PRAMANA
— journal of
physics

Vol. 80, No. 1
January 2013
pp. 61–68

Unpolarized coupled DGLAP evolution equation at small-x

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MS received 10 October 2011; revised 22 June 2012; accepted 24 July 2012

Abstract. In this paper, we have obtained the solution of the unpolarized coupled Dokshitzer–Gribove–Lipatov–Alterelli–Parisi (DGLAP) evolution equation in leading order at the small-x limit. Here, we have used a Taylor series expansion, separation of functions and then the method of characteristics to solve the evolution equations. We have also calculated t-evolution of singlet and gluon distribution functions and the results are compared with E665 and NNPDF data for singlet structure function and GRV1998 and MRST2004 gluon parametrizations. It is shown that our results are in good agreement with the parametrizations especially at small-x and high-Q^2 region. From global parametrizations and our results, we have seen that the singlet and gluon distribution functions increase when Q^2 increases for fixed values of x.

Keywords. Deep inelastic scattering; Dokshitzer–Gribove–Lipatov–Alterelli–Parisi equation; small-x; method of characteristics; structure function.

PACS Nos 12.35.Eq; 12.38.t; 12.39.x; 13.60.Hb

1. Introduction

The internal structure of the hadrons, as probed in hard interactions, is determined particularly in lepton nucleon scattering experiments in deep inelastic scattering (DIS) process. The differential cross-section in terms of proton structure functions is given by density functions for different partons, i.e., $q(x, Q^2)$ and $g(x, Q^2)$ for quarks and gluons, where x is the Bjorken scaling and Q^2 is the lepton four-momentum transfer. These experiments provide an important testing ground for quantum chromodynamics (QCD). The structure function $F_2(x, Q^2)$ dominantly contributes to the cross-section,

DOI: 10.1007/s12043-012-0460-9; ***e*Publication: 8 January 2013**

while the gluon density contribution comes from logarithmic Q^2 dependence in perturbative QCD (pQCD). In standard analysis of QCD over the DIS data, a parametrization of the parton distribution functions at a starting scale Q_0 is assumed, which is further evolved to higher Q^2 using the next-to-leading (NLO) Dokshitzer–Gribove–Lipatov–Alterelli–Parisi (DGLAP) [1–4] equation. HERA's striking discovery of the steep rise in $F2$ at small x which is driven by $g(x)$ has no agreed fundamental explanation. The Q^2 dependence of the data is well-described by DGLAP evolution equation without further parameters at small-x, where $\alpha s \ln(1/x)$ terms are neglected. Experimentally, the rise towards small x was first observed in 1993 in the HERA data. Now, the improved precision of the data allows detailed study of the rise. H1 and ZEUS [5–7] have performed DGLAP-based (pQCD) analyses, which describe their data very well and provide parton distribution functions including uncertainties. However, this rise may be limited at very low-x by unitarity constraints. Perturbative quantum chromodynamics (pQCD) provides a rigorous and successful theoretical description of the Q^2 dependence of $F2(x, Q^2)$ in deep inelastic scattering. In the double asymptotic limit, the DGLAP evolution equations can be solved and $F2$ is expected to rise approximately as a power of x towards low-x. The low-x behaviour of DGLAP evolution equation for $Q^2 > 3$ GeV^2 is driven solely by the gluon field $G(x, t)$, since quarks to the scaling violations of $F_2(x, t)$ are negligible and gluons being the dominating partons.

At small-x limit, the DGLAP evolution equation can be solved analytically [8–10] with a Taylor approximated expansion of the structure functions for the parameter x and hence obtain a general solution for them. The DGLAP evolution equations [10–12] for singlet and gluon structure functions have the same form of derivative with respect to t in LO (leading order). Since, at the small-x limit as mentioned earlier, gluons become the dominating parton, the solution thus obtained is based on an assumption of the relation between them, which makes it possible to obtain a solution for either singlet or gluon distribution function.

