

Applications of the AdS/CFT correspondence

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Abstract. In this writeup of a talk delivered at the Lepton Photon Conference 2011, applications of the AdS/CFT correspondence in diverse areas of physics are reviewed.

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1. The AdS/CFT correspondence

Quantum systems often admit classical limits in which quantum fluctuations are parametrically suppressed. The most familiar such limit arises when the loop counting parameter of the system is taken as zero. Special systems however also admit a less familiar classical limit. Consider a system with a very large number of (let us say N^2) degrees of freedom, together with a rule that asserts that these degrees of freedom can be measured only in appropriate averages. Quantum fluctuations of the observable averages are suppressed simply by the law of large numbers, and it is sometimes possible to define a large N limit in which the observable averages are effectively classical even at finite values of the loop counting parameter.

The scenario outlined in the previous paragraph arises very naturally in diverse problems in theoretical physics. Consider, for instance, large N $SU(N)$ pure Yang–Mills theory (a system that plausibly approximates the physics of $SU(3)$ theory of strong interactions to reasonable accuracy). This theory has $(N^2 - 1)$ gluonic degrees of freedom. These degrees of freedom are, however, observable only through their gauge invariant ‘averages’ like $(1/N)(\text{Tr} F_{\mu\nu} F_{\alpha\beta})$. The discussion of the previous paragraph suggests that Yang–Mills theory may be effectively classical at large N ; it is possible to show that this is indeed the case when the large N limit is taken in an appropriate manner [1]. The same conclusions apply to all $U(N)$ gauge theories coupled to matter in the adjoint representation.

In the simpler $\hbar \rightarrow 0$ classical limit of a quantum system, the effective equations of classical dynamics easily deduced; they assert stationarity of the phase that weights paths in the path integral, i.e. the action. On the other hand, the classical equations that

emerge in the t'Hooft large N limit of gauge theories cannot directly be read off from the classical action of the theory. The determination of these equations requires the summation of all planar graphs in the theory [1], a process that is usually impossible to carry out in practice. The resultant equations govern the nonlinear nonlocal dynamics of an infinite number of degrees of freedom (the single trace operators of the theory). As a consequence, the analysis of these equations is a daunting and unfamiliar task, even if they can somehow be determined.

The remarkable AdS/CFT correspondence of the string theory [2] (see [3–6] and references therein for reviews) may be thought of as a set of conjectures for the effective classical description of the large N limits of classes of (usually supersymmetric) d -dimensional gauge theories. Quite remarkably, classical systems obtained in this process are far from messy. In particular when the field theory loop counting parameter is taken to infinity, these equations turn out simply to be Einstein's equations coupled to familiar matter fields. String theory manages to reword the unfamiliar messy nonlocal equations for trace fields in a simple local form using a remarkable maneuver; it turns out that metric and other classical fields of the string description propagate in a $D > d$ dimensional space time that is different from the d -dimensional manifold on which the original gauge theory is formulated. The original field theory resides on the d -dimensional conformal boundary of the bulk D -dimensional space-time.

According to the AdS/CFT correspondence, the single trace operators of the d -dimensional gauge theory completely encodes the D -dimensional gravitational dynamics of the bulk theory, analogous to the manner in which a two-dimensional hologram encodes a three-dimensional image. For this reason the gravitational description of classical large N field theory dynamics is sometimes said to be 'holographic'. The 'bulk' (i.e. D -dimensional) gravitational system is holographically 'dual' to the boundary (i.e. d -dimensional) field theory.

The AdS/CFT correspondence is equipped with a dictionary that relates the boundary values of the bulk fields like the graviton to the expectation values of the corresponding boundary operators like the stress tensor. Using this dictionary it is in principle possible to reformulate the familiar local equations of Einstein gravity in D dimensions as unfamiliar and nonlocal equations for expectation values of the corresponding operators on the d -dimensional boundary. We re-emphasize that the AdS/CFT correspondence does more than determine the equations that govern the trace dynamics of large N field theories; it determines these equations in a beautiful local and familiar form by adopting a holographic representation. One of the fields of the dual holographic theory is always a dynamical metric, and so the holographic system is a theory of gravity.

