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Antisynchronization of a novel hyperchaotic system with parameter mismatch and external disturbances

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Abstract. A novel hyperchaotic system is proposed. It is particularly interesting that the hyperchaotic system has a nonlinear term in the form of an exponential function and has only one equilibrium. Basic dynamical properties of the hyperchaotic system are investigated. Moreover, antisynchronization of the new hyperchaotic system with parameter mismatch and external disturbances is also studied in this paper by using adaptive control. Numerical simulation results further demonstrate that the proposed methods are effective and robust.

Keywords. Hyperchaotic system; exponential function; antisynchronization; parameter mismatch; external disturbances.

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1. Introduction

Chaos has attracted wide attention after Lorenz [1] found the first chaotic system during his studies of the atmospheric convection in 1963. Many new chaotic attractors, such as the Rössler system [2], Chen system [3], Lü system [4], Liu system [5], and the generalized Lorenz system family [6] have been proposed. But, there is a possibility that messages in secure communication masked by these chaotic systems can be easily extracted when they were intercepted, since they have only a single positive Lyapunov exponent [7]. As is well known, hyperchaotic systems are characterized by at least two positive Lyapunov exponents for typical trajectories in the arbitrarily high dimension phase space [8] and have the characteristics of high capacity, high security and high efficiency. Many hyperchaotic systems have been presented since the first hyperchaos was reported by Rössler in 1979 [9]. Wang *et al* [10] proposed a four-dimensional (4D) hyperchaotic Lorenz system by adding a nonlinear controller to Lorenz chaotic system. Pang and Liu [11] presented a 4D hyperchaotic system which was constructed by adding a linear controller to a 3D Lü system. It can be seen that the most common method to construct a new hyperchaotic system is to add an additional nonlinear state feedback controller into these Lorenz-like chaotic systems and the complicated dynamic properties of all these hyperchaotic systems are obtained by some quadratic cross-product nonlinear terms. Therefore, it is very interesting to ask whether there exists a hyperchaotic system that contains a nonlinear term. This paper gives a positive answer to this question by showing a new hyperchaotic system equipped with a nonlinear term in the form of exponential function. This new system is autonomous, and can display complicated and unusual dynamical behaviours.

Chaos synchronization is one of the main features of chaos applied in practical engineering. Thus, chaos and hyperchaos synchronizations have been active research topics. Since Pecora and Carroll first introduced the notation of chaos synchronization in 1990 [12], various kinds of synchronization schemes such as sliding mode control [13], adaptive control [14], observer-based control [15], hybrid synchronization [16], robust gain scheduling synchronization [17], H_∞ observer-based synchronization [18], backstepping approach [19], active and passive control [20,21] and so on have been successfully applied to the chaos and hyperchaos synchronizations. Recently, with the development of nonlinear control theory, adaptive antisynchronization (AS) which belongs to projective synchronization becomes an effective method to resolve the control and synchronization of chaotic and hyperchaotic systems [22]. AS phenomenon is a noticeable phenomenon in periodic oscillators. In fact, the first observation of synchronization of two oscillators by Huygens in the seven-tenth century was, AS between two pendulum clocks. Kim *et al* [23] have found AS phenomenon in mutually coupled identical Lorenz chaotic systems. AS phenomena have been observed experimentally in salt-water oscillators [24], semiconductor lasers [25] and so on. Recently, using different adaptive control methods, the AS for some typical chaotic and hyperchaotic systems has been discussed [22,26,27]. Further analysis found that these synchronization schemes only concern some dynamic systems that all the complicated dynamic properties are obtained by some quadratic cross-product nonlinear terms. To the author's knowledge, AS of a hyperchaotic system with a nonlinear term in the form of exponential function has never been reported in the literature. What is more, these proposed techniques assume that the involved systems are free from external perturbations. In practice, we have to take parameter mismatch and external disturbances into account. The effect of these uncertainties will destroy synchronization and even break it. So, it would be very instructive and significant to study AS in systems both with unknown parameters and external disturbances.

2. A novel hyperchaotic system and its basic properties

Consider a novel hyperchaotic system generated from a modified Lorenz system with a nonlinear term in the form of exponential function

$$\begin{cases} \dot{x} = a(y - x)\,, \\ \dot{y} = bx - xz + cy + u, \\ \dot{z} = e^{xy} - dz, \\ \dot{u} = -kx, \end{cases} \tag{1}$$

where a, b, c, d, k are system parameters and x, y, z, u are state variables. It is easy to see the invariance of the system under the coordinate transformation $(x, y, z, u) \rightarrow$

$(-x, -y, z, -u)$, i.e., the system has rotation symmetry around the z-axis. Note that $\nabla V = (\partial\dot{x}/\partial x) + (\partial\dot{y}/\partial y) + (\partial\dot{z}/\partial z) + (\partial\dot{u}/\partial u) = -a + c - d$. So the system is dissipative as long as $-a + c - d < 0$. That means the volume element V_0 is contracted by the flow into a volume element $V_0 e^{-a+c-d}$ in time t.

2.1 *Phase portraits*

Take the parameters $a = 10$, $b = 40$, $c = 1$, $d = 3$ and $k = 8$. The phase portraits are displayed in figures 1a–1d. It appears that the new hyperchaotic attractor exhibits a very interesting, complex and chaotic dynamical behaviour. The Lyapunov exponents of system (1) are found to be $l_1 = 1.6877$, $l_2 = 0.1214$, $l_3 = 0$ and $l_4 = -13.7271$. There are two positive Lyapunov exponents and it is obvious that the system is really a hyperchaotic system.

