PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 79, No. 3
September 2012
pp. 393–402

Exact solutions of some coupled nonlinear diffusion-reaction equations using auxiliary equation method

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MS received 1 January 2012; revised 29 February 2012; accepted 10 May 2012

Abstract. Travelling and solitary wave solutions of certain coupled nonlinear diffusion-reaction equations have been constructed using the auxiliary equation method. These equations arise in a variety of contexts not only in biological, chemical and physical sciences but also in ecological and social sciences.

Keywords. Nonlinear diffusion equation; auxiliary equation method; solitary wave solution.

PACS Nos 05.45.Yv; 02.30.Ik; 02.30.Jr

1. Introduction

Lotka-Volterra model and its variants are used to model a large variety of prey–predator problems [1]. Interestingly, the same set of equations are also used to model the population flow between urban and rural areas mainly on the basis of analogy [2]. A fundamental characteristic of this model is that the two populations interact in a nonlinear fashion resulting in a pair of coupled partial differential equations. If the analytic solution of such equations becomes available, then the dependence of the solution on the parameter involved can be studied in a rather more transparent manner. While such a system has been of great interest for more than 80 years now, its modified version consisting of diffusion terms has been studied only empirically. In the present work, we shall investigate certain coupled diffusion-reaction (D-R) equations of very general nature.

In recent years, various direct methods have been proposed to find the exact solutions not only of nonlinear partial differential equations but also of their coupled versions. These methods include unified ansatz approach [3], extended hyperbolic function method [4], (G'/G)-expansion method [5], generalized (G'/G)-expansion method [6], generalized hyperbolic function method [7], variational iteration method [8,9], exponential function method [10,11], auxiliary equation method [12–14], generalized auxiliary equation method [15,16], and so on. Here, we plan to employ the auxiliary equation method to obtain an exact solution of the following coupled D-R equations [17]:

$$\begin{aligned} u_t - c_1 u_x &= D_1 u_{xx} + \alpha u - \beta u^2 - \gamma u v, \\ v_t - c_2 v_x &= D_2 v_{xx} - \mu v + \chi u v \end{aligned} \tag{1}$$

and

$$\begin{aligned} u_t - c_1 u_x &= D_1 u v_{xx} + l_1 u^2 v, \\ v_t - c_2 v_x &= l_2 u v^2 - D_2 v u_{xx}. \end{aligned} \tag{2}$$

These pair of equations describe the cases when both the predator and the prey disperse by diffusion. In particular, eqs (2) arise when the diffusion coefficient becomes density-dependent, and the same is indicated by the presence of $D_1 u$ and $D_2 v$ coefficients in the diffusion terms. In the above equations u and v respectively represent the populations of the prey and the predator; α, β, γ, μ, χ, l_1 and l_2 are positive constants; D_1 and D_2 are diffusion coefficients and c_1 and c_2 are convective velocities of the prey and the predator. We first transform the pairs of partial differential equations (1) and (2) into the following coupled total differential equations by defining a variable $\xi = x - wt$, viz.,

$$\begin{aligned} &(c_1 + w)u' - D_1 u'' - \alpha u + \beta u^2 + \gamma u v = 0, \\ &(c_2 + w)v' - D_2 v'' + \mu v - \chi u v = 0 \end{aligned} \tag{3}$$

and

$$\begin{aligned} &(c_1 + w)u' - D_1 u v'' - l_1 u^2 v = 0, \\ &(c_2 + w)v' + D_2 v u'' - l_2 v^2 u = 0, \end{aligned} \tag{4}$$

and then look for solutions of these equations by making the ansatz [12]

$$\begin{aligned} u(\xi) &= \sum_{i=0}^{M} a_i z^i, \\ v(\xi) &= \sum_{i=0}^{N} b_i z^i, \end{aligned} \tag{5}$$

where a_i and b_i are real constants to be determined, M and N are positive integers which can be determined by balancing the highest order derivative term with the highest order nonlinear term in these equations and $z(\xi)$ satisfies the following auxiliary equation:

$$\left(\frac{\mathrm{d}z}{\mathrm{d}\xi}\right)^2 = A z^4(\xi) + B z^3(\xi) + C z^2(\xi) + D, \tag{6}$$

where A, B, C and D are real arbitrary constants to be determined later.

