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SUSY formalism for the symmetric double well potential

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Abstract. Using first- and second-order supersymmetric Darboüx formalism and starting with symmetric double well potential barrier we have obtained a class of exactly solvable potentials subject to moving boundary condition. The eigenstates are also obtained by the same technique.

Keywords. SUSY; moving boundary condition; exactly solvable; symmetric double well; NH_3 molecule.

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1. Introduction

Exactly solvable potentials (here by the terminology 'exactly solvable' we mean that the eigenstates for the potential can be given in closed form) are very few in number in quantum mechanics [1–3]. So, it is always interesting to construct some exactly solvable potentials. Supersymmetric (SUSY) Darboüx method [1,4,5] is one of the elegant methods to construct integrable (exactly solvable) potentials. In SUSY quantum mechanics one has to start with a known potential and its eigenstates (the seed solutions) and following a very much well-defined route several exactly solvable potentials can be obtained. Various work has been done to construct the integrable potentials by SUSY Darboüx formalism. If the boundary conditions are time-dependent, complication increases (for a general treatment for SUSY quantum mechanics in time-dependent boundary condition, see [1] and the references therein).

In this paper, starting with the symmetric double well potential [2,3,6] under time-dependent boundary condition, we have obtained a class of exactly solvable potentials under time-dependent boundary condition. The reason behind choosing the double well potential is the following. For the specific tetrahedron structure of NH_3 molecule [7], the plane formed by three hydrogen atoms can be moved continuously from one side to the other side (because of quantum tunnelling effect [2,3,6,8,9]) of nitrogen atom with some definite frequency, called inversion frequency [3,8,9], it can absorb or emit the electromagnetic radiation of frequency equal to its inversion frequency [3,8,9] and this phenomenon is used in maser theory [9,10], radioastronomy [9–11] and many other branches [3,11]. Much work have been done by many authors [2,3,6–10] to determine the eigenstates and eigenvalues of NH_3 molecule. In many of the cases the symmetric double well potential [2,3,12] with time-independent boundary condition is used as the approximation of the potential of that system [13] and this simplified model successfully describes the existence of the inversion frequency of NH_3 molecule.

The organization of the article is as follows. Section 2 describes about the SUSY formalism in moving boundary case. Then the exactly solvable potentials obtained are given along with their eignestates.

2. Solvability of Schödinger equation in moving boundary case and SUSY formalism

To solve the problem for the moving boundary condition, one usually transform it to the fixed boundary problem by redefining the variables and solve the system for fixed boundary case and then easily transform the solutions for the moving boundary case [1].

To solve the time-dependent Schrödinger equation

$$\left[-\frac{\partial^2}{\partial x^2} + V(x,t)\right]\psi(x,t) = i\frac{\partial}{\partial t}\psi(x,t) \tag{1}$$

with the moving boundary condition

$$\psi(0,t) = 0, \quad \psi(L(t),t) = 0 \tag{2}$$

if we transform it in fixed boundary problem as $q = x/L(t)$ and transform the wave function as $\psi(q,t) \longmapsto \mathrm{e}^{\phi(q,t)}\chi(q,t)$, then for the potential $V(q,t) = g(t)\tilde{V}(q) + U(q,t) + g_0(t)$ the condition to apply the separation of variable technique to solve eq. (1) yields the following conditions [4]:

$$\phi(q,t) = a(t)\frac{q^2}{2} + b(q) + c(t), \quad \text{where} \quad a(t) = \frac{i}{2}\ddot{L}(t)L(t). \tag{3}$$

If we choose $b(q) = 0$ and $c(t) = -i\int_0^t g_0(s) - \frac{1}{2}\log L(t)$ and if we use $\chi(q,t) = Q(q)T(t)$; then we obtain $g(t) = 1/L(t)$ and $T(t) = \mathrm{e}^{-i\epsilon\tau(t)}$, where $\tau(t) = \int_0^t (1/L^2(s))\mathrm{d}s$. If we choose $U(q,t) = -\frac{1}{4}L(t)\ddot{L}(t)q^2$, then we can easily obtain the solutions for the moving boundary case by reusing $q = x/L(t)$.