2.2 *Equilibria*

To analyse the system, a good start is to find its equilibria, and the equilibria of system (1) can be found by solving the following algebraic equations simultaneously:

$$\begin{cases} a(y-x) = 0, \\ bx - xz + cy + u = 0, \\ e^{xy} - dz = 0, \\ -kx = 0. \end{cases} \tag{2}$$

We operate the above nonlinear algebraic equations and find that the system has only one equilibrium point, which is described as $E(0, 0, 1/d, 0)$. In this case, system (1) is a

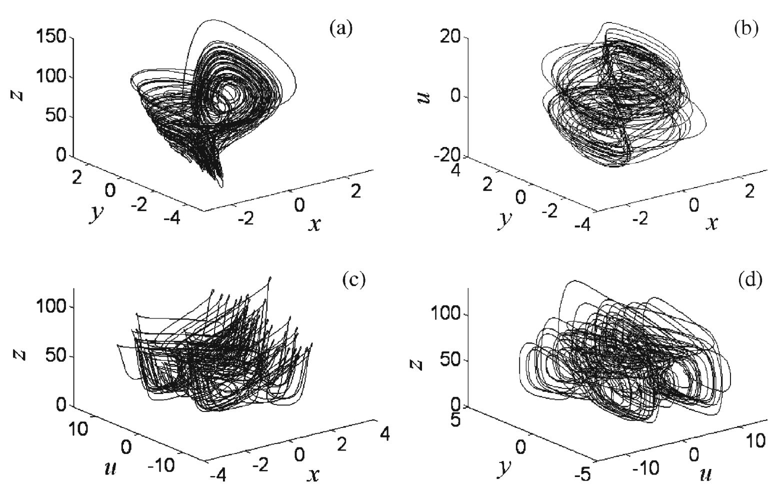

Figure 1. Phase portraits of system (1). (**a**) x–y–z view, (**b**) x–y–u view, (**c**) x–u–z view, (**d**) u–y–z view.

hyperbolic system. Linearizing the system (1) at $E(0, 0, 1/d, 0)$ now yields the Jacobian matrix

$$J = \begin{pmatrix} -a & a & 0 & 0 \\ b - 1/d & c & 0 & 1 \\ 0 & 0 & -d & 0 \\ -k & 0 & 0 & 0 \end{pmatrix},$$

and its characteristic equation

$$\Delta(\lambda) = (\lambda + d)\left[\lambda^3 + (a - c)\,\lambda^2 + a\left(b - c - \frac{1}{d}\right)\lambda + ak\right] = 0, \tag{3}$$

which gives $\lambda_1 = -d$, and

$$\Delta_0(\lambda) = \lambda^3 + (a - c)\,\lambda^2 + a\left(b - c - \frac{1}{d}\right)\lambda + ak = 0. \tag{4}$$

According to the Routh–Hurwitz criterion, the real parts of all the roots λ in $\Delta_0(\lambda) = 0$ are negative if and only if $a - c > 0$, $ak > 0$ and $a(a - c)\,(b - c - 1/d) - ak > 0$. From these inequalities, one obtains $a > 0$, $a > c$ and $(a - c)\,(b - c - 1/d) > k > 0$. Based on the above discussion, the following property is verified.

Theorem 1. *System* (1) *has a unique equilibrium* $E(0, 0, 1/d, 0)$. *Furthermore, the necessary and sufficient condition for equilibrium E to be locally stable is* $a > 0$, $a > c$ *and* $(a - c)\,(b - c - 1/d) > k > 0$.

2.3 *Hopf bifurcation*

Theorem 2. *Suppose that* $a > 0$, $k > 0$ *and* $a > c$ *holds. Then, as k varies and passes through the critical value* $k_0 = (a - c)\,(b - c - 1/d)$, *system* (1) *undergoes a Hopf bifurcation at the equilibrium* $E(0, 0, 1/d, 0)$.

Proof. Suppose that (3) has a pure imaginary root $\lambda = i\omega$, $(\omega \in R^+)$. Substituting it into (3) yields

$$ak - (a - c)\,\omega^2 + i\omega\left[a\left(b - c - \frac{1}{d}\right) - \omega^2\right] = 0. \tag{5}$$

It follows that

$$ak - (a - c)\,\omega^2 = 0, \quad a\left(b - c - \frac{1}{d}\right) - \omega^2 = 0.$$

Solving the above equations gives

$$\omega = \sqrt{\frac{ak}{a - c}}, \quad k = (a - c)\left(b - c - \frac{1}{d}\right),$$

under the condition $a > c$. Substituting $k = (a - c)(b - c - 1/d)$ into (3), one obtains

$$\lambda_1 = i\omega, \quad \lambda_2 = -i\omega, \quad \lambda_3 = -d, \quad \lambda_4 = -(a - c),$$

where $\omega = \sqrt{ak/(a - c)}$. Thus, when $a > c$, $k = k_0$. So the first condition for Hopf bifurcation [28] is satisfied.