We pause here for an aside. Although we have emphasized the strict large N limit in this introduction, it is of course also possible to study the same field theories at large but finite values of N . The parameter $1/N$ controls the fluctuations of the trace fields. Retreating away from the strict large N consequently quantizes the bulk gravitational description. It follows that only quantizable gravitational theories that can arise as the dual description of large N systems, and that the finite N behaviour of the duals provides a nonperturbative definition of the quantization of these gravitational theories. In these notes we emphasize field theory lessons learnt from the AdS/CFT correspondence. However, perhaps the deepest future lessons of this correspondence lie in the lessons it contains for the quantum theory of gravity.

The best studied example of the AdS/CFT correspondence is the duality between $\mathcal{N} = 4$ Yang–Mills theory in four dimensions and IIB theory on $AdS_5 \times S^5$. I now pause to review this correspondence in a little more detail (see the early review [6] for more details and references). $\mathcal{N} = 4$ Yang–Mills theory is the maximally supersymmetric cousin of pure Yang–Mills theory. Apart from the Yang–Mills gauge field, this theory has six adjoint (i.e. matrix valued) scalar fields and four adjoint Weyl fermions. The requirement of supersymmetry completely determines the Lagrangian of the theory (and so the interactions of these basic fields) up to a single overall dimensional coupling constant, g_{YM}^2 . The high degree of supersymmetry of this theory may be exploited to demonstrate that β function of this coupling constant vanishes. It follows that any value of the rank N , $\mathcal{N} = 4$ Yang–Mills theory defines a line of fixed points parametrized by the coupling constant g_{YM} . In other words, this system defines a set of conformal field theories labelled by the two parameters N and g_{YM}^2 . The effective loop counting parameter of this theory is the 't Hooft coupling $\lambda = g_{YM}^2 N$ (Feynman diagram perturbation theory is good at small λ). In the so called t'Hooft limit $N \rightarrow \infty$ and $g_{YM}^2 \rightarrow 0$ with λ held fixed, the theory reduces to a classical system at all values of λ .

The AdS/CFT correspondence asserts that $\mathcal{N} = 4$ Yang–Mills is dual to IIB string theory on a particular background that solves the equations of motion of the theory, namely $AdS_5 \times S^5$ with string coupling $g \propto g_{YM}^2$ and N units of 5 form flux threading the S^5 (the number of flux units is the same as the rank of the gauge group of the dual field theory). In the t'Hooft limit of the previous paragraph g is taken as zero; consequently the string theory is effectively classical in the traditional sense. It turns out that the radius of S^5 and AdS_5 are equal and of order $(gN)^{1/4} = \lambda^{1/4}$ in units of the string length l_s . In other words, the $AdS_5 \times S^5$ background is strongly curved in string units at small λ but weakly curved at large λ . The equations of classical IIB string theory on weakly curved manifolds are very well understood; in particular they reduce, at low energies in string units to the equations of IIB supergravity; a set of well-studied classical equations that describe a ten-dimensional graviton coupled in a supersymmetric fashion to a few other matter fields.

In summary, the effective classical description of large N , $\mathcal{N} = 4$ Yang–Mills theory in the t'Hooft limit is the classical IIB string theory at every value of λ . In the limit $\lambda \rightarrow \infty$, IIB string theory can be replaced by IIB supergravity propagating on $AdS_5 \times S^5$ for many purposes, and we have a simpler dual classical description in this limit.

2. The fluid gravity map

In this section we review one of the most interesting applications of the AdS/CFT duality, namely the fluid gravity correspondence ([7,8], see [9–11] for reviews) which asserts a tight connection between the equations of Einstein gravity and those of relativistic generalizations of the Navier–Stokes equations. While the fluid gravity correspondence is strongly suggested by the AdS/CFT correspondence, it has now been independently derived by direct analysis of the equations of gravity [8]. The fluid gravity correspondence may be thought of as a small subsector of the AdS/CFT correspondence which has been independently verified.

This section is organized as follows. We begin this section with a review of the structure of the equations of hydrodynamics with emphasis on open questions, and explain how some of these questions have been addressed by AdS/CFT-inspired techniques. We then provide a brief description of the construction of the fluid gravity map that was used to generate these results.

Hydrodynamics may be thought of as a long-distance effective theory for real-time thermal physics. In the simplest contexts, the variables of hydrodynamics are local values of thermodynamical fields (like the temperature, chemical potential and local fluid velocities). The equations of hydrodynamics are simply the equations of conservation of the stress tensor and charge currents, once these conserved currents have been expressed in terms of local thermodynamical fields. The dynamical content in the equations of hydrodynamics lies in the so-called constitutive relations that express the stress tensor and charge current as functions of local thermodynamical fields in a systematic expansion in derivatives of these fields.