Various analytical solutions of the DGLAP evolution equations have one common discrepancy with their partial differential form, and hence their general solutions are not unique. Using the method of characteristics, it can be overcome. The method of characteristics [13–18] is an important technique for solving initial value problems of first-order partial differential equation (PDE). It turns out that if we change coordinates from (x, t) to suitable new coordinates (S, τ), then the PDE becomes an ordinary differential equation (ODE). Then, we can solve ODE by the standard method and obtain an exact solution.

In the present method, we have coupled the Taylor approximated form of both DGLAP evolution equations for quark and gluon distribution functions in LO and then applied the method of characteristic and finally used separation of variable to obtain a simultaneous solution for them, instead of solving them separately as earlier. Also, in the present work, we have not considered any such relation between singlet and gluon distribution functions on ad hoc basis, but obtained general solution for the same compatibility parameter at a time. We have compared our solution for the quark and singlet structure function with experimental results; this provides a justification to the solutions we have obtained. The result of the singlet structure functions are compared with E665 [19] and NNPDF [20,21] parametrizations and gluon structure functions are compared with GRV98 [22,23] and MRST 2001 [24] parametrizations.

2. Theory

The gluon distribution $g(x, Q^2)$ is a part of the singlet densities $\sum(x, Q^2)$ and remains coupled with the singlet quark density as

$$\sum(x, Q^2) = \sum_{i=1}^{N_f} [q_i(x, Q^2) + \bar{q}_i(x, Q^2)].$$

Here, x is the fractional momentum carried by the partons, Q^2 is the energy imparted by the lepton beam to the hadron and q_is are the valence quarks and antiquarks flavours.

The Mellin convolution [24,25] between two functions is given by

$$a(x) \otimes b(x) = \int_x^1 \frac{\mathrm{d}y}{y} a(y) b\left(\frac{x}{y}\right).$$

Using the above convolution, DGLAP evolution equation [1–4] can be written in momentum space as

$$\frac{\partial}{\partial \ln Q^2} \begin{bmatrix} \sum(x, Q^2) \\ g(x, Q^2) \end{bmatrix} = \frac{\alpha_S(Q^2)}{2\pi} \sum_j \int_x^1 \frac{\mathrm{d}\xi}{\xi} \times \begin{pmatrix} P_{q_i q_j}\left(\frac{x}{\xi}, \alpha_S(Q^2)\right) & P_{q_i g}\left(\frac{x}{\xi}, \alpha_S(Q^2)\right) \\ P_{g q_j}\left(\frac{x}{\xi}, \alpha_S(Q^2)\right) & P_{gg}\left(\frac{x}{\xi}, \alpha_S(Q^2)\right) \end{pmatrix} \times \begin{bmatrix} \sum(\xi, Q^2) \\ g(\xi, Q^2) \end{bmatrix}, \tag{1}$$

where $\alpha_S(Q^2)$ is the fine structure constant, P_{ij} are the splitting functions related to the interaction between the partons, $g(x, Q^2)$ is the gluon density.

Splitting functions can be expanded in power series of $\alpha_S(Q^2)$ and is given by

$$P_{ij}\left(\frac{x}{\xi}, \alpha_S(Q^2)\right) = \left(\frac{\alpha_S(Q^2)}{2\pi}\right) p_{ij}^0\left(\frac{x}{\xi}\right) + \left(\frac{\alpha_S(Q^2)}{2\pi}\right)^2 p_{ij}^1\left(\frac{x}{\xi}\right) + \left(\frac{\alpha_S(Q^2)}{2\pi}\right)^3 p_{ij}^2\left(\frac{x}{\xi}\right) + \cdots, \tag{2}$$

where p_{ij}^0, p_{ij}^1 and p_{ij}^2 are the splitting functions of LO, NLO and NNLO, respectively.

