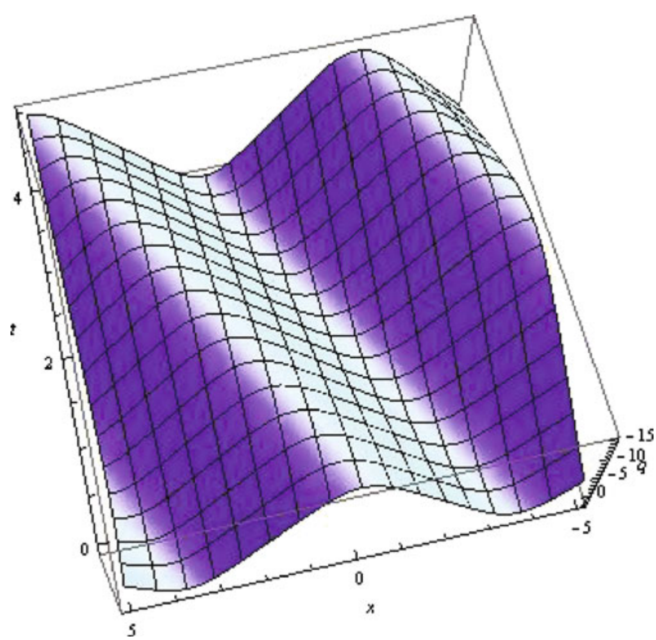

Figure 2. Profile of solution (8).

Let us consider the solution of (4) in the form

$$F(z) = \sum_{i=0}^{M} A_i (G(z))^i, \tag{10}$$

where $G(z)$ satisfies the Bernoulli and Riccati equations, M is a positive integer that can be determined by balancing procedure as in [32] and $A_0, \ldots, A_M$ are parameters to be determined. We note that the Bernoulli and Riccati equations are well-known nonlinear ODEs whose solutions can be expressed in terms of elementary functions.

We consider the Bernoulli equation

$$G'(z) = dG(z) + eG^s(z), \tag{11}$$

where s is an integer with $s > 1$ and for simplicity we let $s = 2$. As a result the solutions of the Bernoulli equation are [33]

$$G(z) = d\left\{\frac{\cosh[d(z+C)] + \sinh[d(z+C)]}{1 - e\cosh[d(z+C)] - e\sinh[d(z+C)]}\right\}$$

for $d > 0$, $e < 0$ and

$$G(z) = -d\left\{\frac{\cosh[d(z+C)] + \sinh[d(z+C)]}{1 + e\cosh[d(z+C)] + e\sinh[d(z+C)]}\right\}$$

for $d < 0$ and $e > 0$. Here C is a constant of integration. For the Riccati equation

$$G'(z) = dG^2(z) + eG(z) + f \tag{12}$$

we shall use the solutions [33]

$$G(z) = -\frac{e}{2d} - \frac{\theta}{2d}\tanh\left[\frac{1}{2}\theta(z+C)\right]$$

and

$$G(z) = -\frac{e}{2d} - \frac{\theta}{2d}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\theta\{1 + \tanh(z\theta/2)\}}{2\{d + 2C\theta\cosh^2(z\theta/2) + C\theta\sinh(z\theta)\}},$$

where $\theta^2 = e^2 - 4df > 0$ and C is a constant of integration.

2.2.1 *Solutions of* (1) *using the simplest equation method.*

Solutions of (1) *using Bernoulli equation as the simplest equation.* The balancing procedure with $s = 2$ [32] yields $M = 2$ and so the solutions in (10) are of the form

$$F(z) = A_0 + A_1 G + A_2 G^2. \tag{13}$$

Substituting (13) into (4) and making use of (11) and then equating all coefficients of the functions G^i to zero, we obtain an algebraic system of equations in terms of A_0, A_1 and A_2. These algebraic equations are

$$
\begin{aligned}
&aA_1d + A_0A_1bd + A_1(-c)d^3\nu - A_1d\nu = 0,\\
&2aA_2d + aA_1e + A_1^2bd + 2A_0A_2bd + A_0A_1be - 8A_2cd^3\nu\\
&\qquad - 7A_1cd^2e\nu - 2A_2d\nu - A_1e\nu = 0,\\
&2aA_2e + 3A_1A_2bd + A_1^2be + 2A_0A_2be - 38A_2cd^2e\nu\\
&\qquad - 12A_1cde^2\nu - 2A_2e\nu = 0,\\
&2A_2^2bd + 3A_1A_2be - 54A_2cde^2\nu - 6A_1ce^3\nu = 0,\\
&2A_2^2be - 24A_2ce^3\nu = 0.
\end{aligned}
$$