The constitutive relations for the fluid of any given field theory, like QCD, are determined in principle by the microscopic dynamics of the system. In a strongly coupled system, however, the required computations are very difficult to perform either by analytic techniques (because of strong coupling) or by numerical lattice techniques (because some of the needed quantities are inherently Lorentzian, and difficult to deduce, in a practical manner, by analytic continuation from Euclidean computations). Consequently, the hydrodynamical description of a strongly coupled field theory like QCD is best regarded as an autonomous effective field theory whose parameters should be deduced from experiment. In order to implement this process it is very important to have a clear parametrization of the most general form of the constitutive relations of hydrodynamics that are allowed on general grounds (i.e. without reference to the detailed dynamics of the parent quantum field theory), at each order in an expansion in space–time derivatives.

At zero order, in the derivative expansion, the hydrodynamical constitutive relations are completely specified by thermodynamics of the underlying system. In the classic text on fluid dynamics by Landau and Lifshitz, the constitutive relations of a charged fluid was parametrized by three functions – the shear viscosity, bulk viscosity and conductivity – at first order in the derivative expansion. Although the form of the Landau–Lifshitz constitutive relations were unchallenged for over 70 years, the explicit computation of constitutive relations in examples of the AdS/CFT correspondence [12–14] demonstrated that they are incomplete. The most general constitutive relations are now known; they include extra terms (like the so-called chiral vorticity term and the chiral magnetic term) [15–18]. It may also be demonstrated that these extra terms do appear in the QCD fluid.

In a similar manner, the complete characterization of the constitutive relations of an uncharged fluid at second order in the derivative expansion has only been obtained in the last year [16,19,20], spurred by results obtained from the fluid gravity map of the AdS/CFT correspondence. Further, the most general constitutive relations for a relativistic superfluid, at first order in the derivative expansion, was also only recently obtained (correcting errors in the superfluidity literature) by trying to make sense of the results obtained from the AdS/CFT correspondence [21–25]. Moreover, [16,20,25] provide a clean method to separate the parameters that enter hydrodynamical constitutive relations into two classes; nondissipative and dissipative parameters. Nondissipative constitutive

parameters may be determined from Euclidean partition functions in modified stationary backgrounds and so are effectively computable using Euclidean lattice techniques. On the other hand, dissipative constitutive parameters, like the viscosity, cannot be computed in equilibrium; they are intrinsically Lorentzian quantities which are harder to compute using lattice techniques.

The AdS/CFT correspondence has also yielded interesting information about the values of hydrodynamical parameters, in particular the shear viscosity. The shear viscosity controls the transfer of momentum between layers of fluid undergoing a shear type motion. Momentum is transferred very efficiently when ‘molecules’ from one layer can freely stream into another layer with very different momentum. As a consequence, the shear viscosity is infinite in a free theory (e.g. at $\lambda = 0$ in $\mathcal{N} = 4$ Yang–Mills theory) and decreases upon increasing the coupling. Using the AdS/CFT correspondence it was discovered that the dimensionless ratio of the shear viscosity to the entropy of $\mathcal{N} = 4$ Yang–Mills theory tends to $1/4\pi$ in the opposite strong coupling limit ($\lambda \rightarrow \infty$) [7,26,27]. In the absence of any other guidance, this result suggests a ballpark figure for this ratio in strongly coupled systems.

Several of the results obtained above have already proved practically useful. The value of the shear viscosity-to-entropy ratio suggests a ballpark figure for the ratio of the same quantity in the intermediate fluid state in RHIC and LHC heavy-ion collisions, one that is considerably lower than that suggested by extrapolation from perturbative computations. The AdS/CFT inspired estimate appears to be in qualitative agreement with data. It has also been suggested that the new chiral magnetic and chiral vorticity terms in the constitutive relations of charged fluids (described above) play a crucial role in some aspects of RHIC fluid dynamics. Finally, the correct parametrization of second-order terms in the constitutive relations of uncharged fluids helps me to guide the choice of equations to be used in computer simulations of the RHIC fluid. These applications are likely to be the tip of a large iceberg; as fluids permeate diverse aspects of physics, it is likely that these results will find increasingly diverse applications in the coming years.