2. Exact solution of eqs (1)

Note that for eq. (1), the balancing procedure immediately leads to $M = N = 2$. This suggests the choice of $u(\xi)$ and $v(\xi)$ in eq. (5) as

$$\begin{aligned} u(\xi) &= a_0 + a_1 z(\xi) + a_2 z^2(\xi), \\ v(\xi) &= b_0 + b_1 z(\xi) + b_2 z^2(\xi), \end{aligned} \tag{7}$$

where the constants a_0, a_1, a_2, b_0, b_1 and b_2 are yet to be determined. Substituting (7) along with (6) in eq. (3) and then setting the coefficients of $z^j(\xi)$ $(j = 0, 1, ..., 4)$, $z'(\xi)$ and $z(\xi)z'(\xi)$ to zero in the resultant expression, one obtains a set of algebraic equations involving a_0, a_1, a_2, b_0, b_1, b_2, w, A, B, C and D as

$$
\begin{aligned}
&(w + c_1)\, a_1 = 0,\\
&2\,(w + c_1)\, a_2 = 0,\\
&2DD_1a_2 + \alpha a_0 - \beta a_0^2 - \gamma a_0 b_0 = 0,\\
&D_1 a_1 C + \alpha a_1 - 2\beta a_0 a_1 - \gamma a_0 b_1 - \gamma a_1 b_0 = 0,\\
&\frac{3}{2} B D_1 a_1 + 4C D_1 a_2 + \alpha a_2 - \beta a_1^2 - 2\beta a_0 a_2 - \gamma a_0 b_2 - \gamma a_1 b_1 - \gamma a_2 b_0 = 0,\\
&2AD_1 a_1 + 5BD_1 a_2 - 2\beta a_1 a_2 - \gamma a_1 b_2 - \gamma a_2 b_1 = 0,\\
&6AD_1 a_2 - \beta a_2^2 - \gamma a_2 b_2 = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&(w + c_2)\, b_1 = 0,\\
&2\,(w + c_2)\, b_2 = 0,\\
&2Db_2D_2 + \chi a_0 b_0 - \mu b_0 = 0,\\
&D_2 b_1 C + \chi a_0 b_1 + \chi a_1 b_0 - \mu b_1 = 0,\\
&\frac{3}{2} BD_2 b_1 + 4CD_2 b_2 + \chi a_0 b_2 + \chi a_1 b_1 + \chi a_2 b_0 - \mu b_2 = 0,\\
&2AD_2 b_1 + 5BD_2 b_2 + \chi a_1 b_2 + \chi a_2 b_1 = 0,\\
&6AD_2 b_2 + \chi a_2 b_2 = 0.
\end{aligned}
\tag{8}
$$

We solve the above set of algebraic equations for $B = b_2 = 0$ and one obtains

$$
\begin{aligned}
&a_0 = \frac{\mu}{\chi}, \quad a_1 = \pm\sqrt{\frac{6aD_2}{5\chi\beta}\left(\frac{\beta\mu}{\chi} + \frac{4ADD_1D_2}{\mu} - \alpha\right)},\\
&a_2 = \frac{6aD_1}{\beta}, \quad b_0 = \frac{1}{\gamma}\left(\alpha - \frac{\beta\mu}{\chi} + \frac{12ADD_1^2\chi}{\beta\mu}\right), \quad D_2\beta = -3D_1\chi,\\
&b_1 = \mp\sqrt{\frac{10\beta AD_2}{3\gamma^2\chi}\left(\frac{\beta\mu}{\chi} + \frac{4ADD_1D_2}{\mu} - \alpha\right)},\\
&C = \frac{2\mu^2 - 4ADD_2^2}{D_2\mu}, \quad D = \mu\left(\frac{\alpha}{24AD_1D_2} + \frac{13\mu}{24AD_2^2}\right),\\
&w = -c_1 = -c_2,
\end{aligned}
\tag{9}
$$