Solving the above system of algebraic equations with the aid of Mathematica, we obtain the following values of A_0, A_1 and A_2:

$$A_0 = \frac{-a + cd^2\nu + \nu}{b}, \quad A_1 = \frac{12cde\nu}{b}, \quad A_2 = \frac{12ce^2\nu}{b}.$$

Therefore, when $d > 0$ and $e < 0$, the solution of (1) with $n = 1$ is given by

$$
\begin{aligned}
q(t,x) \;=\; & A_0 + A_1d\left\{\frac{\cosh[d(z+C)] + \sinh[d(z+C)]}{1 - e\cosh[d(z+C)] - e\sinh[d(z+C)]}\right\}\\
& + A_2d^2\left\{\frac{\cosh[d(z+C)] + \sinh[d(z+C)]}{1 - e\cosh[d(z+C)] - e\sinh[d(z+C)]}\right\}^2
\end{aligned}
\tag{14}
$$

and when $d < 0$ and $e > 0$ the solution of (1) is

$$
\begin{aligned}
q(t,x) \;=\; & A_0 - A_1d\left\{\frac{\cosh[d(z+C)] + \sinh[d(z+C)]}{1 + e\cosh[d(z+C)] + e\sinh[d(z+C)]}\right\}\\
& + A_2d^2\left\{\frac{\cosh[d(z+C)] + \sinh[d(z+C)]}{1 + e\cosh[d(z+C)] + e\sinh[d(z+C)]}\right\}^2,
\end{aligned}
\tag{15}
$$

where $z = x - \nu t$ and C is a constant of integration.

Solutions of (1) *using Riccati equation as the simplest equation.* The balancing procedure yields $M = 2$ and so the solutions in (10) are of the form

$$F(z) = A_0 + A_1G + A_2G^2. \tag{16}$$

Substituting (16) into (4) and using (12), we obtain algebraic system of equations in terms of A_0, A_1, A_2 by equating all coefficients of the functions G^i to zero. The corresponding algebraic equations are

$$
\begin{aligned}
&aA_1f + A_0A_1bf - 2A_1cdf^2\nu\\
&\quad -A_1ce^2f\nu - 6A_2cef^2\nu - A_1f\nu = 0,\\
&aA_1e + 2aA_2f + A_0A_1be + A_1^2bf + 2A_0A_2bf - 8A_1cdef\nu\\
&\quad -16A_2cdf^2\nu + A_1(-c)e^3\nu - 14A_2ce^2f\nu - A_1e\nu - 2A_2f\nu = 0,\\
&aA_1d + 2aA_2e + A_0A_1bd + A_1^2be + 2A_0A_2be + 3A_1A_2bf - 8A_1cd^2f\nu\\
&\quad -7A_1cde^2\nu - 52A_2cdef\nu - 8A_2ce^3\nu - A_1d\nu - 2A_2e\nu = 0,
\end{aligned}
$$

$$
\begin{aligned}
&2aA_2d + A_1^2bd + 2A_0A_2bd + 3A_1A_2be + 2A_2^2bf - 12A_1cd^2e\nu \\
&\quad -40A_2cd^2f\nu - 38A_2cde^2\nu - 2A_2d\nu = 0, \\
&3A_1A_2bd + 2A_2^2be - 6A_1cd^3\nu - 54A_2cd^2e\nu = 0, \\
&\quad 2A_2^2bd - 24A_2cd^3\nu = 0.
\end{aligned}
$$