In the rest of this section we briefly explain how the AdS/CFT correspondence was used to obtain or motivate the results pertaining to constitutive relations and values of transport coefficients described above. A useful starting point is the map between the boundary and bulk dynamics in global thermal equilibrium. In the field theory thermal equilibrium is characterized by a choice of static frame, a temperature and a chemical potential for every conserved charge. The gravity dual of static equilibrium is the stationary asymptotically AdS black brane solution (a black brane is similar to a black hole, except for the fact that its event horizon is planar rather than spherical, and in fact translationally invariant). The temperature and chemical potential of the fluid is given by the Hawking temperature and chemical potential of the black hole, while the fluid dynamical velocity is captured by the horizon boost velocity of the black brane.

It is a conceptually straightforward exercise to solve the linearized equations of small perturbations around the uniform black brane background (subject to infalling boundary conditions at the black hole horizon, and the condition of normalizability at the black hole boundary). The eigenmodes of this linear problem are called the quasinormal modes of the black brane. Every eigenmode is governed by its dispersion relation, a relationship that expresses the frequency ω of the mode as a function of the spatial wave vector k of that mode. It was demonstrated in [7,26,27] that precisely four of these eigenmodes

are massless (in the sense that $\omega \rightarrow 0$ when $k \rightarrow 0$), and that the dispersion relations of these modes precisely match the dispersion relations of the linearized fluctuations of fluid dynamics (shear and sound modes) provided one chooses the ratio of the viscosity to the entropy density to be $1/4\pi$. These computations gave the first direct hints of the connection between hydrodynamics and gravity.

The connection between hydrodynamics and gravity was established at the nonlinear level as follows [8,28] (see [10,11] for reviews). Starting with the stationary black hole (namely the boosted planar Schwarzschild- AdS_{d+1}) solution, it is possible to build solutions of Einstein's equations dual to slowly varying temperature and velocity configurations on the boundary. We start with an ansatz that describes the patching together of tubes of black holes with slightly different temperatures and boosts in a smooth way. It turns out that Einstein's constraint equations constrain the velocity and temperature fields to obey an equation; this equation turns out to be the equation of perfect boundary fluid dynamics. In order to obtain a complete solution of Einstein's equations order by order in a derivative expansion, we are forced to correct our starting ansatz. It turns out to be possible to explicitly implement this process order by order in a derivative expansion (see [8] for the original reference). This process generates corrections to the constraint equations, order by order in derivatives, effectively generating the constituent relations order by order in this expansion. In the simplest context of the fluid of $\mathcal{N} = 4$ Yang-Mills theory all the steps outlined above have been implemented in a completely explicit manner. The end result is a one-to-one map between long wavelength solutions of negative cosmological constant Einstein-Maxwell gravity and the equations of fluid dynamics in one dimension lower. As we have emphasized, this map is completely explicit; given any solution to the equations of hydrodynamics with the appropriate constitutive relations, the fluid gravity map produces an explicit corresponding solution of Einstein's equations.

One may regard the construction as formulation of the 'Chiral Lagrangian' for black hole horizons. The presence of the black hole respecting thermal boundary conditions breaks the symmetries of the vacuum geometry (for conformal systems we describe these would be dilatations and boosts). The fluid/gravity construction effectively promotes these parameters to 'Goldstone fields' (or perhaps more accurately collective coordinate fields) and determines the effective dynamics of these collective coordinate fields, order by order in the derivative expansion, but making no assumption about amplitudes. As reviewed above, the constitutive relations obtained using this method, have thrown up some surprises that have resulted in a significantly improved understanding of the 'framework' of hydrodynamics.

3. Scattering amplitudes from AdS/CFT

In this section we review the applications of the AdS/CFT correspondence to the computation of gluon scattering amplitudes.

The computation of gluon scattering amplitudes in QCD is of practical importance for the precise characterization of Standard Model processes and backgrounds in collider experiments like LHC. Although the computation of these amplitudes is a finite task at any order in the QCD loop expansion, the complexity of this task grows exponentially

with the number of external gluons and with loop order using Feynman diagrams. Even summing all Feynman diagrams for the six-gluon tree amplitude is a formidable task; better methods of computation are clearly called for.

Over the last twenty years several different groups have developed onshell techniques to compute perturbation amplitudes (see for instance [29] for a review). These unitarity-based methods yield recursion relations that relate higher point to lower point amplitudes, or higher loop to lower loop amplitudes. These methods are much more efficient than the simple Feynman diagram expansion, and permit the evaluation of scattering processes that would be practically impossible to evaluate using Feynman diagrams.