along with a constraining relation $D_2\alpha = 29\mu D_1$. In view of this constraining relation, the above values of a_0, b_0, a_1, b_1 and C now take the forms

$$a_0 = -\frac{3\alpha}{29\beta}, \quad b_0 = \frac{25\alpha}{29\gamma}, \quad a_1 = \pm\sqrt{\frac{90A\alpha^2 D_2}{841\mu\beta^2}},$$
$$b_1 = \mp\sqrt{\frac{250A\alpha^2 D_2}{841\mu\gamma^2}},$$
$$C = -\frac{20\alpha}{87D_1}.$$

In what follows we discuss some special cases for certain choices of the unknowns A, C and D in eq. (6).

Case 1a: Let us take $A = m^2$, $C = -(1+m^2)$ and $D = 1$, where $0 < m^2 < 1$, in eq. (6). Then the solution of (6) turns out to be [18] $z(\xi) = \mathrm{sn}(\xi)$, which, from (7), leads to the solution of (3) as

$$\begin{aligned} u(\xi) &= \frac{D_1}{\beta}\left(-\frac{9(1+m^2)}{20} \pm \sqrt{\frac{27m^2(1+m^2)}{2}}\mathrm{sn}(\xi) + 6m^2\mathrm{sn}^2(\xi)\right), \\ v(\xi) &= \frac{D_1}{\gamma}\left(\frac{75(1+m^2)}{20} \mp \sqrt{\frac{75m^2(1+m^2)}{2}}\mathrm{sn}(\xi)\right), \end{aligned} \tag{10}$$

which is a periodic wave solution of eq. (3). In the limit when $m \to 1$, $\mathrm{sn}(\xi) \to \tanh(\xi)$, the solitary wave solutions of eq. (3) become

$$u(\xi) = \frac{D_1}{\beta}\left(-\frac{9}{10} \pm \sqrt{27}\tanh(\xi) + 6\tanh^2(\xi)\right)$$

and

$$v(\xi) = \frac{D_1}{\gamma}\left(\frac{75}{10} \mp \sqrt{75}\tanh(\xi)\right).$$

Case 1b: If $A = -m^2$, $C = 2m^2 - 1$ and $D = 1 - m^2$, then the solution of (6) becomes [18], $z(\xi) = \mathrm{cn}(\xi)$. Thus, from (7), we have

$$\begin{aligned} u(\xi) &= \frac{D_1}{\beta}\left(\frac{9(2m^2-1)}{20} \pm \sqrt{\frac{27m^2(2m^2-1)}{2}}\mathrm{cn}(\xi) - 6m^2\mathrm{cn}^2(\xi)\right), \\ v(\xi) &= \frac{D_1}{\gamma}\left(-\frac{75(2m^2-1)}{20} \mp \sqrt{\frac{75m^2(2m^2-1)}{2}}\mathrm{cn}(\xi)\right). \end{aligned} \tag{11}$$

When $m \to 1$, leading to $\mathrm{cn}(\xi) \to \mathrm{sech}(\xi)$, the solitary wave solutions of eq. (3) become

$$u(\xi) = \frac{D_1}{\beta}\left(\frac{9}{20} \pm \sqrt{\frac{27}{2}}\,\mathrm{sech}(\xi) - 6\,\mathrm{sech}^2(\xi)\right)$$

and

$$v(\xi) = \frac{D_1}{\gamma}\left(\frac{-75}{20} \mp \sqrt{\frac{75}{2}}\,\mathrm{sech}(\xi)\right).$$