Solving the above equations one obtains

$$
A_0 = \frac{-a + 8cdf\nu + ce^2\nu + \nu}{b}, \quad A_1 = \frac{12cde\nu}{b}, \quad A_2 = \frac{12cd^2\nu}{b}
$$

and hence the solutions of (1) with $n = 1$ are

$$
\begin{aligned}
q(t,x) \;=\; & A_0 + A_1\left\{ -\frac{e}{2d} - \frac{\theta}{2d}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\} \\
& + A_2\left\{ -\frac{e}{2d} - \frac{\theta}{2d}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^2
\end{aligned}
\tag{17}
$$

and

$$
\begin{aligned}
q(t,x) \;=\; & A_0 + A_1\left\{ -\frac{e}{2d} - \frac{\theta}{2d}\tanh\left(\frac{1}{2}\theta z\right)\right. \\
& \left. \qquad + \frac{\theta\{1+\tanh(z\theta/2)\}}{2\{d + 2C\theta\cosh^2(z\theta/2) + C\theta\sinh(z\theta)\}}\right\} \\
& + A_2\left\{ -\frac{e}{2d} - \frac{\theta}{2d}\tanh\left(\frac{1}{2}\theta z\right)\right. \\
& \left. \qquad + \frac{\theta\{1+\tanh(z\theta/2)\}}{2\{d + 2C\theta\cosh^2(z\theta/2) + C\theta\sinh(z\theta)\}}\right\}^2,
\end{aligned}
\tag{18}
$$

where $z = x - \nu t$ and C is a constant of integration.

By taking $a = 1$, $b = 1$, $c = 1$, $d = 1$, $e = 3$, $f = 1$, $C = 1$, $k = 1$ and $\nu = 1$ in (18) we have the following profile of the solution of (18) (see figure 3).

2.2.2 *Solutions of* (2) *using the simplest equation method.*

Solutions of (2) *using Bernoulli equation as the simplest equation.* The balancing procedure in this case yields $M = 1$, and so the solutions of (4) are of the form

$$
F(z) = A_0 + A_1G.
$$

Now following the above procedure, we obtain

$$
\begin{aligned}
& a = \frac{b^2 + 4c\nu - 2cd^2k\nu}{4c}, \quad A_0 = \frac{-b \pm \sqrt{b^2 - 4ac + 4c\nu + 4cd^2k\nu}}{2c}, \\
& A_1 = \frac{2\left(2ae - 2e\nu + d^2ek\nu + beA_0\right)}{bd}.
\end{aligned}
$$

Therefore, when $d > 0$ and $e < 0$ the solution of (2) with $n = 1$ is given by

$$
q(t,x) = A_0 + A_1 d\left\{\frac{\cosh[d(z+C)] + \sinh[d(z+C)]}{1 - e\cosh[d(z+C)] - e\sinh[d(z+C)]}\right\}
\tag{19}
$$

and when $d < 0$ and $e > 0$ the solution is

$$q(t,x) = A_0 - A_1 d \left\{ \frac{\cosh[d(z+C)] + \sinh[d(z+C)]}{1 + e\cosh[d(z+C)] + e\sinh[d(z+C)]} \right\}, \tag{20}$$

where $z = x - \nu t$ and C is a constant of integration.

Solutions of (2) *using Riccati equation as the simplest equation.* Here again the balancing procedure yields $M = 1$, and so the solutions of (4) are of the form

$$F(z) = A_0 + A_1 G.$$

In this case, we obtain

$$a = \frac{b^2 + 4c\nu - 2ce^2k\nu + 8cdfk\nu}{4c},$$

$$A_0 = \frac{-b \pm \sqrt{b^2 - 4ac + 4c\nu + 4ce^2k\nu + 8cdfk\nu}}{2c},$$

$$A_1 = \frac{2\left(2ad - 2d\nu + de^2k\nu - 4d^2fk\nu + bdA_0\right)}{be}$$

and hence the solutions of (2) with $n = 1$ are given by

$$q(t,x) = A_0 + A_1 \left\{ -\frac{e}{2d} - \frac{\theta}{2d}\tanh\left[\frac{1}{2}\theta(z+C)\right] \right\} \tag{21}$$

and

$$q(t,x) = A_0 + A_1 \left\{ -\frac{e}{2d} - \frac{\theta}{2d}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\theta\{1 + \tanh(z\theta/2)\}}{2\{d + 2C\theta\cosh^2(z\theta/2) + C\theta\sinh(z\theta)\}} \right\}, \tag{22}$$

where $z = x - \nu t$ and C is a constant of integration.