Now, once we have a potential and its eigenstates by applying Darboüx transformation we can generate new types of potentials along with their eigenstates. The Darboüx transformation [5,14,15] method is based on the existence of an operator L and its adjoint $L^\dagger$ which act as transformation operators between a pair of self-adjoint Hamiltonians H and $\tilde{H}$ [15] and they are intertwined through $LH = \tilde{H}L \Rightarrow HL^\dagger = L^\dagger H$. For first-order SUSY,

$$L = \frac{\mathrm{d}}{\mathrm{d}q} + w(q).$$

So

$$L^\dagger = -\frac{\mathrm{d}}{\mathrm{d}q} + w(q)$$

$w(q)$ being the superpotential. If

$$H = -\frac{\partial^2}{\partial q^2} + V(q)$$

then V and $\tilde{V}$ can be expressed as

$$V(q) = w^2(q) - \frac{\mathrm{d}w}{\mathrm{d}q}, \quad \tilde{V}(q) = w^2(q) + \frac{\mathrm{d}w}{\mathrm{d}q}.$$

H and $\tilde{H}$ are iso-spectral except possibly for the ground state (ground state admits normalization problem; as for the first-order transformation L is something like the generalization of famous annihilation or creation operator for the angular momentum algebra or for the harmonic oscillator; this is not so much surprising) [1,5], and in second-order SUSY

$$L_{j,j+1} = \frac{\mathrm{d}^2}{\mathrm{d}q^2} + \beta_j(q)\frac{\mathrm{d}}{\mathrm{d}q} + \gamma(\beta_j).$$

So,

$$L^{\dagger}_{j,j+1} = \frac{\mathrm{d}^2}{\mathrm{d}q^2} - \beta_j \frac{\mathrm{d}}{\mathrm{d}q} + \gamma(-\beta_j),$$

where

$$\beta_j(q) = -\frac{\mathrm{d}}{\mathrm{d}q}\log W_{j,j+1}(x)$$

and

$$\gamma(\beta_j) = -\frac{\beta_j''}{2\beta_j} + \left(\frac{\beta_j'}{2\beta_j}\right)^2 + \frac{\beta_j'}{2} + \frac{\beta_j^2}{4} - \left(\frac{\epsilon_{j+1} - \epsilon_j}{2\beta_j}\right)^2$$

ϵ_j being the energy eigenvalue for the jth level. So the obtained potential [1,4,5] is

$$V(q) = V_0(q) - 2\frac{\mathrm{d}^2}{\mathrm{d}q^2}\log W_{j,j+1}(q),$$

$$W_{j,j+1}(q) = Q_j^0 Q_{j+1}^{0'} - Q_j^{0'} Q_{j+1}^0 \tag{4}$$

and the wave functions

$$Q_k(q) = \frac{1}{W_{j,j+1}}(q)\det\begin{bmatrix} Q_j^0 & Q_{j+1}^0 & Q_k^0 \\ Q_j^{0'} & Q_{j+1}^{0'} & Q_k^{0'} \\ Q_j^{0''} & Q_{j+1}^{0''} & Q_k^{0''} \end{bmatrix}, \quad j, j+1 \neq k. \tag{5}$$

First we note that for the approximate version of the potential of NH_3 molecule, i.e., symmetric double well potential barrier

$$\begin{aligned} V(q) &= V_0^2 \quad \text{at} -\left(b - \frac{a}{2}\right) \le q \le \left(b - \frac{a}{2}\right) \\ &= \infty \quad \text{at} \quad q = \pm b \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{6}$$

The eigenstates are

$$\begin{aligned} Q_k^0(q) &= \sin\left[k\left(b+\frac{a}{2}-q\right)\right], \quad \text{if } b-\frac{a}{2} \le q \le b+\frac{a}{2} \\ &= \sin\left[k\left(b+\frac{a}{2}+q\right)\right], \quad \text{if } b-\frac{a}{2} \le -q \le b+\frac{a}{2} \\ &= \cosh[\alpha_{\mathrm{s}} q], \quad \text{if } -\left(b-\frac{a}{2}\right) \le q \le b-\frac{a}{2} \ \text{(symmetric case)} \\ &= \sinh[\alpha_{\mathrm{a}} q], \quad \text{if } -\left(b-\frac{a}{2}\right) \le q \le b-\frac{a}{2} \ \text{(antisymmetric case)} \end{aligned} \tag{7}$$

with the corresponding eigenvalues $E_{\mathrm{s,a}} = h^2 k_{\mathrm{s,a}}^2/2m$, where $k_{\mathrm{s,a}}$ is to be determined from the solutions of the following transidental equation:

$$\begin{aligned} \tan(k_{\mathrm{s}} a) &= -\frac{k_{\mathrm{s}}}{\sqrt{\alpha^2-k_{\mathrm{s}}^2}} \coth\left[\sqrt{\alpha^2-k_{\mathrm{s}}^2}\left(b-\frac{a}{2}\right)\right] \\ \tan(k_{\mathrm{a}} a) &= -\frac{k_{\mathrm{a}}}{\sqrt{\alpha^2-k_{\mathrm{a}}^2}} \tanh\left[\sqrt{\alpha^2-k_{\mathrm{a}}^2}\left(b-\frac{a}{2}\right)\right], \quad \alpha^2 = \frac{2mV_0}{\hbar^2} \end{aligned} \tag{8}$$

which have countable number of solutions which confirms the energy quantization for the system and that is why we can safely use the index $k, k+j\,(j \in \aleph)$ to denote the various energy levels and the index s and a are for symmetric and antisymmetric respectively. Now we can easily construct the new types of potentials using SUSY.

3. New potentials by SUSY

One should usually start with the superpotential and can obtain new types of exactly solvable potentials. For our case we start with the superpotential

$$\begin{aligned} w(q) &= -V_0 \coth(V_0 q) \quad \text{at } -\left(b-\frac{a}{2}\right) \le q \le \left(b-\frac{a}{2}\right) \\ &= \infty \quad \text{at } \ q = \pm b \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{9}$$

The corresponding partner potentials

$$V(q) = w^2(q) - \frac{\mathrm{d}w}{\mathrm{d}q}$$

and

$$\tilde{V}(q) = w^2(q) + \frac{\mathrm{d}w}{\mathrm{d}q}$$

are the following:

$$\begin{aligned} V(q) &= V_0^2 \quad \text{at } -\left(b-\frac{a}{2}\right) \le q \le \left(b-\frac{a}{2}\right) \\ &= \infty \quad \text{at } \ q = \pm b \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{10}$$

$$\begin{aligned}\tilde{V}(q) &= V_0^2[1+2\,\mathrm{cosech}^2(V_0 q)] && \text{at } -\left(b-\frac{a}{2}\right)\le q\le\left(b-\frac{a}{2}\right)\\ &= \infty && \text{at} \quad q=\pm b \\ &= 0 && \text{otherwise.}\end{aligned} \tag{11}$$

Using eqs (3) and (4) we now can construct new solvable potentials along with their eigenstates under moving boundary conditions are given by

$$\begin{aligned}V(x,t) &= -\frac{\ddot{L}(t)}{L(t)}x^2, \quad \text{for reg}(1)\\ &= -\frac{\ddot{L}(t)}{L(t)}x^2, \quad \text{for reg}(2)\\ &= \frac{V_0^2\left(1+2\,\mathrm{cosech}^2\left(V_0x/L(t)\right)\right)}{L^2(t)}-\frac{\ddot{L}(t)}{L(t)}x^2, \quad \text{for reg}(3).\end{aligned} \tag{12}$$

For the sake of simplicity we denote $b-\frac{a}{2}\le q\le b+\frac{a}{2}$ as region (1), $b-\frac{a}{2}\le -q\le b+\frac{a}{2}$ as region (2) and $-(b-\frac{a}{2})\le q\le b-\frac{a}{2}$ as region (3), and the corresponding eigenstates are:

For regions (1) *and* (2):

$$\psi(x,t) = \mp\frac{k+1}{\sqrt{L(t)}}\exp(\theta(x,t))\cos\left[(k+1)X\mp\right].$$

For region (3) *and for symmetric condition*:

$$\begin{aligned}\psi(x,t) &= \frac{1}{\sqrt{L(t)}}\exp(\theta(x,t))\\ &\quad\times\left[\frac{L(t)}{x}\alpha_{\mathrm{s}-}\sinh\alpha_{\mathrm{s}-}-V_0\coth\left(\frac{V_0x}{L}\right)\cosh\alpha_{\mathrm{s}-}\right].\end{aligned} \tag{13}$$