Case 1c: If $A = -1$, $C = 2 - m^2$ and $D = m^2 - 1$, then one finds [18] $z(\xi) = \mathrm{dn}(\xi)$, and, from (7), we have

$$\begin{aligned} u(\xi) &= \frac{D_1}{\beta}\left(\frac{9(2-m^2)}{20} \pm \sqrt{\frac{27(2-m^2)}{2}}\mathrm{dn}(\xi) - 6\,\mathrm{dn}^2(\xi)\right), \\ v(\xi) &= \frac{D_1}{\gamma}\left(-\frac{75(2-m^2)}{20} \mp \sqrt{\frac{75(2-m^2)}{2}}\mathrm{dn}(\xi)\right). \end{aligned} \tag{12}$$

Here, when $m \to 1$, then $\mathrm{dn}(\xi) \to \mathrm{sech}(\xi)$, and the solitary wave solution of eq. (3) is given by

$$u(\xi) = \frac{D_1}{\beta}\left(\frac{9}{20} \pm \sqrt{\frac{27}{2}}\mathrm{sech}(\xi) - 6\,\mathrm{sech}^2(\xi)\right)$$

and

$$v(\xi) = \frac{D_1}{\gamma}\left(\frac{-75}{20} \mp \sqrt{\frac{75}{2}}\,\mathrm{sech}(\xi)\right).$$

Case 1d: If $A = 1$, $C = 2 - m^2$ and $D = 1 - m^2$, then [18] $z(\xi) = \mathrm{cn}(\xi)/\mathrm{sn}(\xi)$, and, from (7), we have

$$\begin{aligned} u(\xi) &= \frac{D_1}{\beta}\left(\frac{9(2-m^2)}{20} \pm \sqrt{\frac{27(m^2-2)}{2}}\,\frac{\mathrm{cn}(\xi)}{\mathrm{sn}(\xi)} + 6\left(\frac{\mathrm{cn}(\xi)}{\mathrm{sn}(\xi)}\right)^2\right), \\ v(\xi) &= \frac{D_1}{\gamma}\left(-\frac{75(2-m^2)}{20} \mp \sqrt{\frac{75(m^2-2)}{2}}\,\frac{\mathrm{cn}(\xi)}{\mathrm{sn}(\xi)}\right). \end{aligned} \tag{13}$$

As before, when $m \to 1$, then $(\mathrm{cn}(\xi)/\mathrm{sn}(\xi)) \to \cosh(\xi)$, and the solutions are given by

$$u(\xi) = \frac{D_1}{\beta}\left(\frac{9}{20} \pm \iota\sqrt{\frac{27}{2}}\,\cosh(\xi) + 6\cosh^2(\xi)\right)$$

and

$$v(\xi) = \frac{D_1}{\gamma}\left(-\frac{75}{20} \mp \iota\sqrt{\frac{75}{2}}\,\cosh(\xi)\right).$$

Note that $u(\xi)$ and $v(\xi)$ become imaginary for this case.

Case 1e: If $A = 1$, $C = -(1+m^2)$ and $D = m^2$, then [18] $z(\xi) = \mathrm{dn}(\xi)/\mathrm{cn}(\xi)$, and from (7), we have

$$\begin{aligned} u(\xi) &= \frac{D_1}{\beta}\left(-\frac{9(1+m^2)}{20} \pm \sqrt{\frac{27(1+m^2)}{2}}\frac{\mathrm{dn}(\xi)}{\mathrm{cn}(\xi)} + 6\left(\frac{\mathrm{dn}(\xi)}{\mathrm{cn}(\xi)}\right)^2\right), \\ v(\xi) &= \frac{D_1}{\gamma}\left(\frac{75(1+m^2)}{20} \mp \sqrt{\frac{75(1+m^2)}{2}}\frac{\mathrm{dn}(\xi)}{\mathrm{cn}(\xi)}\right), \end{aligned} \tag{14}$$

which represents a divergent solution of eq. (3).