From (3) and $a > c$, it follows that

$$\begin{aligned} \lambda'(k_0)\big|_{\lambda=i\omega} &= \left.\frac{-(a-c)}{3\lambda^2 + 2\lambda(a-c) + a(b-c-1/d)}\right|_{\lambda=i\omega} \\ &= \frac{a-c}{2}\left[\frac{a(b-c-1/d) + i(a-c)\sqrt{a(b-c-1/d)}}{a^2(b-c-1/d)^2 + a(a-c)^2(b-c-1/d)}\right], \end{aligned} \tag{6}$$

implying

$$\mathrm{Re}(\lambda'(k_0)|_{\lambda=i\omega}) = \frac{a(b-c-1/d)(a-c)^2}{2a^2(b-c-1/d)^2 + 2a(a-c)^2(b-c-1/d)} \neq 0. \tag{7}$$

Therefore, the second condition for a Hopf bifurcation [28] is also met. Consequently, Hopf bifurcation exists.

In order to validate Theorem 2, we take parameters $a = 10$, $b = 36$, $c = 1$ and $d = 3$ while k varies on the closed interval [1,323]. According to Theorem 2, system (1) undergoes a Hopf bifurcation at $k = k_0 = (a - c)(b - c - 1/d) = 312$. Figure 2 shows the bifurcation diagram vs. increasing k. It is clear that the system is undergoing a Hopf bifurcation at $k = k_0 = 312$. □

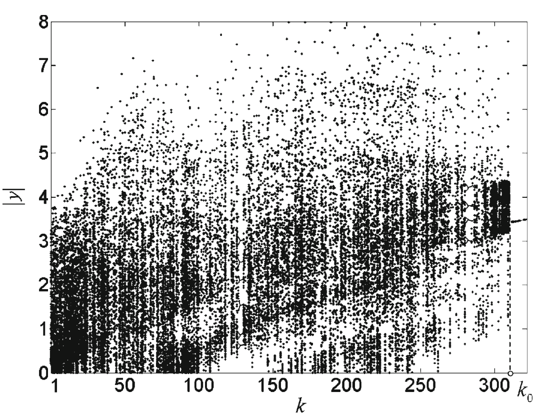

Figure 2. Bifurcation diagram of system (1) vs. parameter k.

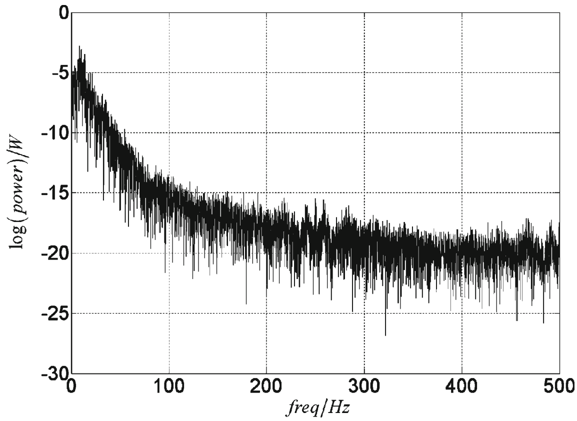

Figure 3. Spectrum map of $\log|x|$.

2.4 *Spectrum map, Lyapunov exponent spectrum and time domain waveform*

Figure 3 shows the spectrum map of system (1) which exhibits a continuous broadband feature. To investigate the impact of parameters on the dynamics of the hyperchaotic system, here we take parameter c as an example and extend the range of c to an interval $[-6, 6]$. The variation of two largest Lyapunov exponents for different values of c is given in figure 4 by Wolf algorithm [29]. It is found that the new hyperchaotic system possesses two positive Lyapunov exponents within a wide range of parameter c. Figure 5 shows the time domain waveform, and it can be observed that the time domain waveform has non-cyclical characteristics.

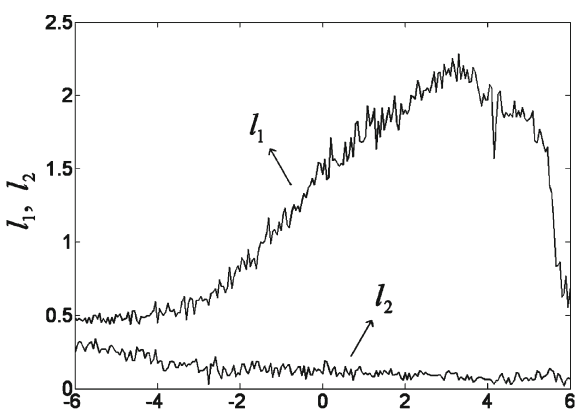

Figure 4. Lyapunov exponents spectrum of system (1) vs. parameter c.

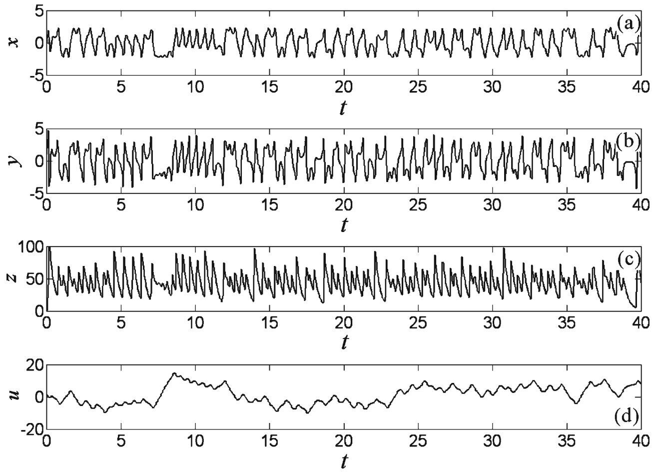

Figure 5. Time domain waveform of system (1). (**a**) t–x wave, (**b**) t–y wave, (**c**) t–z wave, (**d**) t–u wave.