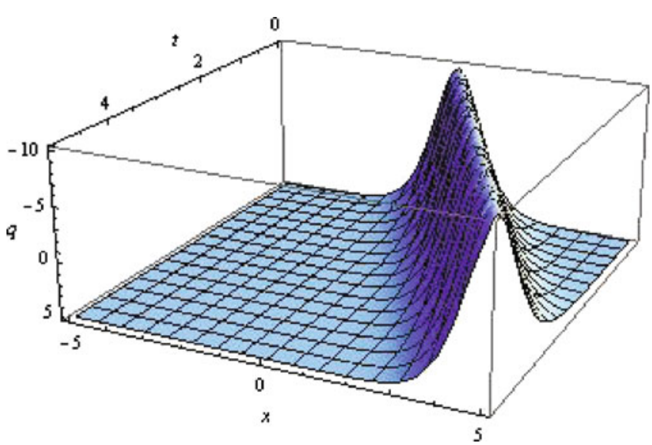

Figure 3. Profile of solution (18).

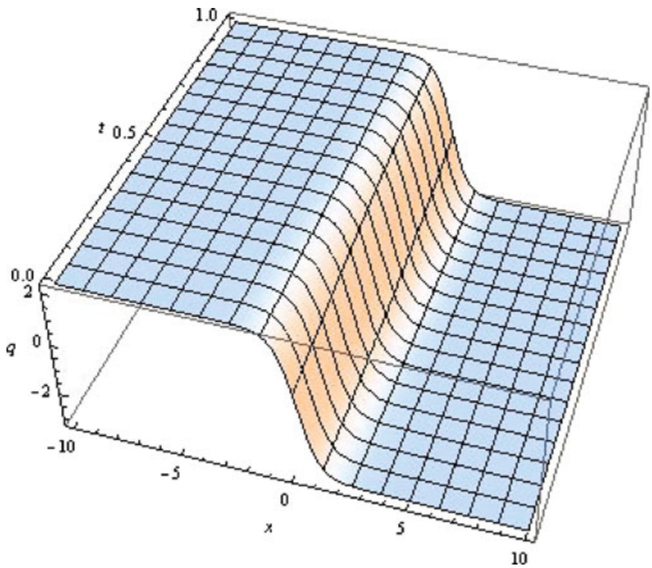

Figure 4. Profile of solution (22).

By taking $b = 1, c = 1, d = 1, e = 3, f = 1, C = 1, k = 1$ and $\nu = 1$ in (22) we have the following profile of the solution of (22) (see figure 4).

3. Conservation laws

In this section we construct conservation laws for (1) and (2). The multiplier method will be used [26,34–36]. See also [37].

Consider a kth-order system of PDEs of n independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$, viz.,

$$E_\alpha(x, u, u_{(1)}, \ldots, u_{(k)}) = 0, \qquad \alpha = 1, \ldots, m, \tag{23}$$

where $u_{(1)}, u_{(2)}, \ldots, u_{(k)}$ denote the collections of all first, second, ..., kth-order partial derivatives, that is, $u_i^\alpha = D_i(u^\alpha), u_{ij}^\alpha = D_j D_i(u^\alpha), \ldots$ respectively, with the *total derivative operator* with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \qquad i = 1, \ldots, n, \tag{24}$$

and the summation convention is used whenever appropriate [28].

The following are known (see for example, [28] and the references therein).

The Euler–Lagrange operator, for each α, is given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} (-1)^s D_{i_1} \ldots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_s}}, \qquad \alpha = 1, \ldots, m. \tag{25}$$

The n-tuple vector $T = (T^1, T^2, \ldots, T^n)$, $T^j \in \mathcal{A}$, $j = 1, \ldots, n$, is a conserved vector of (23) if T^i satisfies

$$D_i T^i|_{(23)} = 0. \tag{26}$$

Equation (26) defines a local conservation law of system (23).

A multiplier $\Lambda_\alpha(x, u, u_{(1)}, \ldots)$ has the property that

$$\Lambda_\alpha E_\alpha = D_i T^i \tag{27}$$

hold identically. Here we shall consider multipliers of the second order, i.e., $\Lambda_\alpha = \Lambda_\alpha(t, x, q, q_t, q_x, q_{tt}, q_{tx}, q_{xx})$. The right-hand side of (27) is a divergence expression. The determining equation for the multiplier Λ_α is

$$\frac{\delta(\Lambda_\alpha E_\alpha)}{\delta u^\alpha} = 0. \tag{28}$$

Once the multipliers are obtained, the conserved vectors are calculated via a homotopy formula [35].