Case 1f: If $A = 1$, $C = 2m^2 - 1$ and $D = -m^2(1-m^2)$, then [18] $z(\xi) = \mathrm{dn}(\xi)/\mathrm{sn}(\xi)$, and from (7), we have

$$\begin{aligned} u(\xi) &= \frac{D_1}{\beta}\left(\frac{9(2m^2-1)}{20} \pm \sqrt{\frac{27(1-2m^2)}{2}}\frac{\mathrm{dn}(\xi)}{\mathrm{sn}(\xi)} + 6\left(\frac{\mathrm{dn}(\xi)}{\mathrm{sn}(\xi)}\right)^2\right), \\ v(\xi) &= \frac{D_1}{\gamma}\left(\frac{75(1-2m^2)}{20} \mp \sqrt{\frac{75(1-2m^2)}{2}}\frac{\mathrm{dn}(\xi)}{\mathrm{sn}(\xi)}\right), \end{aligned} \tag{15}$$

which again represents a divergent solution of eq. (3).

3. Exact solution of eqs (2)

If one applies the balancing procedure to eq. (4), then one obtains $M = N = 2$, which in turn leads to the choices of $u(\xi)$ and $v(\xi)$ as

$$\begin{aligned} u(\xi) &= a_0 + a_1 z(\xi) + a_2 z^2(\xi), \\ v(\xi) &= b_0 + b_1 z(\xi) + b_2 z^2(\xi). \end{aligned} \tag{16}$$

As before, using (16) and (6) in eq. (4) and then setting the coefficients of $z^j(\xi)$ ($j = 0, 1, ..., 6$), $z'(\xi)$ and $z(\xi)z'(\xi)$ equal to zero, one obtains the following set of algebraic equations for the unknowns, namely a_0, a_1, a_2, b_0, b_1, b_2, w, A, B, C and D as

$$\begin{aligned} &(w + c_1)\, a_1 = 0, \\ &2\,(w + c_1)\, a_2 = 0, \\ &l_1 a_0^2 b_0 + 2b_2 D_1 D a_0 = 0, \\ &l_1 a_0^2 b_1 + 2l_1 a_0 a_1 b_0 + D_1 b_1 C a_0 + 2b_2 D_1 D a_1 = 0, \\ &l_1 a_0^2 b_2 + l_1 a_1^2 b_0 + 2l_1 a_0 a_1 b_1 + 2l_1 a_0 a_2 b_0 \\ &\quad + \frac{3}{2} D_1 b_1 B a_0 + D_1 b_1 C a_1 + 4b_2 D_1 C a_0 + 2b_2 D_1 D a_2 = 0, \end{aligned}$$

$$l_1a_1^2b_1 + 2l_1a_0a_1b_2 + 2l_1a_0a_2b_1 + 2l_1a_1a_2b_0 + 2Aa_0D_1b_1 + \frac{3}{2}Ba_1D_1b_1 + D_1b_1Ca_2 + 5Ba_0b_2D_1 + 4Ca_1b_2D_1 = 0,$$
$$l_1a_1^2b_2 + l_1a_2^2b_0 + 2l_1a_0a_2b_2 + 2l_1a_1a_2b_1 + 2Aa_1D_1b_1 + \frac{3}{2}Ba_2D_1b_1 + 6Aa_0b_2D_1 + 5Ba_1b_2D_1 + 4Ca_2b_2D_1 = 0,$$
$$l_1a_2^2b_1 + 2l_1a_1a_2b_2 + 2Aa_2b_1D_1 + 6Aa_1b_2D_1 + 5Ba_2b_2D_1 = 0,$$
$$l_1a_2^2b_2 + 6b_2D_1Aa_2 = 0,$$