Gluon scattering amplitudes in $\mathcal{N} = 4$ Yang–Mills theory are equal to the same amplitudes in pure Yang–Mills theory at tree level. At loop level the $\mathcal{N} = 4$ amplitudes are very nontrivial; nonetheless they are considerably simpler than the equivalent amplitudes in pure Yang–Mills theory. For this reason $\mathcal{N} = 4$ Yang–Mills theory has proved to be a natural playground for the development of onshell techniques for the computation of Feynman diagrams (these techniques have already been successfully applied to the computation of nonsupersymmetric amplitudes relevant for LHC). For instance, the BCFW recursion relations [30] that allow the construction of higher point scattering amplitudes in terms of lower point amplitudes had their genesis in the study of a conjectured twistor space string theory reformulation of $\mathcal{N} = 4$ Yang–Mills theory [31].

The AdS/CFT correspondence has played a key role in uncovering structural features of scattering amplitudes in the $\mathcal{N} = 4$ Yang–Mills theory. It was argued by Alday and Maldacena [32] that the AdS/CFT correspondence relates gluon scattering amplitudes to a geometrical minimal surface problem at strong coupling. This minimal surface problem enjoys a new symmetry – called dual conformal invariance – that is not manifest in the field theory. The string theory construction of scattering amplitudes may be used to show that dual conformal invariance continues to hold even at weak coupling [33] though only at infinite N . Dual conformal invariance for gluon scattering amplitudes in planar $\mathcal{N} = 4$ Yang–Mills theory is thus a prediction of the AdS/CFT correspondence.

Dual conformal invariance had in fact been noticed in perturbation computations at low orders in the loop expansion even before the AdS/CFT prediction [34,35]. It has since been extensively verified in perturbation computations. This invariance plays a key role in organizing the structure of allowed terms in the perturbation expansion; for instance it implies that 4- and 5-point gluon scattering amplitudes in $\mathcal{N} = 4$ Yang–Mills are constrained to take the BDS form [36]. It follows that these amplitudes are completely determined – including momentum and helicity dependence – by four unknown functions of λ . Six and higher point scattering amplitudes are less constrained [36a], but also end up taking a remarkably simple form [37]. Moreover the AdS/CFT correspondence supplies explicit – and simple – answers for scattering amplitudes at strong coupling (see e.g. [38]).

Given all that is already known about $\mathcal{N} = 4$ scattering amplitudes – in part from the AdS/CFT correspondence – it does not seem implausible that these amplitudes will one day be computed exactly as a function of λ at least for small numbers of scattering gluons (see e.g. [39] for related work). Advances in our understanding of the structure of scattering amplitudes in $\mathcal{N} = 4$ Yang–Mills have already gone hand in hand with significant advances in practical methods of computation that apply to QCD. It may be possible to obtain exact results for scattering amplitudes in the $\mathcal{N} = 4$ theory at all values

of the couplings. This will surely have an important application to understand the structure of gluon scattering amplitudes in QCD.

4. Compressible matter from AdS/CFT

In condensed matter physics, a phase of matter is said to be ‘incompressible’ at zero temperature if the charge of the system remains unchanged when the corresponding chemical potential is increased infinitesimally. A band structure insulator is an example of an incompressible system. As the chemical potential of an insulator lies in the middle of a band gap, an infinitesimal increase of the chemical potential does not modify any electronic occupation numbers and so the charge of the system.

A system is compressible if it is not incompressible. A simple example of a compressible system is a Fermi sea; in such a system an increase in the chemical potential increases the number of occupied Fermi states and so increases the system charge. Superfluids supply another class of examples of compressible fluids; one in which the $U(1)$ symmetry is spontaneously broken. Quite remarkably, these two examples are the only two theoretically well understood compressible phases of matter in condensed matter physics. Moreover, the Fermi liquid is the only well understood compressible phase of matter with an unbroken $U(1)$ symmetry.

Nature, however, abounds with experimental systems – most notably the so-called ‘strange metals’ – that are compressible but cannot be described by Fermi liquid theory (see e.g. [40] for some details and references). The theoretical description of these systems is an outstanding puzzle for condensed matter physicists. It is thus interesting that the AdS/CFT correspondence also provides many examples of non-Fermi liquid compressible field theories with unbroken $U(1)$ symmetry. As we have explained above, within the AdS/CFT correspondence, the gravitational description of a finite temperature and finite chemical potential system is a homogeneous charged black brane. Systems at finite chemical potential but zero temperature are described by ‘extremal’ branes. The near horizon geometry of extremal branes changes qualitatively depending on the details of the bulk gravitational equations (see the reviews [40–42] and papers [43–46] for examples) (see also [47,48] and references therein for compressible phases that preserve different symmetries). In all the examples the chemical potential of such branes is a continuous function of their charge. It follows that extremal branes are dual to compressible states of matter.