3. Adaptive AS with external disturbances

3.1 *Principle*

Consider the drive chaotic system in the form

$$\dot{x} = f(x) + \mathbf{F}(x)\boldsymbol{\alpha} + \mathbf{d}', \tag{8}$$

where $x = (x_1, x_2, \ldots, x_{n1})^T \in \Omega_1 \subset \mathbf{R}^n$ is the state vector, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)^T \in \mathbf{R}^m$ is the uncertain parameter vector of the drive system, which is to be asymptotically estimated finally. $f(x)$ is an $n1 \times 1$ matrix without the uncertain parameter vector $\boldsymbol{\alpha}$. $\mathbf{F}(x)$ is an $n1 \times m$ matrix, and the element $\mathbf{F_{ij}}(x)$ in matrix $\mathbf{F}(x)$ satisfies $\mathbf{F_{ij}}(x) \in L_\infty$ for $x \in \Omega_1 \subset \mathbf{R}^n$. $\mathbf{d}' = \left(d'_1, d'_2, \ldots, d'_n\right)^T \subset \mathbf{R}^n$ is the exotic disturbance of system (8), which satisfies the bounded condition $\|d'_n\| \leq \sigma_n$ for all t, where σ_n are known positive constants. On the other hand, the response system is assumed by

$$\dot{y} = g(y) + \mathbf{G}(y)\boldsymbol{\beta} + \mathbf{U} + \mathbf{d}'', \tag{9}$$

where $y = (y_1, y_2, \ldots, y_{n2})^T \in \Omega_2 \subset \mathbf{R}^n (n2 \leq n1)$ is the state vector, $\boldsymbol{\beta} = \left(\beta_1, \beta_2, \ldots, \beta_p\right)^T \in \mathbf{R}^p$ is the uncertain parameter vector of the slave system, which is to be asymptotically estimated finally. $g(y)$ is an $n2 \times 1$ matrix without the uncertain parameter vector $\boldsymbol{\beta}$. $\mathbf{G}(y)$ is an $n2 \times p$ matrix, and the element $\mathbf{G_{ij}}(y)$ in matrix $\mathbf{G}(y)$ satisfies $\mathbf{G_{ij}}(y) \in L_\infty$ for $y \in \Omega_2 \subset \mathbf{R}^n$. $\mathbf{d}'' = \left(d''_1, d''_2, \ldots, d''_n\right)^T \subset \mathbf{R}^n$ is the exotic disturbance of system (9), which satisfies the bounded condition $\|d''_n\| \leq \rho_n$ for all t, where ρ_n are known positive constants. The control input vector $\mathbf{U} = (U_1, U_2, \ldots, U_{n2})^T \in \mathbf{R}^p$ is an $n2 \times 1$ matrix, which is used to realize synchronization of systems (8) and (9).

Let $e = (e_1, e_2, \ldots, e_{n2})^T = x + y$ is the AS error vector. Our goal is to design an appropriate controller $\mathbf{U}$ such that the trajectory of the response system (9) with initial conditions y_0 can asymptotically approach the drive system (8) with initial conditions x_0.

In this sense, we have $\lim_{t\to\infty}\|e\| = \lim_{t\to\infty}\|x(t, x_0) + y(t, y_0)\| = 0$, where $\|\cdot\|$ is the Euclidean norm. At this point, it means the drive system (8) and the response system (9) are antisynchronized under the controller $\mathbf{U}$ as time t tends to infinity.

3.2 *Adaptive AS controller design*

Theorem 3. *If the nonlinear controller* $\mathbf{U}$ *is taken as*

$$\mathbf{U} = -f(x) - \mathbf{F}(x)\boldsymbol{\alpha} - \mathbf{d}' - g(y) - \mathbf{G}(y)\boldsymbol{\beta} - \mathbf{d}'' - \mathbf{k}e, \tag{10}$$

where the control amplitude $\mathbf{k} = (k_1, k_2, \ldots, k_n)^T$ *is an* $n \times 1$ *positive constant matrix, and the adaptive laws of parameters are taken as*

$$\dot{\hat{\boldsymbol{\alpha}}} = [\mathbf{F}(x)]^T e, \quad \dot{\hat{\boldsymbol{\beta}}} = [\mathbf{G}(y)]^T e, \tag{11}$$

where $\dot{\hat{\boldsymbol{\alpha}}}$ *and* $\dot{\hat{\boldsymbol{\beta}}}$ *are estimations of the unknown parameters* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$*, respectively, then the response system* (8) *can synchronize the drive system* (9) *globally and asymptotically.*

Proof. From eqs (8)–(11), the error dynamical system is

$$\dot{e} = \mathbf{F}(x)(\boldsymbol{\alpha} - \hat{\boldsymbol{\alpha}}) + \mathbf{G}(y)(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}) + \mathbf{d}' + \mathbf{d}'' - \mathbf{k}e. \tag{12}$$