After simplifying eq. (1), the singlet and gluon distribution functions using the LO splitting function [16] can be written as

$$\frac{\partial F_2^S(x, t)}{\partial t} - \frac{A_f}{t}\left[\{3 + 4\ln(1-x)\} F_2^S(x, t) + I_1^S(x, t) + I_2^S(x, t)\right] = 0 \tag{3}$$

$$\frac{\partial G(x, t)}{\partial t} - \frac{\alpha_S}{2\pi}\frac{2}{3}\left[\left\{\frac{11}{12} - \frac{N_f}{18} + \ln(1-x)\right\} G(x, t) + I_G\right] = 0, \tag{4}$$

where $F_2^S(x,t)$ and $G(x,t)$ are the singlet and gluon distribution functions, respectively, and $t=\ln(Q^2/\Lambda^2)$, Λ is the QCD cut-off parameter. Here $A_{\mathrm{f}}=36/(33-N_{\mathrm{f}})$ where N_{f} is the number of flavours and

$$I_1^S(x,t) = 2\int_x^1 \frac{\mathrm{d}\omega}{1-\omega}\left[(1+\omega^2)F_2^S\left(\frac{x}{\omega},t\right) - 2F_2^S(x,t)\right], \tag{5a}$$

$$I_2^S(x,t) = 2N_{\mathrm{f}}\int_x^1 \left\{\omega^2+(1-\omega)^2\right\}G\left(\frac{x}{\omega},t\right)\mathrm{d}\omega, \tag{5b}$$

$$\begin{aligned} I_G(x,t) = \int_x^1 \mathrm{d}\omega\Bigg[&\frac{\omega G\left((x/\omega),t\right)-G(x,t)}{1-\omega} \\ &+\left\{\omega(1-\omega)+\frac{1-\omega}{\omega}\right\}G\left(\frac{x}{\omega},t\right) \\ &+\frac{2}{9}\left(\frac{1+(1-\omega)^2}{\omega}\right)F_2^S\left(\frac{x}{\omega},t\right)\Bigg]. \end{aligned} \tag{6}$$

Now, let us introduce a variable $u=1-\omega$, and since $x<\omega<1$, we have $0<u<1-x$; hence the series is convergent for $|u|<1$ and we can use Taylor's expansion series [26]

$$\begin{aligned} F_2^S\left(\frac{x}{\omega},t\right) &= F_2^S\left(x+\frac{xu}{1-u},t\right) \\ &= F_2^S(x,t)+\frac{xu}{1-u}\frac{\partial F_2^S(x,t)}{\partial x}+\frac{1}{2}\left(\frac{xu}{1-u}\right)^2\frac{\partial^2 F_2^S(x,t)}{\partial x^2}+\cdots. \end{aligned}$$

Since x is small in our region of discussion, the terms containing x^2 can be neglected. So, we can write,

$$F_2^S\left(\frac{x}{\omega},t\right) \approx F_2^S(x,t)+\frac{xu}{1-u}\frac{\partial F_2^S(x,t)}{\partial x}. \tag{7}$$

Similarly for gluon, we can write

$$G(x,t) \approx G(x,t)+\frac{xu}{1-u}\frac{\partial G(x,t)}{\partial x}. \tag{8}$$

Using the relation from eq. (7) in eqs (5a) and (5b) and eq. (8) in eq. (6) and finally performing u-integration, we get eqs (3) and (4) in the following forms respectively:

$$\begin{aligned} t\frac{\partial F_2^S(x,t)}{\partial t} - \frac{A_{\mathrm{f}}B(x)}{t}\frac{\partial F_2^S(x,t)}{\partial x} - \frac{A_{\mathrm{f}}D(x)}{t}\frac{\partial G(x,t)}{\partial x} \\ = \frac{A_{\mathrm{f}}A(x)}{t}F_2^S(x,t)+\frac{A_{\mathrm{f}}C(x)}{T}G(x,t) \end{aligned} \tag{9a}$$

$$\begin{aligned} t\frac{\partial G(x,t)}{\partial t} - \frac{A_{\mathrm{f}}A_2^g(x)}{t}\frac{\partial G(x,t)}{\partial x} - \frac{A_{\mathrm{f}}A_4^g(x)}{t}\frac{\partial F_2^S(x,t)}{\partial x} \\ = \frac{A_{\mathrm{f}}A_3^g(x)}{t}F_2^S(x,t)+\frac{A_{\mathrm{f}}A_1^g(x)}{t}G(x,t), \end{aligned} \tag{9b}$$

where $A(x)$, $B(x)$, $C(x)$, $D(x)$, $A_2^g(x)$, $A_3^g(x)$ and $A_4^g(x)$ are functions of x.