The gravitational description of extremal black branes is completely explicit, and permits explicit computations; using these results it is often easy to verify that the correlators of the system are often not those of a standard Fermi liquid. In other words, the AdS/CFT correspondence provides several examples of compressible ‘non-Fermi liquids’. The low-energy properties of these systems is governed by universal aspects of the ‘near horizon geometry’ of extremal black branes, and can be computed quite robustly. The AdS/CFT correspondence has thus provided a large number of examples of compressible states of matter in which analytic computations are possible. Given that traditional condensed matter physics techniques yield no examples of such systems, this sounds like progress.

Of course the problem of real interest to a condensed matter physicist is the description of the particular non-Fermi liquids that appear in experimental systems like strange

metals. It still remains to be seen whether the developments reviewed above prove to be a useful stepping stone towards that goal.

The AdS/CFT correspondence also supplies several examples of compressible phases of matter with a broken $U(1)$ symmetry. The dual description of these systems is a charged black brane in the presence of a bulk charged scalar condensate [48a]. At finite temperature the low energy degrees of freedom of the corresponding solutions are simply the traditional excitations of superfluidity. However, the solutions in question possess new low energy degrees of freedom at zero temperature (these degrees of freedom are a consequence of a throat in the near horizon geometry of the solution [50]). It follows that these holographic phases are not traditional superfluids (whose only low energy degrees of freedom are the Goldstone modes at zero temperature). It is possible that these new phases describe a qualitatively new broken symmetry phase of matter that will one day turn up in a laboratory experiment [50a].

5. Toy models for QCD

5.1 Dynamics at high temperatures

$\mathcal{N} = 4$ Yang–Mills theory is qualitatively different from QCD at low temperatures. While the $\mathcal{N} = 4$ theory is conformal, QCD confines and has a mass gap. Upon heating up, however, QCD undergoes a deconfinement phase transition. The nonconfining dynamics of QCD in the high-temperature phase no longer qualitatively differs from that of hot $\mathcal{N} = 4$ Yang–Mills. These observations suggest that strongly coupled $\mathcal{N} = 4$ can perhaps provide qualitative guidance for high-temperature QCD phenomena. Several computations in the AdS dual of (usually hot) $\mathcal{N} = 4$ Yang–Mills have been performed with this philosophy in mind.

The strong coupling version of the Linhard formula for the energy loss of an accelerating quark at zero temperature was computed in a remarkable paper by Mikhialov [51]; the energy loss formula has the same dependence on the path of the quarks at strong and weak coupling but differs by an overall factor. This result was later generalized to arbitrary coupling (and the coefficient determined as a function of λ) in [52].

The drag force on quarks in hot $\mathcal{N} = 4$ Yang–Mills, as well as various Wilson loops in the hot theory (that characterize the motion of quarks in this system) have been computed. Several attempts have been made to relate these computations to the observed phenomenology of jet quenching in AdS/CFT. As reviewed in [53,54], these attempts have met with some successes, but have also encountered some obstacles.

The results of the RHIC experiment indicate that hydrodynamics takes over as the effective description of the fireball formed by heavy-ion collisions very soon (at a time of order one Fermi) after the collision. This suggests that the system ‘thermalizes’ extremely rapidly. Gedanken experiments in $\mathcal{N} = 4$ Yang–Mills theory also display similar behaviour; hydrodynamics takes over as the effective description more rapidly than might naively have been estimated (for reasons having to do with the causal structure of the bulk dual description of the system) (see [55–57] and references therein for early work; see [58] and references therein for more recent work). It is possible that

further study of such Gedanken experiments can help better understanding of the apparently rapid thermalization in QCD.

5.2 Hadron physics

Clearly the conformal $\mathcal{N} = 4$ Yang–Mills cannot be used to model hadronic physics of QCD. Nonetheless it is possible to obtain a plausible AdS/CFT model of hadronic physics as we now explain.