Let $\tilde{\boldsymbol{\alpha}} = \boldsymbol{\alpha} - \hat{\boldsymbol{\alpha}}$, $\tilde{\boldsymbol{\beta}} = \boldsymbol{\beta} - \hat{\boldsymbol{\beta}}$. The Lyapunov function candidate can be taken as

$$\mathbf{V}(e, \tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}) = \frac{1}{2}(e^2 + \tilde{\boldsymbol{\alpha}}^2 + \tilde{\boldsymbol{\beta}}^2) = \frac{1}{2}(e^T e + \tilde{\boldsymbol{\alpha}}^T \tilde{\boldsymbol{\alpha}} + \tilde{\boldsymbol{\beta}}^T \tilde{\boldsymbol{\beta}}). \tag{13}$$

The time derivative of $\mathbf{V}$ along the trajectory of system (9) is as follows:

$$\begin{aligned}
\dot{\mathbf{V}}(e, \tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}) &= \dot{e}^T e + \tilde{\boldsymbol{\alpha}}^T \dot{\hat{\boldsymbol{\alpha}}} + \tilde{\boldsymbol{\beta}}^T \dot{\hat{\boldsymbol{\beta}}} \\
&= \left[\mathbf{F}(x)\tilde{\boldsymbol{\alpha}} + \mathbf{G}(y)\tilde{\boldsymbol{\beta}} + \mathbf{d}' + \mathbf{d}'' - \mathbf{k}e\right]^T e \\
&\quad - \tilde{\boldsymbol{\alpha}}^T \left[\mathbf{F}(x)\right]^T e - \tilde{\boldsymbol{\beta}}^T [\mathbf{G}(y)]^T e \\
&= -\left(\mathbf{k} - \left(\mathbf{d}' + \mathbf{d}''\right)\right) e^T \\
&\le -\mathrm{diag}\{k_n - (\|d_n'\| + \|d_n''\|)\} e^T \\
&= -\mathrm{diag}\{k_n - (\sigma_n + \rho_n)\} e^T \\
&= -\mathbf{L}e^T,
\end{aligned} \tag{14}$$

where $\mathbf{L} = \mathrm{diag}\{k_1 - (\sigma_1 + \rho_1), k_2 - (\sigma_2 + \rho_2), \ldots, k_n - (\sigma_n + \beta_n)\}$, from the theorem of Lyapunov on asymptotic stability, as long as $e \neq 0$ and $\mathbf{L}$ is a positive-definite matrix. Thus, $\dot{\mathbf{V}} < \mathbf{0}$ for $\mathbf{V} > \mathbf{0}$, and this completes the proof. □

Remark 1. Note that most of the chaotic or hyperchaotic systems, such as the generalized Lorenz system family and many hyperchaotic systems including the above proposed hyperchaotic system, as well as the Duffing oscillator and some variants of Chuas circuits can be described by (8).

Remark 2. If the drive system (8) and the response system (9) satisfy $f(\cdot) = g(\cdot)$, $\mathbf{F}(\cdot) = \mathbf{G}(\cdot)$, $(n1 = n2, m = p)$, the proposed schemes can be extended to the adaptive AS of two identical chaotic or hyperchaotic systems with fully uncertain parameters.

4. Adaptive AS between two non-identical hyperchaotic systems

In this section, we study the AS between two non-identical hyperchaotic systems. Our aim is to design an adaptive controller and force the response system's trajectory to have antiamplitude to the drive system's trajectory and adjust the unknown parameters and suppress disturbances simultaneously. System (1) is the drive system which is redescribed by

$$\begin{cases} \dot{x}_1 = a(y_1 - x_1) + d_{11}, \\ \dot{y}_1 = bx_1 - x_1 z_1 + c y_1 + f w_1 + d_{12}, \\ \dot{z}_1 = e^{x_1 y_1} - d z_1 + d_{13}, \\ \dot{w}_1 = -k x_1 + d_{14}, \end{cases} \tag{15}$$

and here, we take the Lü hyperchaotic system [30] as the response system given by

$$\begin{cases} \dot{x}_2 = a_2(y_2 - x_2) + d_{21} + u_1, \\ \dot{y}_2 = c_2 y_2 - x_2 z_2 + w_2 + d_{22} + u_2, \\ \dot{z}_2 = x_2 y_2 - b_2 z_2 + d_{23} + u_3, \\ \dot{w}_2 = z_2 - w_2 + d_{24} + u_4, \end{cases} \tag{16}$$

where d_{ij} $(i = 1, 2,\ j = 1, 2, 3, 4)$ are the disturbances of systems (15) and (16), respectively, and $\|d_{1j}\| \le \alpha_{1j}, \|d_{2j}\| \le \beta_{2j}$, where α_{1j}, β_{2j} are known positive constants. $\mathbf{U} = (u_1, u_2, u_3, u_4)^T$ is the controller, which determines the control functions to realize the adaptive AS between systems (15) and (16). We add (15) to (16), and yield the following error dynamical system:

$$\begin{cases} \dot{e}_1 = a_1(y_1 - x_1) + a_2(y_2 - x_2) + d_{21} + d_{11} + u_1, \\ \dot{e}_2 = b_1 x_1 - x_1 z_1 - x_2 z_2 + c_1 y_1 + c_2 y_2 + e_4 + d_{22} + d_{12} + u_2, \\ \dot{e}_3 = e^{x_1 y_1} - d_1 z_1 - b_2 z_2 + x_2 y_2 + d_{23} + d_{13} + u_3, \\ \dot{e}_4 = -k_1 x_1 + z_2 - w_2 + d_{24} + d_{14} + u_4, \end{cases} \tag{17}$$