Summing up the left-hand side and right-hand side of eqs (9a) and (9b), we get as follows:

$$\begin{aligned}\frac{\partial F_2^S(x,t)}{\partial t} &- \left(\frac{A_{\mathrm{f}}B(x)}{t}+\frac{A_{\mathrm{f}}A_4^g(x)}{t}\right)\frac{\partial F_2^S(x,t)}{\partial x}+\frac{\partial G(x,t)}{\partial t}\\ &-\left(\frac{A_{\mathrm{f}}D(x)}{t}+\frac{A_{\mathrm{f}}A_2^g(x)}{t}\right)\frac{\partial G(x,t)}{\partial x}\\ &=\left(\frac{A_{\mathrm{f}}A(x)}{t}+\frac{A_{\mathrm{f}}A_3^g(x)}{t}\right)F_2^S(x,t)\\ &\quad+\left(\frac{A_{\mathrm{f}}C(x)}{t}+\frac{A_{\mathrm{f}}A_1^g(x)}{t}\right)G(x,t)\,. \end{aligned}\tag{10}$$

Now, separating the singlet and gluon terms from both sides of eq. (10), we get

$$\begin{aligned}&-t\frac{\partial F_2^S(x,t)}{\partial t}+A_{\mathrm{f}}(B(x)+A_4^g(x))\frac{\partial F_2^S(x,t)}{\partial x}\\ &+A_{\mathrm{f}}(A(x)+A_3^g(x))F_2^S(x,t)\\ &\quad=t\frac{G(x,t)}{\partial t}-A_{\mathrm{f}}(D(x)+A_2^g(x))\frac{\partial G(x,t)}{\partial x}\\ &\qquad-A_{\mathrm{f}}(C(x)+A_1^g(x))G(x,t)=m\ (\mathrm{say})\,,\end{aligned}\tag{11}$$

where m may be a constant or functions of x.

To introduce the method of characteristics [13–18], let us consider two new variables S and τ instead of x and t such that,

$$\frac{\mathrm{d}t}{\mathrm{d}S}=-t \qquad \text{and} \qquad \frac{\mathrm{d}x}{\mathrm{d}S}=A_{\mathrm{f}}(B(x)+A_4^g(x)). \tag{12a, b}$$

This leads to the left-hand side of eq. (11) in the following form:

$$\frac{\mathrm{d}F_2^S(S,\tau)}{\mathrm{d}S}+A_{\mathrm{f}}(A(x)+A_3^g(x))F_2^S(S,\tau)=m. \tag{13}$$

Now eq. (13) can be solved with initial condition, $F_2^S(S,\tau)=F_2^S(\tau)$, such that $S=0$, $t=t_0$. Using the relation (12a) and replacing the coordinate system (S,τ) to (x,t) with input function $F_2^S(\tau)=F_2^S(x,\ t_0)$, we shall get the t-evolution of the singlet structure function in LO as

$$\begin{aligned}F_2^S(x,t)\ =\ &\frac{m}{A_{\mathrm{f}}(A(x)+A_3^g(x))}\\ &+\left\{F_2^S(x,t_0)-\frac{m}{A_{\mathrm{f}}(A(x)+A_3^g(x))}\right\}\left(\frac{t}{t_0}\right)^{A_{\mathrm{f}}\left(A(x)+A_3^g(x)\right)}.\end{aligned}\tag{14}$$