Consider the five-dimensional Yang–Mills theory compactified on a Scherk Schwarz circle, i.e. a circle around which fermions/bosons have antiperiodic/periodic boundary conditions (these boundary conditions are allowed for fermions as all physical observables are constructed in terms of fermion bilinears, and so are periodic). Now let us study the low-energy effective theory of this system at long distances compared to the compactification circle scale. The boundary conditions ensure that all fermions acquire a tree-level mass. This fact destroys supersymmetric cancellations, and all bosons acquire masses at one loop. It follows that the only ‘massless’ fields in the theory, at long distances compared to the compactification scale, are the four-dimensional gauge fields. In other words, maximally supersymmetric five-dimensional Yang–Mills theory compactified on a Scherk Schwarz circle reduces to four-dimensional pure Yang–Mills theory coupled to towers of massive fields. The dynamics of the resultant system depends on the ratio of the mass gap m of the additional fields to the four-dimensional Λ_{QCD} . This ratio, $r = (m/\Lambda_{\text{QCD}})$ is a free parameter that may be constructed out of the (dimensionful) Yang–Mills coupling of the five-dimensional theory and the radius of compactification. When $r \gg 1$ the dynamics of the resultant system is identical to that of four-dimensional pure Yang–Mills theory at energies much smaller than m . On the other hand, when r is of order unity or less, the dynamics of this system differs from that of the pure Yang–Mills theory at energies of order the Yang–Mills mass gap; in this case the detailed dynamics does not approximate that of pure Yang–Mills theory in any energy regime.

Just as maximally supersymmetric Yang–Mills theory in four dimensions has a conjectured string dual description, a particular UV completion of five-dimensional maximally supersymmetric Yang–Mills theory has a conjectured string dual description. Performing a Scherk Schwarz compactification of both sides of this duality yields dual description of maximally supersymmetric five-dimensional Yang–Mills theory on a circle [59]. It turns out, however, that when $r \gg 1$ the dual string description is highly curved. In the opposite limit $r \ll 1$, however, the dual string description is weakly curved, and the low-energy equations of this system reduce to those of supergravity on an appropriate background manifold. Thus while we have an in-principle string dual description of the system at all values of r , we have a useful computable description of this system only for small r ; i.e. in a parameter regime in which the system differs significantly from pure Yang–Mills theory.

The bulk dual description of pure Yang–Mills theory described above is easily upgraded to a description of a one-parameter set of large N QCD-like theories by adding three D8 branes (and three corresponding anti-D8 branes) [60]. As emphasized above, with the current state of string world sheet technology this description is computationally useful only in the large r limit where the detailed dynamics of the system differs significantly

from that of large N QCD. Even at large r , however, the behaviour of this theory (as extracted from gravitational computations) is qualitatively similar to standard expectations for QCD. The theory has a mass gap; its excitations are gluballs and mesons. Upon heating up, the theory undergoes a deconfinement transition followed by a phase transition for chiral symmetry restoration [60a].

The qualitative similarity of the large r system to real QCD has led to the proposal that the large r gravitational model – or similar models with some couplings allowed to vary as fitting parameters – be treated as an effective phenomenological model for QCD, somewhat in the spirit of the MIT bag model. Models obtained with this philosophy have worked surprisingly well at reproducing experimental data on the masses of hadronic states, and may give qualitative guidance for other aspects of hadronic physics (see [63] for a review and references). Of course if one can solve the dual string theory at small r we would have the bulk dual of actual large N QCD not just a one-parameter deformation away from it; however the required solution of the world sheet sigma model appears to be a difficult problem whose solution may not lie just around the corner.

6. Application to diverse areas of mathematical physics

In this section I very briefly survey the large impact that the AdS/CFT correspondence has had on diverse areas of mathematical physics.

To start with, the attempts to understand results obtained from this correspondence have stimulated extensive investigations of $\mathcal{N} = 4$ Yang–Mills theory as a field theory. It has been established that the spectrum of operators of this theory, as a function of the coupling constant λ , is given by the spectrum of an integrable spin chain. A large effort by several groups over an eight-year period has led to the formulation of a set of algebraic ‘Y system’ equations whose solution plausibly yields the scaling dimension of all operators of this theory as a function of the coupling constant λ (see [64] for a review with references). The dependence of these scaling dimensions on the coupling λ is intricate and has several points of interest. If the proposed equations are indeed correct then large N $\mathcal{N} = 4$ Yang–Mills theory has been ‘solved’ as far as the determination of the spectrum of operators of the theory goes. As $\mathcal{N} = 4$ Yang–Mills is a complicated, interacting, four-dimensional field theory, this solution is an impressive accomplishment in mathematical physics. This solution used crucial inputs from the dual bulk description of the theory, and would have been impossible without the AdS/CFT correspondence.