3.1 *Conservation laws of* (1)

For the BBMP equation with power-law nonlinearity (1), after some lengthy calculations, we obtain the following three second-order multipliers, i.e., $\Lambda = \Lambda(t, x, q, q_t, q_x, q_{tt}, q_{tx}, q_{xx})$ that are given by

$$\Lambda_1 = 1, \quad \Lambda_2 = q, \quad \Lambda_3 = \frac{bq^{n+1}}{c(n+1)} + q_{tx}. \tag{29}$$

Corresponding to the above three multipliers we have the following conserved vectors of (1):

$$T_1^t = \frac{1}{3}\{3q + cq_{xx}\}, \tag{30}$$

$$T_1^x = \frac{1}{3(n+1)}\{3anq + 3aq + 3bq^{n+1} + 2cnq_{tx} + 2cq_{tx}\}, \tag{31}$$

$$T_2^t = \frac{1}{6}\{2cq_{xx}q + 3q^2 - cq_x{}^2\}, \tag{32}$$

$$T_2^x = \frac{1}{6(n+2)}\{4cnqq_{tx} + 8cqq_{tx} + 3anq^2 + 6aq^2 + 6bq^{n+2} - 2cnq_tq_x - 4cq_tq_x\}, \tag{33}$$

$$
\begin{aligned}
T_3^t = &\ \frac{1}{12c(n+1)(n+2)}\{-3acn^2q_{xx}q - 9acnq_{xx}q - 6acq_{xx}q \\
&- 2bcn^2{q_x}^2q^n + 2bc{q_x}^2q^n + 2bcq_{xx}q^{n+1} - 2bcnq_{xx}q^{n+1} \\
&- c^2n^2qq_{txxx} - 3c^2nqq_{txxx} - 2c^2qq_{txxx} + 3cn^2qq_{tx} + 9cnqq_{tx} \\
&+ 6cqq_{tx} + 12bq^{n+2} + 3acn^2{q_x}^2 + 9acn{q_x}^2 + 6ac{q_x}^2 \\
&+ 2c^2n^2q_{xx}q_{tx} + c^2n^2q_xq_{txx} + 6c^2nq_{xx}q_{tx} + 3c^2nq_xq_{txx} \\
&+ 4c^2q_{xx}q_{tx} + 2c^2q_xq_{txx} + 3cn^2q_tq_x + 9cnq_tq_x + 6cq_tq_x\}, \qquad (34)
\end{aligned}
$$

$$
\begin{aligned}
T_3^x = &\ \frac{1}{12c(n+1)^2(n+2)}\{3acn^3qq_{tx} + 12acn^2qq_{tx} + 15acnqq_{tx} \\
&+ 6acqq_{tx} + 2bcn^3q_tq_xq^n + 2bcn^2q_tq_xq^n + 14bcn^2q_{tx}q^{n+1} \\
&- 2bcq_tq_xq^n - 2bcnq_tq_xq^n + 22bcq_{tx}q^{n+1} + 36bcnq_{tx}q^{n+1} \\
&+ c^2n^3qq_{ttxx} + 4c^2n^2qq_{ttxx} + 5c^2nqq_{ttxx} + 2c^2qq_{ttxx} - 3cn^3q_{tt}q \\
&- 12cn^2q_{tt}q - 15cnq_{tt}q - 6cq_{tt}q + 12abq^{n+2} + 12abnq^{n+2} \\
&+ 12b^2q^{2n+2} + 6b^2nq^{2n+2} + 3acn^3q_tq_x + 12acn^2q_tq_x \\
&+ 15acnq_tq_x + 6acq_tq_x + 4c^2n^3{q_{tx}}^2 + c^2n^3q_tq_{txx} \\
&- 2c^2n^3q_xq_{ttx} + 16c^2n^2{q_{tx}}^2 + 4c^2n^2q_tq_{txx} - 8c^2n^2q_xq_{ttx} \\
&+ 20c^2n{q_{tx}}^2 + 5c^2nq_tq_{txx} - 10c^2nq_xq_{ttx} + 8c^2{q_{tx}}^2 + 2c^2q_tq_{txx} \\
&- 4c^2q_xq_{ttx} + 3cn^3{q_t}^2 + 12cn^2{q_t}^2 + 15cn{q_t}^2 + 6c{q_t}^2\}. \qquad (35)
\end{aligned}
$$