where $e_1 = x_1 + x_2, e_2 = y_1 + y_2, e_3 = z_1 + z_2, e_4 = w_1 + w_2$. Our goal is to find proper controller $\mathbf{U}$ and parameter update rule, such that system (16) globally antisynchronizes system (15) asymptotically. When controls are applied, the two systems will approach AS for any initial conditions by an appropriate controller. For this end, we propose the following adaptive control law for system (15):

$$\begin{cases} u_1 = -\hat{a}_1(y_1 - x_1) - \hat{a}_2(y_2 - x_2) - (d_{21} + d_{11})\, e_1 - p_1 e_1, \\ u_2 = -\hat{b}_1 x_1 + x_1 z_1 + x_2 z_2 - \hat{c}_1 y_1 - \hat{c}_2 y_2 - e_4 \\ \qquad - (d_{22} + d_{12})\, e_2 - p_2 e_2, \\ u_3 = -e^{x_1 y_1} + \hat{d}_1 z_1 + \hat{b}_2 z_2 - x_2 y_2 - (d_{23} + d_{13})\, e_3 - p_3 e_3, \\ u_4 = k_1 x_1 - z_2 + w_2 - (d_{24} + d_{14})\, e_4 - p_4 e_4, \end{cases} \tag{18}$$

and the adaptive laws of parameters are described as

$$\begin{cases} \dot{\hat{a}}_1 = (y_1 - x_1)\, e_1, \dot{\hat{b}}_1 = x_1 e_2, \dot{\hat{c}}_1 = y_1 e_2, \dot{\hat{d}}_1 = -z_1 e_3, \\ \dot{\hat{k}}_1 = -x_1 e_4, \dot{\hat{a}}_2 = (y_2 - x_2)\, e_1, \dot{\hat{b}}_2 = -z_2 e_3, \dot{\hat{c}}_2 = y_2 e_2, \end{cases} \tag{19}$$

where $\dot{\hat{a}}_1, \dot{\hat{b}}_1, \dot{\hat{c}}_1, \dot{\hat{d}}_1, \dot{\hat{k}}_1, \dot{\hat{a}}_2, \dot{\hat{b}}_2, \dot{\hat{c}}_2$ are the estimates of a_1, b_1, c_1, d_1, k_1, a_2, b_2, c_2 respectively. p_1, p_2, p_3 and p_4 are four positive control coefficients, with which we can control the convergence speed of the scheme.

Theorem 4. *The two non-identical hyperchaotic systems* (15) *and* (16) *are globally asymptotically antisynchronized by the adaptive control law in system* (15) *and the parameter update rule in eq.* (19) *with any initial conditions.*

Proof. Applying control law in system (18) to system (19) yields the resulting error dynamics as follows:

$$\begin{cases} \dot{e}_1 = \tilde{a}_1(y_1 - x_1) + \tilde{a}_2(y_2 - x_2) + (d_{21} + d_{11})\, e_1 - p_1 e_1, \\ \dot{e}_2 = \tilde{b}_1 x_1 + \tilde{c}_1 y_1 + \tilde{c}_2 y_2 + e_4 + (d_{22} + d_{12})\, e_2 - p_2 e_2, \\ \dot{e}_3 = -\tilde{d}_1 z_1 - \tilde{b}_2 z_2 + (d_{23} + d_{13})\, e_3 - p_3 e_3, \\ \dot{e}_4 = -\tilde{k}_1 x_1 + z_2 - w_2 + (d_{24} + d_{14})\, e_4 - p_4 e_4, \end{cases} \tag{20}$$

where $\tilde{a}_1 = a_1 - \hat{a}_1, \tilde{b}_1 = b_1 - \hat{b}_1, \tilde{c}_1 = c_1 - \hat{c}_1, \tilde{d}_1 = d_1 - \hat{d}_1, \tilde{k}_1 = k_1 - \hat{k}_1, \tilde{a}_2 = a_2 - \hat{a}_2, \tilde{b}_2 = b_2 - \hat{b}_2$ and $\tilde{c}_2 = c_2 - \hat{c}_2$. Consider the following Lyapunov function

$$\mathbf{V} = \frac{1}{2}(e^T e + \tilde{a}_1^2 + \tilde{b}_1^2 + \tilde{c}_1^2 + \tilde{d}_1^2 + \tilde{k}_1^2 + \tilde{a}_2^2 + \tilde{b}_2^2 + \tilde{c}_2^2). \tag{21}$$

The time derivative of $\mathbf{V}$ along the solution of error dynamical system gives that