For region (3) *and for antisymmetric condition*:

$$\begin{aligned}\psi(x,t) &= \frac{1}{\sqrt{L(t)}}\exp(\theta(x,t))\\ &\quad\times\left[\frac{L(t)}{x}\alpha_{\mathrm{a}+}\cosh\alpha_{\mathrm{a}+}-V_0\coth\left(\frac{V_0x}{L}\right)\sinh\alpha_{\mathrm{a}+}\right],\end{aligned} \tag{14}$$

where

$$\theta(x,t) = \left(\frac{i\dot{L}(t)}{4L(t)}x^2 - i\epsilon\tau\right),$$

$$\alpha_{\mathrm{s}-} = \sqrt{\frac{x^2}{L^2}\left(V_0-(k_{\mathrm{s}}+1)^2\right)},$$

$$\alpha_{\mathrm{a}+} = \sqrt{\frac{x^2}{L^2}(V_0-(k_{\mathrm{a}}+1)^2)}, \quad X\mp = \left(b+\frac{a}{2}\mp\frac{x}{L}\right).$$

And for the second-order Darboüx if one is equipped with the form of $\gamma(\beta)$, then easily new types of potentials can be written. As their expressions are slightly tedious, the γ are given in the Appendix.

In this section the potentials for only the special case $j = 1$ and the corresponding eigenstates are given.

For $\mathbf{j} = \mathbf{1}$ the obtained potential is

$$V_2(x,t) = \frac{1}{L^2(t)}\left[V_0^2 + 6\,\mathrm{cosec}^2 X_{\mp}\right] - \frac{\ddot{L}(t)}{4L(t)}x^2. \tag{15}$$

$\mp$ for region (1) and region (2) respectively, and the corresponding eigenstates are

$$\begin{aligned}\psi_k^{0,2}(x,t) = {}& \frac{1}{\sqrt{L}}\exp(\theta(x,t))[-k^2 \sin kX_- - 3k\cot X_- \cos kX_- \\ & + \sin kX_-(1 + 3\cot^2 X_-)], \qquad \text{for region (1)}\end{aligned} \tag{16}$$

and

$$\begin{aligned}\psi_k^{0,2}(x,t) = {}& \frac{1}{\sqrt{L}}\exp(\theta(x,t))[-k^2 \sin kX_+ - 3k\cot X_+ \cos kX_+ \\ & + \sin X_+(1 + 3\cot^2 X_+)], \qquad \text{for region(2).}\end{aligned} \tag{17}$$

For region (3) and for symmetric condition:

$$\begin{aligned}V_2(x,t) = {}& \frac{-3}{L^2(t)\Delta_s}[(V_0 - 1)(\cosh 2A_s - \cosh 2C_s) - 6\cosh^2 A_s] \\ & - \frac{\ddot{L}(t)}{4L(t)}x^2\end{aligned} \tag{18}$$

and the corresponding eigenstates:

$$\begin{aligned}\psi_k^{0,2}(x,t) = {}& \frac{1}{\Delta_s\sqrt{L}}[\alpha_{s+1}\alpha_{s_k}\cosh A_s(\alpha_{1s}\sinh D_{1s} - \alpha_{2s}\sinh D_{2s}) \\ & + \alpha_{s_k}\alpha_s \cos C_s(\alpha_{3s}\sinh D_{3s} - \alpha_{4s}\sinh D_{4s}) \\ & + \alpha_s\alpha_{s+1}\cosh B_s(\alpha_{5s}\sinh D_{5s} - \alpha_{6s}\sinh D_{6s})],\end{aligned} \tag{19}$$

where

$$\begin{aligned}\alpha_{s_k} &= \sqrt{(V_0 - k_s^2)}; \quad A_s = \alpha_s\frac{x}{L}; \quad B_s = \alpha_{s_k}\frac{x}{L}; \quad C_s = \alpha_{s+1}\frac{x}{L}; \\ D_{1s} &= B_s + C_s; \quad D_{2s} = B_s - C_s; \quad D_{3s} = A_s + B_s; \quad D_{4s} = A_s - B_s; \\ D_{5s} &= C_s + A_s; \quad D_{6s} = C_s - A_s; \quad \alpha_{1s} = \frac{\alpha_{s_k}}{2} - \frac{\alpha_{s+1}}{2};\end{aligned}$$