3.2 *Conservation laws of* (2)

For the BBMP equation with dual power-law nonlinearity (2), we obtain the following three second-order multipliers:

$$
\Lambda_1 = 1, \quad \Lambda_2 = q, \quad \Lambda_3 = \frac{1}{k}\left\{\frac{bq^{n+1}}{n+1} + \frac{cq^{2n+1}}{2n+1}\right\} + q_{tx}. \qquad (36)
$$

In this case the corresponding conserved vectors of (2) are

$$
T_1^t = \frac{1}{3}\{3q + kq_{xx}\}, \qquad (37)
$$

$$
\begin{aligned}
T_1^x = &\ \frac{1}{3(n+1)(2n+1)}\{6an^2q + 9anq + 3aq + 3bq^{n+1} \\
&+ 6bnq^{n+1} + 3cq^{2n+1} + 3cnq^{2n+1} + 4kn^2q_{tx} + 6knq_{tx} + 2kq_{tx}\} \qquad (38)
\end{aligned}
$$

$$
T_2^t = \frac{1}{6}\{2kq_{xx}q + 3q^2 - kq_x^2\}, \qquad (39)
$$

$$
\begin{aligned}
T_2^x = {} & \frac{1}{6(n+1)(n+2)}\{4kn^2qq_{tx} + 12knqq_{tx} + 8kqq_{tx} + 3an^2q^2 \\
& + 9anq^2 + 6aq^2 + 6bq^{n+2} + 6bnq^{n+2} + 6cq^{2n+2} + 3cnq^{2n+2} \\
& - 2kn^2q_tq_x - 6knq_tq_x - 4kq_tq_x\}, \qquad (40)
\end{aligned}
$$

$$
\begin{aligned}
T_3^t = {} & \frac{1}{12k(n+1)(n+2)(2n+1)}\{-6akn^3q_{xx}q - 21akn^2q_{xx}q \\
& - 21aknq_{xx}q - 6akq_{xx}q - 4bkn^3{q_x}^2q^n - 2bkn^2{q_x}^2q^n - 4bkn^2q_{xx}q^{n+1} \\
& + 2bk{q_x}^2q^n + 4bkn{q_x}^2q^n + 2bkq_{xx}q^{n+1} + 2bknq_{xx}q^{n+1} \\
& - 4ckn^3{q_x}^2q^{2n} - 8ckn^2{q_x}^2q^{2n} - 2ckn^2q_{xx}q^{2n+1} \\
& + 2ck{q_x}^2q^{2n} + ckn{q_x}^2q^{2n} + 2ckq_{xx}q^{2n+1} - 3cknq_{xx}q^{2n+1} - 2k^2n^3qq_{txxx} \\
& - 7k^2n^2qq_{txxx} - 7k^2nqq_{txxx} - 2k^2qq_{txxx} + 6kn^3qq_{tx} + 21kn^2qq_{tx} \\
& + 21knqq_{tx} + 6kqq_{tx} + 12bq^{n+2} + 24bnq^{n+2} + 12cq^{2n+2} + 6cnq^{2n+2} \\
& + 6akn^3{q_x}^2 + 21akn^2{q_x}^2 + 21akn{q_x}^2 + 6ak{q_x}^2 + 4k^2n^3q_{xx}q_{tx} \\
& + 2k^2n^3q_xq_{txx} + 14k^2n^2q_{xx}q_{tx} + 7k^2n^2q_xq_{txx} \\
& + 14k^2nq_{xx}q_{tx} + 7k^2nq_xq_{txx} + 4k^2q_{xx}q_{tx} + 2k^2q_xq_{txx} \\
& + 6kn^3q_tq_x + 21kn^2q_tq_x + 21knq_tq_x + 6kq_tq_x\}, \qquad (41)
\end{aligned}
$$