Similarly for gluon, from the right-hand side of eq. (11), we get the t-evolution of gluon structure function in LO as,

$$\begin{aligned}G(x,t)\ =\ &\frac{-m}{A_{\mathrm{f}}(C(x)+A_1^g(x))}\\ &+\left\{G(x,t_0)+\frac{m}{A_{\mathrm{f}}(C(x)+A_1^g(x))}\right\}\left(\frac{t}{t_0}\right)^{A_{\mathrm{f}}\left(C(x)+A_1^g(x)\right)}.\end{aligned}\tag{15}$$

The deuteron structure function in DIS can be written in terms of singlet as,

$$F_2^d(x,t) = \frac{5}{2} F_2^S(x,t)\,. \tag{16}$$

3. Results and discussions

In figures 1a and b, we have compared our result for t-evolution of deuteron structure function in LO, from eq. (16), with the measured data of Fermilab E665 [19] Collaboration on deep inelastic muon scattering having a average beam energy 470 GeV^2. The QCD cut-off parameter [19] for the fit is $\Lambda(N_\mathrm{f} = 4) = 250\,\mathrm{MeV}$.

The red marks represent experimental data and red line shows our matching. Error bar shows the total combined statistical and systematic uncertainty. Figures 1a and b show the best fit at $x = 0.035$ and $x = 0.069$ having $3.82 < Q^2$ (GeV^2) < 13.39 and $9.79 < Q^2$ (GeV^2) < 34.277, respectively for a given separation constant. It is evident from the figure that the steepest rise of the structure function with increasing Q^2 gets a better fit at $x = 0.035$, as compared to $x = 0.069$.

We have also compared the t-evolution of deuteron structure function in LO with NNPDF [20] parametrization. The NNPDF group parametrizes the DIS data of structure functions without any theoretical bias by neural network interpolation with existing data points. This unbiased interpolation provides a measure for all points within a range of x and Q^2, where the sampling provided by the data is fine enough. To obtain the deuteron structure function, NNPDF [21] group used the NMC Collaboration and BCDMS Collaboration data of deuteron structure function for parametrization.

Figures 2c and d show the comparison of our result with the experimental data for a best fit at $x = 0.01$ and $x = 0.05$ having $10 \leq Q^2(\mathrm{GeV}^2) \leq 50$. Here also, the comparison shows a better fit at smaller-x.

We have also compared the result of gluon structure function from eq. (15) with GRV98 [22,23] and MRST2001 [24] parametrizations (figure 3).

We consider GRV1998 parametrization for $x = 10^{-5}$ and $20 \leq Q^2 \leq 90$ GeV^2, where they used H1 [13] and ZEUS [14] high precision data on $G(x, Q^2)$. We have taken

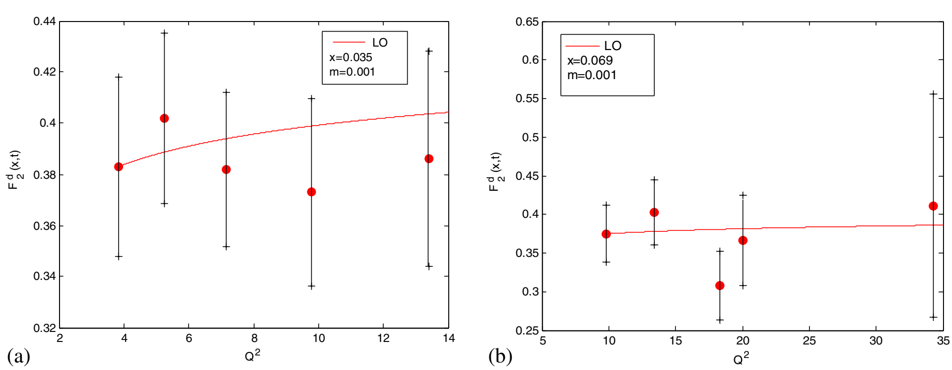

Figure 1. t-evolution of deuteron structure function in LO, compared with E665 data at (**a**) $x = 0.035$ and (**b**) $x = 0.069$, respectively, for $m = \text{constant} = 0.001$.