and

$$(w+c_2)\,b_1 = 0,$$
$$2\,(w+c_2)\,b_2 = 0,$$
$$l_2b_0^2a_0 - 2D_2a_2Db_0 = 0,$$
$$l_2b_0^2a_1 + 2l_2b_0b_1a_0 - a_1D_2Cb_0 - 2D_2a_2Db_1 = 0,$$
$$l_2b_0^2a_2 + l_2b_1^2a_0 + 2l_2b_0b_1a_1 + 2l_2b_0b_2a_0 - \frac{3}{2}Bb_0a_1D_2 - a_1D_2Cb_1 - 4Cb_0a_2D_2 - 2D_2a_2Db_2 = 0,$$
$$l_2b_1^2a_1 + 2l_2b_0b_1a_2 + 2l_2b_0b_2a_1 + 2l_2b_1b_2a_0 - 2Ab_0a_1D_2 - \frac{3}{2}Bb_1a_1D_2 - a_1D_2Cb_2 - 5Bb_0a_2D_2 - 4Cb_1a_2D_2 = 0,$$
$$l_2b_1^2a_2 + l_2b_2^2a_0 + 2l_2b_0b_2a_2 + 2l_2b_1b_2a_1 - 2Ab_1a_1D_2 - \frac{3}{2}Bb_2a_1D_2 - 6Ab_0a_2D_2 - 5Bb_1a_2D_2 - 4Cb_2a_2D_2 = 0,$$
$$l_2b_2^2a_1 + 2l_2b_1b_2a_2 - 5Bb_2a_2D_2 - 6Ab_1a_2D_2 - 2Ab_2a_1D_2 = 0,$$
$$l_2b_2^2a_2 - 6Ab_2a_2D_2 = 0. \tag{17}$$

After solving the set of algebraic eqs (17) for $B = 0$ we get,

$$a_0 = -\frac{6ADD_1}{l_1(C \pm \sqrt{C^2 - 3AD})}, \qquad b_0 = \frac{2D_2(C \pm \sqrt{C^2 - 3AD})}{l_2},$$
$$a_1 = b_1 = 0, \quad a_2 = -\frac{6AD_1}{l_1}, \quad b_2 = \frac{6AD_2}{l_2}, \quad w = -c_1 = -c_2.$$

As before, now we discuss the special cases for certain choices of A, C and D in eq. (6).

Case 2a: For $A = m^2$, $C = -(1+m^2)$ and $D = 1$ we get

$$u(\xi) = \frac{6m^2D_1}{l_1}\left(\frac{1}{(1+m^2) \mp \sqrt{1+m^4-m^2}} - \mathrm{sn}^2(\xi)\right),$$
$$v(\xi) = \frac{2D_2}{l_2}(-(1+m^2) \pm \sqrt{1+m^4-m^2} + 3m^2\,\mathrm{sn}^2(\xi)). \tag{18}$$

Note that for $m \to 1$, $\mathrm{sn}(\xi) \to \tanh(\xi)$, one obtains

$$u(\xi) = \frac{6D_1}{l_1}(1 - \tanh^2(\xi)), \qquad v(\xi) = \frac{2D_2}{l_2}(-1 + 3\tanh^2(\xi)),$$

corresponding to the upper sign in (17) and

$$u(\xi) = \frac{2D_1}{l_1}(1 - 3\tanh^2(\xi)), \qquad v(\xi) = \frac{6D_2}{l_2}(-1 + \tanh^2(\xi)),$$

corresponding to the lower sign in (17). Note that eqs (17) while representing periodic wave solutions, the preceding limiting cases in fact are the solitary wave solutions of eq. (4).

Case 2b: For $A = -m^2$, $C = 2m^2 - 1$ and $D = 1 - m^2$ we get

$$\begin{aligned} u(\xi) &= \frac{6m^2 D_1}{l_1}\left(\frac{1 - m^2}{\left(2m^2 - 1\right) \pm \sqrt{1 + m^4 - m^2}} + \mathrm{cn}^2(\xi)\right), \\ v(\xi) &= \frac{2D_2}{l_2}((2m^2 - 1) \pm \sqrt{1 + m^4 - m^2} - 3m^2\mathrm{cn}^2(\xi)). \end{aligned} \tag{19}$$

In general, these are the periodic wave solutions of eq. (4). When $m \to 1$, $\mathrm{cn}(\xi) \to \mathrm{sech}(\xi)$, one obtains only single solution for $u(\xi)$, namely

$$u(\xi) = \frac{6D_1}{l_1}\,\mathrm{sech}^2(\xi).$$

However, corresponding to upper and lower signs in $v(\xi)$ we have

$$v(\xi) = \frac{2D_2}{l_2}(2 - 3\,\mathrm{sech}^2(\xi)) \qquad \text{and} \qquad v(\xi) = -\frac{6D_2}{l_2}\mathrm{sech}^2(\xi),$$

respectively. The latter represents a solitary wave solutions of eq. (4).