$$\begin{aligned} \dot{\mathbf{V}} &= e^T \dot{e} + \tilde{a}_1 \dot{\tilde{a}}_1 + \tilde{b}_1 \dot{\tilde{b}}_1 + \tilde{c}_1 \dot{\tilde{c}}_1 + \tilde{d}_1 \dot{\tilde{d}}_1 + \tilde{k}_1 \dot{\tilde{k}}_1 + \tilde{a}_2 \dot{\tilde{a}}_2 + \tilde{b}_2 \dot{\tilde{b}}_2 + \tilde{c}_2 \dot{\tilde{c}}_2 \\ &= e_1 \left[\tilde{a}_1 (y_1 - x_1) + \tilde{a}_2 (y_2 - x_2) + (d_{21} + d_{11})\, e_1 - p_1 e_1\right] \\ &\quad + e_2\left[\tilde{b}_1 x_1 + \tilde{c}_1 y_1 + \tilde{c}_2 y_2 + e_4 + (d_{22} + d_{12})\, e_2 - p_2 e_2\right] \\ &\quad + e_3\left[-\tilde{d}_1 z_1 - \tilde{b}_2 z_2 + (d_{23} + d_{13})\, e_3 - p_3 e_3\right] \\ &\quad + e_4\left[-\tilde{k}_1 x_1 + z_2 - w_2 + (d_{24} + d_{14})\, e_4 - p_4 e_4\right] \\ &\quad + \tilde{a}_1 \left[-(y_1 - x_1)\, e_1\right] + \tilde{b}_1 (-x_1 e_2) \\ &\quad + \tilde{c}_1 (-y_1 e_2) + \tilde{d}_1 (z_1 e_3) + \tilde{k}_1 (x_1 e_4) \\ &\quad + \tilde{a}_2 \left[-(y_2 - x_2)\, e_1\right] + \tilde{b}_2 (z_2 e_3) + \tilde{c}_2 (-y_2 e_2) \\ &= -p_1 e_1^2 - p_2 e_2^2 - p_3 e_3^2 - p_4 e_4^2 + (d_{11} + d_{21})\, e_1^2 + (d_{12} + d_{22})\, e_2^2 \\ &\quad + (d_{13} + d_{23})\, e_3^2 + (d_{14} + d_{24})\, e_4^2 \\ &\le -p_1 e_1^2 - p_2 e_2^2 - p_3 e_3^2 - p_4 e_4^2 + (\|d_{11}\| + \|d_{21}\|)\, e_1^2 + (\|d_{12}\| + \|d_{22}\|)\, e_2^2 \\ &\quad + (\|d_{13}\| + \|d_{23}\|)\, e_3^2 + (\|d_{14}\| + \|d_{24}\|)\, e_4^2 \\ &\le -\left[p_1 - (\alpha_{11} + \beta_{21})\right] e_1^2 - \left[p_2 - (\alpha_{12} + \beta_{22})\right] e_2^2 \\ &\quad - \left[p_3 - (\alpha_{13} + \beta_{23})\right] e_3^2 - \left[p_4 - (\alpha_{14} + \beta_{24})\right] e_4^2 \\ &= -e^T \mathbf{L} e, \end{aligned} \tag{22}$$

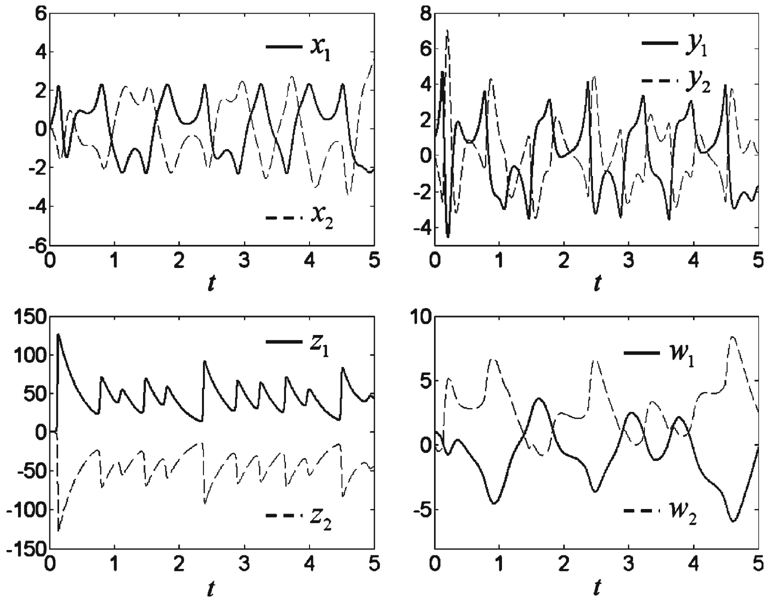

Figure 6. State trajectories of the drive system (15) and the response system (16).

where $\mathbf{L} = \mathrm{diag}\{p_1 - (\alpha_{11} + \beta_{21}), p_2 - (\alpha_{12} + \beta_{22}), p_3 - (\alpha_{13} + \beta_{23}), p_4 - (\alpha_{14} + \beta_{24})\}$. As long as $e \neq 0$ and $\mathbf{L}$ is a positive-definite matrix, then $\dot{\mathbf{V}} < 0$, we have $e, \dot{\hat{a}}_1, \dot{\hat{b}}_1, \dot{\hat{c}}_1, \dot{\hat{d}}_1, \dot{\hat{k}}_1, \dot{\hat{a}}_2, \dot{\hat{b}}_2, \dot{\hat{c}}_2 \in L_\infty$. From the error dynamical system (13), we also have $\dot{e} \in L_\infty$. From the fact that

$$\int_0^t \lambda_{\min}(\mathbf{L}) \|e\|^2 \,\mathrm{d}t \le \int_0^t e^T \mathbf{L} e \,\mathrm{d}t = \int_0^t -\dot{\mathbf{V}} \mathrm{d}t = \mathbf{V}(0) - \mathbf{V}(t) \le \mathbf{V}(0), \quad (23)$$

where $\lambda_{\min}(\mathbf{L})$ is the minimal eigenvalue of the positive-definite matrix $\mathbf{L}$. Therefore, the response system (16) can globally antisynchronize the drive system (15) asymptotically.