$$
\begin{aligned}
T_3^x = {} & \frac{1}{12k(n+1)^2(n+2)(2n+1)^2}\{8bkn^5q_xq_tq^n + 16bkn^4q_xq_tq^n \\
& + 2bkn^3q_xq_tq^n - 14bkn^2q_xq_tq^n - 2bkq_xq_tq^n - 10bknq_xq_tq^n \\
& + 8ckn^5q_xq_tq^{2n} + 28ckn^4q_xq_tq^{2n} + 26ckn^3q_xq_tq^{2n} + ckn^2q_xq_tq^{2n} \\
& - 2ckq_xq_tq^{2n} - 7cknq_xq_tq^{2n} + 56bkn^4q_{tx}q^{n+1} + 200bkn^3q_{tx}q^{n+1} \\
& + 246bkn^2q_{tx}q^{n+1} + 22bkq_{tx}q^{n+1} + 124bknq_{tx}q^{n+1} + 48abn^3q^{n+2} \\
& + 96abn^2q^{n+2} + 12abq^{n+2} + 60abnq^{n+2} + 28ckn^4q_{tx}q^{2n+1} \\
& + 120ckn^3q_{tx}q^{2n+1} + 175ckn^2q_{tx}q^{2n+1} + 22ckq_{tx}q^{2n+1} + 105cknq_{tx}q^{2n+1} \\
& + 24b^2n^3q^{2n+2} + 12acn^3q^{2n+2} + 12b^2q^{2n+2} + 72b^2n^2q^{2n+2} \\
& + 42acn^2q^{2n+2} + 12acq^{2n+2} + 54b^2nq^{2n+2} + 42acnq^{2n+2} \\
& + 24bcn^3q^{3n+2} + 84bcn^2q^{3n+2} + 24bcq^{3n+2} + 84bcnq^{3n+2} \\
& + 6c^2n^3q^{4n+2} + 12c^2q^{4n+2} + 24c^2n^2q^{4n+2} + 30c^2nq^{4n+2} \\
& + 12akn^5q_{tx}q + 60akn^4q_{tx}q + 111akn^3q_{tx}q + 96akn^2q_{tx}q \\
& + 6akq_{tx}q + 39aknq_{tx}q - 12kn^5q_{tt}q - 60kn^4q_{tt}q \\
& - 111kn^3q_{tt}q - 96kn^2q_{tt}q - 6kq_{tt}q - 39knq_{tt}q \\
& + 4k^2n^5q_{ttxx}q + 20k^2n^4q_{ttxx}q + 37k^2n^3q_{ttxx}q + 2k^2q_{ttxx}q \\
& + 32k^2n^2q_{ttxx}q + 13k^2nq_{ttxx}q + 12kn^5{q_t}^2 + 60kn^4{q_t}^2 \\
& + 111kn^3{q_t}^2 + 96kn^2{q_t}^2 + 6k{q_t}^2 + 39kn{q_t}^2 + 16k^2n^5{q_{tx}}^2 \\
& + 80k^2n^4{q_{tx}}^2 + 148k^2n^3{q_{tx}}^2 + 8k^2{q_{tx}}^2 + 128k^2n^2{q_{tx}}^2 \\
& + 52k^2n{q_{tx}}^2 + 12akn^5q_xq_t + 60akn^4q_xq_t + 111akn^3q_xq_t + 96akn^2q_xq_t \\
& + 6akq_xq_t + 39aknq_xq_t + 4k^2n^5q_tq_{txx} + 20k^2n^4q_tq_{txx}
\end{aligned}
$$

$$+\ 37k^2n^3q_tq_{txx} + 2k^2q_tq_{txx} + 32k^2n^2q_tq_{txx} + 13k^2nq_tq_{txx}$$
$$-\ 8k^2n^5q_xq_{ttx} - 40k^2n^4q_xq_{ttx} - 74k^2n^3q_xq_{ttx} - 4k^2q_xq_{ttx}$$
$$-\ 64k^2n^2q_xq_{ttx} - 26k^2nq_xq_{ttx}\}. \quad (42)$$

4. Concluding remarks

In this paper, exact solutions of the Benjamin–Bona–Mahony–Peregrine equation were obtained with power-law and dual power-law nonlinearities. The integration of these equations was performed by the Lie group as well as the simplest equation methods. The solutions obtained include the cnoidal waves, periodic solutions and non-topological soliton solution. It was verified that the solutions found are indeed solutions to the original equations. In general, the exact solutions obtained here could be helpful in the numerical study of the underlying equations. Subsequently, the conservation laws of the Benjamin–Bona–Mahony–Peregrine equation with power-law and dual power-law nonlinearities were derived using second-order multipliers.

Acknowledgement

CMK would like to thank the North-West University, Mafikeng Campus, for its continued support.