Case 2c: For $A = -1$, $C = 2 - m^2$ and $D = m^2 - 1$ we get

$$\begin{aligned} u(\xi) &= \frac{6D_1}{l_1}\left(\frac{m^2 - 1}{(2 - m^2) \pm \sqrt{1 + m^4 - m^2}} + \mathrm{dn}^2(\xi)\right), \\ v(\xi) &= \frac{2D_2}{l_2}((2 - m^2) \pm \sqrt{1 + m^4 - m^2} - 3\,\mathrm{dn}^2(\xi)). \end{aligned} \tag{20}$$

Not only the nature of the above solutions in this case is the same as of case (2b) but also the limiting solutions turn out to be identical.

Case 2d: For $A = 1$, $C = 2 - m^2$ and $D = 1 - m^2$ we get

$$\begin{aligned} u(\xi) &= \frac{6D_1}{l_1}\left(\frac{(m^2 - 1)}{(2 - m^2) \pm \sqrt{1 + m^4 - m^2}} - \left(\frac{\mathrm{cn}(\xi)}{\mathrm{sn}(\xi)}\right)^2\right), \\ v(\xi) &= \frac{2D_2}{l_2}\left((2 - m^2) \pm \sqrt{1 + m^4 - m^2} + 3\left(\frac{\mathrm{cn}(\xi)}{\mathrm{sn}(\xi)}\right)^2\right). \end{aligned} \tag{21}$$

These are the divergent solutions of eq. (4) for large ξ. For $m \to 1$, $(\mathrm{cn}(\xi)/\mathrm{sn}(\xi)) \to \cosh(\xi)$, one obtains

$$u(\xi) = -\frac{6D_1}{l_1}\cosh^2(\xi)$$

and

$$v(\xi) = \frac{2D_2}{l_2}(2 + 3\cosh^2(\xi)) \qquad \text{and} \qquad v(\xi) = \frac{6D_2}{l_2}\cosh^2(\xi),$$

corresponding to the upper and lower signs in $v(\xi)$.

Case 2e: For $A = 1$, $C = -\left(1 + m^2\right)$ and $D = m^2$ we get

$$\begin{aligned} u(\xi) &= \frac{6D_1}{l_1}\left(\frac{m^2}{(1+m^2) \mp \sqrt{1+m^4-m^2}} - \left(\frac{\mathrm{dn}(\xi)}{\mathrm{cn}(\xi)}\right)^2\right), \\ v(\xi) &= -\frac{2D_2}{l_2}\left((1+m^2) \mp \sqrt{1+m^4-m^2} - 3\left(\frac{\mathrm{dn}(\xi)}{\mathrm{cn}(\xi)}\right)^2\right), \end{aligned} \tag{22}$$

which in general represent divergent solutions of eq. (4).

Case 2f: For $A = 1$, $C = 2m^2 - 1$ and $D = -m^2\left(1 - m^2\right)$ we get

$$\begin{aligned} u(\xi) &= \frac{6D_1}{l_1}\left(\frac{m^2(1-m^2)}{\left(2m^2-1\right) \pm \sqrt{1+m^4-m^2}} - \left(\frac{\mathrm{dn}(\xi)}{\mathrm{sn}(\xi)}\right)^2\right), \\ v(\xi) &= \frac{2D_2}{l_2}\left((2m^2-1) \pm \sqrt{1+m^4-m^2} + 3\left(\frac{\mathrm{dn}(\xi)}{\mathrm{sn}(\xi)}\right)^2\right), \end{aligned} \tag{23}$$

which again represent divergent solutions of eq. (4) in general.