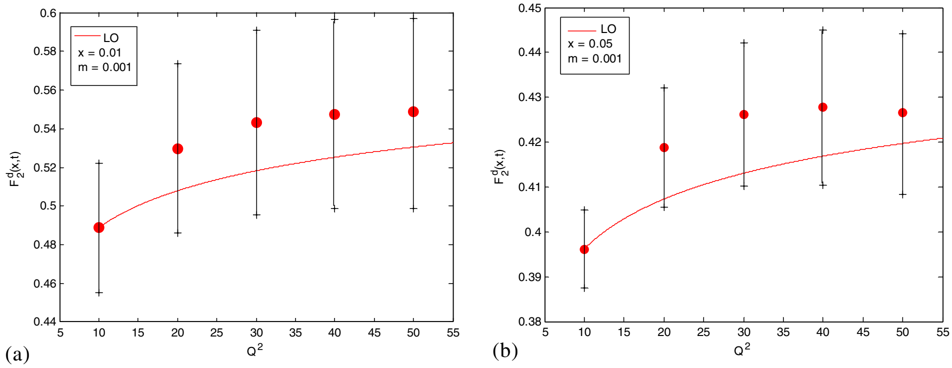

Figure 2. t-evolution of deuteron structure function in LO compared with NNPDF at (**a**) $x = 0.01$ and (**b**) $x = 0.05$ having $m = 0.001$.

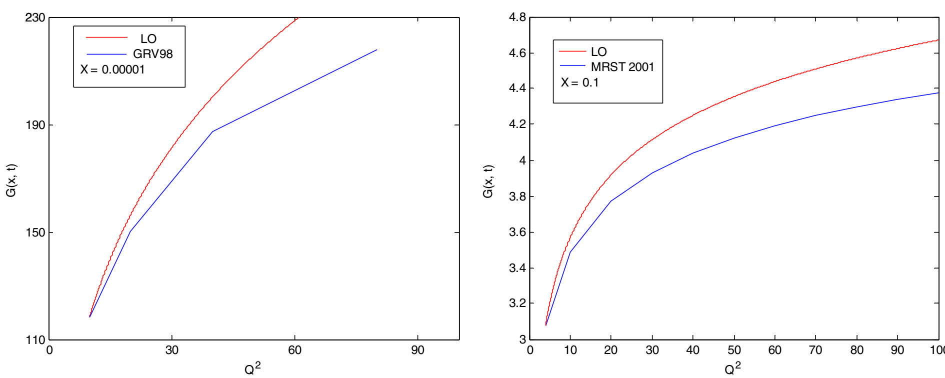

Figure 3. t-evolution of gluon structure function in LO compared with GRV98 and MRST2001 parameterizations for (**a**) $x = 0.00001$ and (**b**) $x = 0.1$ having $m = 0.001$.

MRST2001 fit [24] to initial $Q^2 = 4$ GeV^2 for $2 \leq Q^2 \leq 100$ GeV^2 at $x = 0.1$. It seems that the singlet and gluon structure functions are increasing with increase in energy transferred Q^2, for a given initial value. The figures show a comparison of the parametrizations with our results.

4. Conclusion

Here, we have shown an analytical approach to get a simultaneous solution for quark and gluon structure function in LO for a given input parameter. Using Taylor approximation, we provided the structure functions, a very good phenomenology with the experimental results at low-x. The comparison of our results with E665 and NNPDF data shows a good agreement with experiment as well as a better fit at lower x values between two different values of x for a given parametrization. Gluon structure functions obtained are also in good agreement with experimental data. Structure functions mainly contain the

LO contributions. Higher order contributions from NLO and NNLO are much smaller compared to LO. Though complex, it is also possible to obtain the structure functions for higher orders NLO and NNLO using the same method. As we obtain the solution by separating the LO expression for singlet and gluon structure functions for a given input parameter, it can be anticipated that the same approach provides a better agreement for higher order twist with experimental data. Though there are various methods to solve the DGLAP evolution equation to calculate quark and gluon structure functions, our method to obtain a simultaneous solution as characteristics of a given input parameter is also a viable alternative.

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