To verify the effectiveness of the proposed method, we discuss the simulation result for the AS between the proposed hyperchaotic system and the Lü hyperchaotic system. In the numerical simulations, the fourth-order Runge–Kutta method is used to solve the systems with time step size 0.001. For this numerical simulation, we assume that the initial conditions, $x_1(0) = 0$, $y_1(0) = 1$, $z_1(0) = 0.2$, $w_1(0) = 1$ and $x_2(0) = 0.1$, $y_2(0) = 0.1$, $z_2(0) = 0.1$, $w_2(0) = 0.1$ are employed and the disturbances are set as

$$(d_{11}, d_{12}, d_{13}, d_{14}) = (\sin(20t), -2\cos(10t), 0, 3\sin(30t)),$$
$$(d_{21}, d_{22}, d_{23}, d_{24}) = (2\cos(20t), -3\cos(20t), \sin(20t), \sin(10t)).$$

To ensure that $\mathbf{L}$ is a positive-definite matrix, here we choose the control inputs $(p_1, p_2, p_3, p_4) = (20, 20, 20, 20)$. The unknown parameters chosen are $a_1 = 10$,

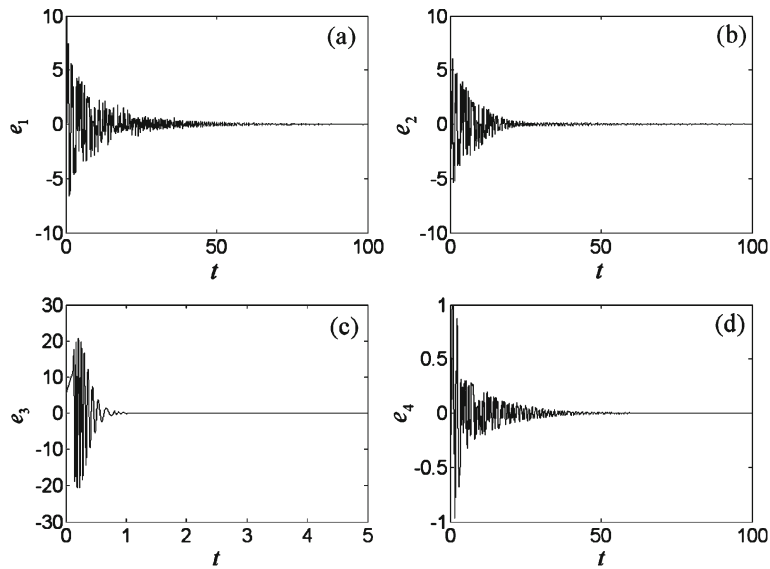

Figure 7. Dynamics of the variables e_1, e_2, e_3 and e_4 for error system (17) with time t. (**a**) e_1, (**b**) e_2, (**c**) e_3, (**d**) e_4.

$b_1 = 40$, $c_1 = 1$, $d_1 = 3$, $k_1 = 8$ and $a_2 = 15$, $b_2 = 5$, $c_2 = 10$ in the simulations such that both systems exhibit hyperchaotic behaviour. In addition, the initial conditions of the adaptive update laws of system parameters are $\hat{a}_1(0) = \hat{b}_1(0) = \hat{c}_1(0) = \hat{d}_1(0) = \hat{k}_1(0) = \hat{a}_2(0) = \hat{b}_2(0) = \hat{c}_2(0) = 0.1$. Figure 6 displays state trajectories of the drive system (15) and the response system (16). Figure 7 displays the AS errors between systems (15) and (16). Figure 8 shows that the estimates $\hat{a}_1(t)$, $\hat{b}_1(t)$, $\hat{c}_1(t)$, $\hat{d}_1(t)$, $\hat{k}_1(t)$, $\hat{a}_2(t)$, $\hat{b}_2(t)$ and $\hat{c}_2(t)$ of the unknown parameters converge to $a_1 = 10$, $b_1 = 40$, $c_1 = 1$, $d_1 = 3$, $k_1 = 8$, $a_2 = 15$, $b_2 = 5$ and $c_2 = 10$ as $t \to \infty$. □

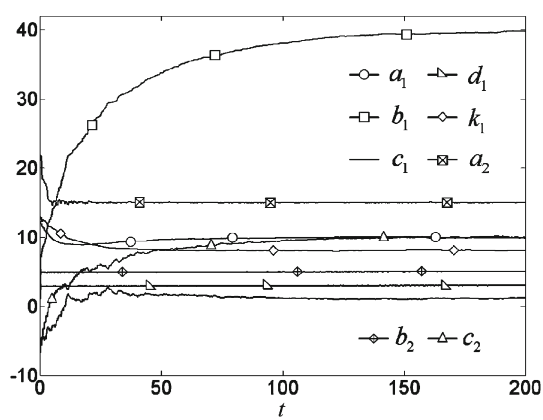

Figure 8. Estimated values of system parameters with parameter updated law.

5. Conclusions

In this paper, a hyperchaotic system with an exponential nonlinear term is presented. Some complex dynamical behaviours such as Hopf bifurcation, Lyapunov exponents and hyperchaotic behaviour of the simple 4D autonomous system are investigated and analysed. Then we propose a novel approach of adaptive AS between two non-identical hyperchaotic systems with parameter mismatch and external disturbances. Finally, numerical simulations are provided to illustrate the effectiveness of our approaches. Some potential engineering applications of the hyperchaotic system, such as in secure communication, will be further investigated in the near future.

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