4. Concluding remarks

With a view of having a deeper understanding of certain problems of population dynamics, particularly when there exists a coupling in the population densities of different species, the exact solution of two pairs of coupled partial differential equations (see eqs (1) and (2)) which frequently occur [17] in the field, is investigated here. Recently, growth of the cancerous cells have been modelled by nonlinear D-R equation [19,20]. Though all these depend on the modelling of the underlying phenomena, some of the results obtained here can be used for studying the growth of cancerous cells [19]. In our case, the two species in eqs (1) and (2) may represent the concentration of normal and cancerous cells. For example, $u(\xi)$ may represent the concentration of cancerous cell and $v(\xi)$ may represent the concentration of normal cell. Then the limiting case of eqs (10), (11), (12) and (18), (19), (20) represent the growth of normal and cancerous cell where, if the concentration $u(\xi)$ of cancerous cell increases then concentration $v(\xi)$ of normal cell decreases [20].

In particular, a variety of periodic and solitary wave solutions are obtained for different choices of the unknown parameters appearing through the ansatz made for the solutions. Periodic solutions obtained in this paper can be used to explain the population dynamics of certain species in ecology. Periodicity of this kind has been observed in the population of hares and lynx [1]. Divergent solutions obtained in this paper are physically not acceptable.

While the solutions of the pair of eqs (4) involve the diffusion coefficients D_1 and D_2 and the corresponding couplings l_1 and l_2 in a symmetrical manner in all the cases, the same is not the case with the solutions of the pair of eqs (3). Interestingly, the solutions of eqs (3) do involve only the diffusion coefficient D_1 along with the coupling parameters β and γ in all the cases. It appears that the diffusion coefficient D_2 does not play any role as far as solutions of eqs (3) are concerned. The case when the coefficients in (1) and (2) become time-dependent could be of much interest. Such studies are in progress.

Acknowledgements

The author would like to thank R S Kaushal for helpful discussion and the referee for many useful suggestions for improving this paper.

References

[1] C S Bertuglia and F Vaio, *Nonlinearity, chaos and complexity (The Dynamics of Natural and Social Systems)* (Oxford University Press, New York, 2005)
[2] Radhey Shyam Kaushal, *Structural analogy in understanding nature* (Anamaya Publishers, New Delhi, 2003)
[3] S A. Khuri, *Chaos, Solitons and Fractals* **36**, 1181 (2008)
[4] Y Shang, *Chaos, Solitons and Fractals* **36**, 762 (2008)
[5] M L Wang, X Z Li and J L Jhang, *Phys. Lett.* **A372**, 417 (2007)
[6] S Jhang, J L Tong and W Wang, *Phys. Lett.* **A372**, 2254 (2008)
[7] E Yomba, *Phys. Lett.* **A372**, 1612 (2008)
[8] J H He, *Comput. Methods Appl. Mech. Eng.* **167**, 57 (1998)
[9] M A Abdou and A A Soliman, *Physica* **D211**, 1 (2005)
[10] J H He and X -H Wu, *Chaos, Solitons and Fractals* **30**, 700 (2006)
[11] S -D Zhu, *Phys. Lett.* **A372**, 654 (2008)
[12] Sirendaoreji, *Phys. Lett.* **A356**, 124 (2006)
[13] Ranjit Kumar, R S Kaushal and Awadhesh Prasad, *Phys. Lett.* **A372**, 1862 (2008)
[14] Ranjit Kumar, R S Kaushal and Awadhesh Prasad, *Phys. Lett.* **A372**, 3395 (2008)
[15] H Zhao, *Chaos, Solitons and Fractals* **36**, 359 (2008)
H Zhao, *Chaos, Solitons and Fractals* **36**, 1283 (2008)
[16] E Yomba, *Phys. Lett.* **A372**, 1048 (2008)
[17] J D Murray, *Mathematical biology* (Springer-Verlag, New York, 1993)
[18] M L Wang and Y B Zhou, *Phys. Lett.* **A318**, 84 (2003)
[19] M L Martins, S C Ferreira Jr and M J Vilela, *Physiol. Rev.* **4**, 128 (2007)
[20] T Alarcón, H M Byrne and P K Maini, *Multiscale Model. Simul.* **3**, 440 (2005)