$$\alpha_{2\mathrm{s}} = \frac{\alpha_{\mathrm{s}_k}}{2} + \frac{\alpha_{\mathrm{s}+1}}{2}; \quad \alpha_{3\mathrm{s}} = \frac{\alpha_{\mathrm{s}}}{2} - \frac{\alpha_{\mathrm{s}_k}}{2}; \quad \alpha_{4\mathrm{s}} = \frac{\alpha_{\mathrm{s}}}{2} + \frac{\alpha_{\mathrm{s}_k}}{2};$$

$$\alpha_{5\mathrm{s}} = \frac{\alpha_{\mathrm{s}+1}}{2} - \frac{\alpha_{\mathrm{s}}}{2}; \quad \alpha_{6\mathrm{s}} = \frac{\alpha_{\mathrm{s}+1}}{2} + \frac{\alpha_{\mathrm{s}}}{2};$$

$$\Delta_{\mathrm{s}} = \left[\alpha_{\mathrm{s}+1}\cosh A_{\mathrm{s}} \sinh C_{\mathrm{s}} - \alpha_{\mathrm{s}} \cosh C_{\mathrm{s}} \sinh A_{\mathrm{s}}\right]^2;$$

$$\alpha_{\mathrm{s}} = \sqrt{V_0 - 1}; \quad \alpha_{\mathrm{s}+1} = \sqrt{V_0 - 4}.$$

For region (3) and for antisymmetric condition:

$$V_2(x,t) = \frac{3}{L^2(t)\Delta_{\mathrm{a}}}[(V_0 - 1)(\cosh 2A_{\mathrm{a}} - \cosh 2C_{\mathrm{a}}) - 6\sinh^2 A_{\mathrm{a}}] - \frac{\ddot{L}(t)}{4L(t)}x^2 \tag{20}$$

and the corresponding eigenstates

$$\psi_k^{0,2}(x,t) = \frac{1}{\Delta_{\mathrm{a}}\sqrt{L}}[\alpha_{\mathrm{a}+1}\alpha_{\mathrm{a}_k} \sinh A_{\mathrm{a}}(\alpha_{1\mathrm{a}} \sinh D_{1\mathrm{a}} - \alpha_{2\mathrm{a}} \sinh D_{2\mathrm{a}}) + \alpha_{\mathrm{a}_k}\alpha_{\mathrm{a}} \sinh C_{\mathrm{a}}(\alpha_{3\mathrm{a}} \sinh D_{3\mathrm{a}} - \alpha_{4\mathrm{a}} \sinh D_{4\mathrm{a}}) + \alpha_{\mathrm{a}}\alpha_{\mathrm{a}+1} \sinh B_{\mathrm{a}}(\alpha_{5\mathrm{a}} \sinh D_{5\mathrm{a}} - \alpha_{6\mathrm{a}} \sinh D_{6\mathrm{a}})], \tag{21}$$

where

$$\alpha_{\mathrm{a}_k} = \sqrt{(V_0 - k_{\mathrm{a}}^2)}; \quad A_{\mathrm{a}} = \alpha_{\mathrm{a}}\frac{x}{L}; \quad B_{\mathrm{a}} = \alpha_{\mathrm{a}_k}\frac{x}{L}; \quad C_{\mathrm{a}} = \alpha_{\mathrm{a}+1}\frac{x}{L};$$

$$D_{1\mathrm{a}} = B_{\mathrm{a}} + C_{\mathrm{a}}; \quad D_{2\mathrm{a}} = B_{\mathrm{a}} - C_{\mathrm{a}}; \quad D_{3\mathrm{a}} = A_{\mathrm{a}} + B_{\mathrm{a}};$$

$$D_{4\mathrm{a}} = A_{\mathrm{a}} - B_{\mathrm{a}}; \quad D_{5\mathrm{a}} = C_{\mathrm{a}} + A_{\mathrm{a}}; \quad D_{6\mathrm{a}} = C_{\mathrm{a}} - A_{\mathrm{a}};$$

$$\alpha_{1\mathrm{a}} = \frac{\alpha_{\mathrm{a}_k}}{2} - \frac{\alpha_{\mathrm{a}+1}}{2}; \quad \alpha_{2\mathrm{a}} = \frac{\alpha_{\mathrm{a}_k}}{2} + \frac{\alpha_{\mathrm{a}+1}}{2}; \quad \alpha_{3\mathrm{a}} = \frac{\alpha_{\mathrm{a}}}{2} - \frac{\alpha_{\mathrm{a}_k}}{2};$$