Other exact results in $\mathcal{N} = 4$ Yang–Mills theory – obtained this time by using the method of supersymmetric localization – include exact results for the expectation values of Wilson and t’Hooft loop operators as a function of λ (see e.g. [65,66]).

Studies motivated by the AdS/CFT $\mathcal{N} = 6$ Chern–Simons matter theories and IIA string theory on $AdS_4 \times CP^3$ [67] have dramatically improved our understanding of the three-dimensional supersymmetric Chern–Simons matter theories [68]. Superconformal Chern–Simons matter theories are much more plentiful than their four-dimensional counterparts. The methods of supersymmetric localization have been used to obtain many exact results in these theories, including results for their partition function on a 3-sphere (see e.g. [69]), results for the renormalization of the scaling dimension of chiral operators

as a function of the Chern–Simons coupling [70,71], the superconformal index [72,73] and conjectures for the three-dimensional counterpart of Cardy’s famous two-dimensional c function that always decreases under RG flows [70].

On a distinct but related note, the AdS/CFT correspondence has uncovered a tight relationship between Vasiliev’s esoteric looking equations describing the interactions of higher spin particles in AdS_4 [74], (see [75] for a similar story in AdS_3), and the physics of large N Chern–Simons coupled gauge theories with fundamental matter or bifundamental matter fields (see [76] for a review). Chern–Simons coupled fundamental matter are effectively solvable in the large N limit [77–79]. The intensive study of these theories and their bulk duals has led to the conjecture of a new qualitatively new bosonization of Chern–Simons coupled fermionic theories in three dimensions [80]. It has also led to the proof of a sort of Coleman–Mandula-type theorem for conformal field theories with higher spin symmetries [81,82]. These studies have also demonstrated that the fundamental string of string theory sometimes breaks up into more fundamental constituents [83] that obey Vasiliev’s higher spin equations.

The last few years has also seen an impressive advance in our precision understanding of the bulk description of supersymmetric black hole entropy. In particular, the AdS/CFT correspondence has been used to conjecture an exact version of the Hawking–Beckenstein formula for black hole entropy (applied to extremal supersymmetric black holes) [84]. This formula has passed very stringent tests and may be precise enough to allow us to pull out integer valued ground state degeneracies from gravitational path integrals for supersymmetric solutions (see [85] for a review).

In another direction, the AdS/CFT duality has been used to geometrize strongly coupled dynamics in theories of technicolour, in order to permit reliable computations in these systems (see e.g. [86–88]).

The concept of entanglement entropy has begun to play an increasingly important role in several areas of theoretical physics. Within the AdS/CFT correspondence there exists a remarkable conjecture for a bulk formula for the entanglement entropy of a region in the boundary theory (see [89] for a review and references). This prescription is given in terms of minimal surfaces in AdS spaces, and has passed several consistency checks. If correct, this conjecture supplies obvious intuitive proofs for several important (but hard to directly establish) properties of entanglement entropy (for instance the property of strong subadditivity) [90], and can prove to be an important source of intuition for the better understanding of entanglement entropy in general. It is also possible that the notion of entanglement entropy has deep connection with the emergence of space–time in AdS/CFT, and that it may eventually play a fundamental role in the formulation of quantum theories of gravity [91].

Recently, ideas from the AdS/CFT correspondence have been used to obtain general constraints on the nature of the non-Gaussianity of the microwave background generated during the (conjectured) approximately De Sitter inflationary expanding phase of the history of the Universe (see e.g. [92]).

From a completely different angle, the AdS/CFT correspondence has provided perhaps the first set of completely nonperturbative definition of any theory of quantum gravity. The existence of such a definition throws light on several conceptual questions in gravity. For instance, it appears to establish that the formation of a black hole by graviton scattering and its subsequent decay by Hawking radiation is a purely unitary process (see e.g.

[93] for a concrete set-up for this experiment), answering a question raised long ago by Hawking.

7. Concluding remarks

In conclusion, the AdS/CFT correspondence has already had applications in an astoundingly wide range of problems in physics, including the theory of hydrodynamics, scattering amplitude physics, the search for new phases of matter in condensed matter physics, the construction of toy models of aspects of QCD physics, and several areas of mathematical physics. Research on this subject continues at a frenetic pace, interesting new results appearing every month; it seems likely that the most interesting applications of these ideas lie in the future.

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