References

[1] A Biswas, *Commun. Nonlin. Sci. Numer. Simulat.* **15**, 2744 (2010)
[2] A Biswas, *Nonlin. Dyn.* **59**, 423 (2010)
[3] A Biswas, *Int. J. Oceans and Oceanography* **4**, 45 (2010)
[4] A Biswas, *Appl. Math. Lett.* **22**, 1775 (2009)
[5] Y Gurefe, A Sonmezoglu and E Misirli, *Pramana – J. Phys.* **77**, 1023 (2011)
[6] S Zhang and H Zhang, *Pramana – J. Phys.* **76**, 561 (2011)
[7] J Lee and R Sakthivel, *Pramana – J. Phys.* **76**, 819 (2011)
[8] H Kheiri, M R Moghaddam and V Vafaei, *Pramana – J. Phys.* **76**, 831 (2011)
[9] A M Wazwaz, *Int. J. Computer Math.* **82**, 699 (2005)
[10] A M Wazwaz, *Commun. Nonlin. Sci. Numer. Simulat.* **11**, 148 (2006)
[11] A M Wazwaz, *Phys. Lett.* **A355**, 358 (2006)
[12] A M Wazwaz, *Appl. Math. Comput.* **217**, 2277 (2010)
[13] A M Wazwaz, *Commun. Nonlin. Sci. Numer. Simulat.* **15**, 2828 (2010)
[14] I Mustafa, *Appl. Math. Comput.* **189**, 508 (2007)
[15] T Roshan, *J. Comput. Appl. Math.* **235**, 1641 (2011)
[16] F Tascan, A Bekir and M Koparan, *Commun. Nonlin. Sci. Numer. Simulat.* **14**, 1810 (2009)
[17] E Yomba, *Phys. Lett.* **A373**, 4001 (2009)
[18] T Xiao-Yan, H Fei and L Sen-Yue, *Chin. Phys. Lett.* **23**, 887 (2006)
[19] S Zhang, *Chaos, Solitons and Fractals* **38**, 270 (2008)
[20] G B Zheng, B Liu, Z J Wang and S K Zheng, *Int. J. Nonlin. Sci. Numer. Simulat.* **10**, 1523 (2009)
[21] T B Benjamin, J L Bona and J J Mahoney, *Philos. Trans. Roy. Soc. London Ser.* **A272**, 47 (1972)

[22] D H Peregrine, *J. Fluid. Mech.* **25**, 321 (1966)
[23] G W Bluman and S Kumei, *Symmetries and differential equations, applied mathematical sciences* (Springer-Verlag, New York, 1989) Vol. 81
[24] E Momoniat, D P Mason and F M Mahomed, *Int. J. Non-Linear Mech.* **36**, 879 (2001)
[25] R Naz, F M Mahomed and D P Mason, *Appl. Math. Comput.* **205**, 212 (2008)
[26] P J Olver, *Applications of Lie groups to differential equations, graduate texts in mathematics*. 2nd edition (Springer-Verlag, Berlin, 1993) Vol. 107
[27] L V Ovsiannikov, *Group analysis of differential equations* (Academic Press, New York, 1982) (English translation by W F Ames)
[28] N H Ibragimov, *CRC Handbook of Lie group analysis of differential equations*, edited by N H Ibrahimov (CRC Press, Boca Raton, Florida, 1994–1996) Vols 1–3
[29] M Abramowitz and I A Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables* (Dover Publications Inc, New York, 1972)
[30] N A Kudryashov, *Chaos, Solitons and Fractals* **24**, 1217 (2005)
[31] N A Kudryashov, *Phys. Lett.* **A342**, 99 (2005)
[32] N K Vitanov, *Commun. Nonlin. Sci. Numer. Simulat.* **15**, 2050 (2010)
[33] N K Vitanov and Z I Dimitrova, *Commun. Nonlin. Sci. Numer. Simulat.* **15**, 2836 (2010)
[34] H Steudel, *Zeit. Naturforsch.* **17A**, 129 (1962)
[35] S C Anco and G W Bluman, *European J. Appl. Math.* **13**, 545 (2002)
[36] G W Bluman, A F Cheviakov and S C Anco, *Applications of symmetry methods to partial differential equations* (Springer-Verlag, New York, 2010)
[37] M Anthonyrajah and D P Mason, *Math. Comput. Appl.* **15**, 529 (2010)