$$\alpha_{4\mathrm{a}} = \frac{\alpha_{\mathrm{a}}}{2} + \frac{\alpha_{\mathrm{a}_k}}{2}; \quad \alpha_{5\mathrm{a}} = \frac{\alpha_{\mathrm{a}+1}}{2} - \frac{\alpha_{\mathrm{a}}}{2}; \quad \alpha_{6\mathrm{a}} = \frac{\alpha_{\mathrm{a}+1}}{2} + \frac{\alpha_{\mathrm{a}}}{2};$$

$$\Delta_{\mathrm{a}} = \left[\alpha_{\mathrm{a}+1} \sinh A_{\mathrm{a}} \cosh C_{\mathrm{a}} - \alpha_{\mathrm{a}} \sinh C_{\mathrm{a}} \cosh A_{\mathrm{a}}\right]^2;$$

$$\alpha_{\mathrm{a}} = \sqrt{V_0 - 1}; \quad \alpha_{\mathrm{a}+1} = \sqrt{V_0 - 4}.$$

It can be checked straightforwardly whether the obtained eigenstates indeed satisfy the eigenvalue equation for the corresponding potential.

4. Conclusion

Applying first- and second-order SUSY formalism to the symmetric double well potential barrier we have obtained a class of exactly solvable potentials with moving right boundary. Also, one can easily realize that after applying first-order Darboüx transformation, the symmetric eigenstates transform into antisymmetric eigenstates and vice-versa and this is expected as the transformation operator can be considered as the generalization of the well-known annihilation and creation operators. As our starting potential is the same as that of the simplified toy model of the potential in NH_3 molecule (the only difference is that the boundary condition we have used is time-dependent) we hope that the obtained potential may be helpful to describe the NH_3 molecule.

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Appendix

In general, the form of $\gamma(\beta)$ are:

$$\begin{aligned}
\gamma(\beta) = {} & \frac{1}{4}\cot^2 p - \frac{j \sin p \cot(j+1)p}{\sin jp}\left[1 - \frac{j \sin p}{\sin jp \sin 2(j+1p)}\right] \\
& + \frac{1}{[j \sin p - \sin jp \cos(j+1)p]^2} \\
& \times \left[j \sin jp + \frac{j}{2}\sin^2 jp - \frac{j^2}{4}(1+12j)\sin p \sin(2j+1)p \right. \\
& \quad + \left(j^2 + j + \frac{1}{4}\right)\frac{\sin^2 jp}{\sin^2(j+1)p} \\
& \quad + \left(j^2 + \frac{1}{4}\right)\sin^2 jp \sin^2(j+1)p + j \sin^2 jp \sin^2(2j+1)p \\
& \quad + \frac{j^2 \cos jp}{\sin(j+1)p}(\sin p - \sin jp) + \frac{j^4 \sin^p \cos(2j+1)p}{\sin jp \sin(j+1)p} \\
& \quad \left. \times \left[1 + \frac{\tan(2j+1)p \sin(2j+1)p}{4 \sin jp \sin(j+1)p}\right]\right]
\end{aligned} \tag{22}$$

for regions (1) and (2) and

$$\gamma(\beta) = \frac{1}{\Theta_{\rm s}}\left[\frac{3}{2}\alpha_{\rm s}^2\alpha_{\rm s+1}^2 + 2\alpha_{\rm s}\alpha_{\rm s+1}\tanh A_{\rm s}\tanh C_{\rm s}(\alpha_{\rm s+1}^2\sinh A_{\rm s}\cosh A_{\rm s} \right.$$
$$- \alpha_{\rm s}^2\sinh C_{\rm s}\cosh C_{\rm s}) - (\alpha_{\rm s+1}^4\cosh^2 A_{\rm s} + \alpha_{\rm s}^4\cosh^2 C_{\rm s})$$
$$+ \frac{1}{4}\cosh^2 C_{\rm s}\,\mathrm{sech}^2 A_{\rm s} + \frac{1}{4}\cosh^2 A_{\rm s}\,\mathrm{sech}^2 C_{\rm s}$$
$$\times\left(j_{\rm s} + \frac{1}{2}\right)(\alpha_{\rm s}^2\cosh^2 C_{\rm s} - \alpha_{\rm s+1}^2\cosh^2 A_{\rm s})$$
$$\left. + \frac{(2j_{\rm s}+1)^2}{4}\cosh^2 A_{\rm s}\cosh^2 C_{\rm s}\right]$$
$$- \frac{1}{4}(\alpha_{\rm s+1}^2\tanh^2 C_{\rm s} + \alpha_{\rm s}^2\tanh^2 A_{\rm s} - 2\alpha_{\rm s}\alpha_{\rm s+1}\tanh A_{\rm s}\tanh C_{\rm s}) \tag{23}$$

for region (3) and symmetric case. Here $\Theta_{\rm s} = [\alpha_{\rm s+1}\cosh A_{\rm s}\sinh C_{\rm s} - \alpha_{\rm s}\cosh C_{\rm s}\sinh A_{\rm s}]$.

For region (3) and antisymmetric case

$$\gamma(\beta) = \frac{1}{\Theta_{\rm a}}\left[2\alpha_{\rm a}^2\alpha_{\rm a+1}^2\right.$$
$$- \alpha_{\rm a}\alpha_{\rm a+1}\coth A_{\rm a}\coth C_{\rm a}(\alpha_{\rm a+1}^2\sinh^2 A_{\rm a} + \alpha_{\rm a}^2\sinh^2 C_{\rm a})$$
$$+ \alpha^4\sinh^2 A_{\rm a} + \alpha^4\sinh^2 A_{\rm a} \times \frac{\alpha_{\rm a+1}^2}{4}\sinh A_{\rm a}\,\mathrm{cosech}\, C_{\rm a}$$
$$- \frac{\alpha_{\rm a}^2}{4}\sinh C_{\rm a}\,\mathrm{cosech}\, A_{\rm a} + \left(j_{\rm a} + \frac{1}{2}\right)(\alpha_{\rm a+1}^2\sinh^2 A_{\rm a}$$
$$\left. - \alpha_{\rm a}^2\sinh^2 C_{\rm a})\left(j_{\rm a} + \frac{1}{2}\right)^2\sinh^2 A_{\rm a}\sinh^2 C_{\rm a}\right]$$
$$- \left(\frac{\alpha^2 a + 1}{4}\coth^2 C_{\rm a} + \frac{\alpha_{\rm a}^2}{4}\coth^2 A_{\rm a} - 2\alpha_{\rm a}\alpha_{\rm a+1}\coth A_{\rm a}\coth C_{\rm a}\right), \tag{24}$$

where $\Theta_{\rm a} = (\alpha_{\rm a+1}\sinh A_{\rm a}\cosh C_{\rm a} - \alpha_{\rm a}\cosh A_{\rm a}\sinh C_{\rm a})^2\alpha$.

References

[1] T K Jana and P Roy, *Phys. Lett.* **A372**, 2368 (2008)
[2] J J Sakurai, *Modern quantum mechanics* (Addison Wesley Publishing Company, 1994) Chapter 4
[3] Claude Cohen-Tannoudji, Bernard Diu and Franck Laloë, *Quantum mechanics* (Wiley Publication, 1977) Vol. 1

[4] B Mielnik, *J. Math. Phys.* **25**, 3387 (1984)
[5] D J C Fernández, V Hussin and B Mielnik, *Phys. Lett.* **A244**, 309 (1998)
[6] Asim Gangopadhyaya, Prasanta K Panigrahi and Uday P Sukhatme, *Phys. Rev.* **A47**, 2720 (1993)
[7] John Sheridan and Walter Gordy, *Phys. Rev.* **79**, 513 (1950)
[8] H Margenau, *Phys. Rev.* **76**, 1423 (1949)
[9] Moshe Elitzur, arXiv:astro-ph/0105205v1, 11 May 2001
[10] Kristen Rohlfs, *Tools of radio astronomy* (Springer-Verlag, Berlin and New York, 1986) p. 332
[11] A C Cheung, D M Rank, C H Townes, D D Thornton and W J Welch, *Phys. Rev. Lett.* **21**, 1701 (1968)
[12] Feng Zhou, Zhuangqi Cao and Qishun Shen, *Phys. Rev.* **A67**, 062112 (2003)
[13] C Quesne, arXiv:1106.1990v1 [math-ph], 10 June 2011
[14] F Finkel, A Gonzalez-Lopez, N Kamran and M A Rodrigues, *J. Math. Phys.* **40**, 3268 (1999)
[15] D J C Fernández, J Negro and L M Nieto, *Phys. Lett.* **A275**, 338 (2000)