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Black holes and groups of type E_7

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Abstract. We report some results on the relation between extremal black holes in locally supersymmetric theories of gravity and groups of type E_7, appearing as generalized electric-magnetic duality symmetries in such theories. Some basics on the covariant approach to the stratification of the relevant symplectic representation are reviewed, along with a connection between special Kähler geometry and a 'generalization' of groups of type E_7.

Keywords. Supergravity; groups of type E_7; black holes; quantum field theory.

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1. From extremal black holes...

Black holes, one of the most stunning consequences of General Relativity, enjoy thermodynamical properties in a generalized phase space whose quantum mechanical attributes are their Arnowitt–Deser–Misner (ADM) mass [1], charge, spin and scalar charges (see [2–6]). They can be regarded as probes of the quantum regime of any fundamental theory of gravity and, as such, they are naturally investigated within the framework of superstring and M-theory. Unlike Schwarzschild black holes, charged (Reissner–Nordström) and/or spin (Kerr–Newman) black holes can be extremal, i.e. with vanishing temperature for non-zero entropy, in which case their event and Cauchy horizons coincide. In formulae, the extremality parameter is given by

$$c = 2ST = \frac{1}{2}(r_+ - r_-) \to 0, \tag{1.1}$$

where c measures the surface gravity and $S = \log \mathcal{N}$ is the black hole entropy which counts the number $\mathcal{N}$ of the microstates. In (semi)classical (super)gravity, S is given by the celebrated Bekenstein–Hawking area–entropy formula [7]

$$S = \frac{1}{2}A_{\mathrm{H}} = \pi R_{\mathrm{H}}^2 = \pi V_{\mathrm{BH,crit}}(Q), \tag{1.2}$$

where R_{H} is the effective radius of the horizon and $V_{\mathrm{BH}}(\phi, Q)$ is the so-called black hole effective potential (a function of the scalar fields ϕ and the electric and magnetic

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charges Q; see below), defined within the ‘attractor mechanism’ [3,4]: its critical points in the scalar manifold correspond to attractive scalar trajectories towards the horizon itself. For extremal charged black holes, R_{H} must respect the symmetries of the theory and in particular it must depend only on the electric and magnetic charges and not on the scalar field values [8]. Therefore, the entropy will depend only on the charges and it will take particular expressions depending on the duality symmetries of the given model.

Another important feature of extremal black holes is that their horizon geometry of space-time is universal, and in four dimensions it is given by the $AdS_2 \times S_2$ Bertotti–Robinson metric. Actually, extremal black holes behave as solitons interpolating between maximally symmetric geometries of (super)space-time: Minkowski at spacial infinity and the conformally flat near-horizon metric [9,10].

In supergravity, it should be recalled that, remarkably, the common radius of $AdS_2 \times S_2$, and therefore the entropy, can actually be computed using the underlying non-compact electric-magnetic duality [8].

In the last few years it has become clear that the scalar field dynamics for the extremal black holes can be entirely encoded into a (charge orbit-dependent) real ‘superpotential’ function $W(\phi, Q)$ whose critical points coincide with a whole class of critical points of V_{BH} itself. For supersymmetric $N = 2$ flows $W = |Z|$, where Z is the central extension of the local supersymmetry algebra; for $N > 2$ $\frac{1}{N}$-BPS flows, $|Z|^2$ is replaced by the highest eigenvalues of $ZZ^{\dagger}$, where $Z \equiv Z_{AB}$ is the central charge matrix [12].

Remarkably, such a function W can be shown to exist also for non-supersymmetric configurations, in which case it is called the ‘fake superpotential’ [11] because of the similarity with the set-up of ‘fake supergravities’, and applications in domain-wall physics [13]. When the attractors are regular, the W function has a minimum for $\phi^i = \phi^i_{\mathrm{H}}$, and its horizon value gives the entropy of the configuration

$$S = \frac{1}{4} A_{\mathrm{H}} = \pi W^2_{\mathrm{H}}(Q) = \pi W^2_{\mathrm{crit}}(\phi^i_h(Q), Q) \tag{1.3}$$

according to the aforementioned Bekenstein–Hawking formula. However, if W has a run-away behaviour in moduli space, $\phi_{\mathrm{H}} \to \infty\ W \to 0$ (which is not acceptable in $D = 4$), the corresponding black hole solutions are singular. Then the scalar fields are never stabilized within the boundaries of moduli space, there are no attractors and the entropy of the extremal configuration vanishes.

In order to describe a static, spherically symmetric, asymptotically flat, extremal dyonic black hole background in the extremal case, $c = 2ST = 0$, the metric ansatz reads [4]

$$\mathrm{d}s^2 = -\mathrm{e}^{2U}\mathrm{d}t^2 + \mathrm{e}^{-2U}\left[\frac{\mathrm{d}\tau^2}{\tau^4} + \frac{1}{\tau^2}(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right], \tag{1.4}$$

with the field strength $F^{\Lambda}_{\mu\nu}$ for n_V vectors ($\Lambda = 1, \ldots, n_V$) and its dual $G_{\Lambda\mu\nu} = (\delta\mathcal{L}/\delta F^{\Lambda}_{\mu\nu})$ given by

$$F = \mathrm{e}^{2U} C\mathcal{M}(\phi^i) Q\mathrm{d}t \wedge \mathrm{d}\tau + Q\sin\theta\mathrm{d}\theta \wedge \mathrm{d}\phi \tag{1.5}$$

$$F = \begin{pmatrix} F^{\Lambda}_{\mu\nu} \\ G_{\Lambda\mu\nu} \end{pmatrix} \frac{\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}}{2}. \tag{1.6}$$

The electric and magnetic charge vector $Q \equiv (p^\Lambda, q_\Lambda)^T$ is defined as

$$q_\Lambda \equiv \frac{1}{4\pi}\int_{S^2_\infty} G_\Lambda\,, \qquad p^\Lambda \equiv \frac{1}{4\pi}\int_{S^2_\infty} F_\Lambda\,. \tag{1.7}$$

$\mathcal{M}(\phi)$ is a $2n_V \times 2n_V$ real, symmetric, negative-definite $Sp(2n_V, \mathbb{R})$ matrix, satisfying $\mathcal{M}\mathbb{C}\mathcal{M} = \mathbb{C}$ ($\mathbb{C}$ denoting the symplectic metric), and given by

$$\mathcal{M}(\phi) = \begin{pmatrix} I + RI^{-1}R & -RI^{-1} \\ -I^{-1}R & I^{-1} \end{pmatrix} \tag{1.8}$$

where $I \equiv \mathrm{Im}\ \mathcal{N}_{\Lambda\Sigma}$ and $R \equiv \mathrm{Re}\ \mathcal{N}_{\Lambda\Sigma}$, with $\mathcal{N}_{\Lambda\Sigma}$ denoting the scalar-dependent vector kinetic matrix appearing in the $D = 4$ Lagrangian density of the Maxwell–Einstein scalar system ($i = 1, ..., n_V - 1$)

$$\mathcal{L} = -\frac{R}{2} + \frac{1}{2} g_{ij}(\phi)\partial_\mu \phi^i \partial^\mu \phi^j + I_{\Lambda\Sigma} F^\Lambda \wedge^* F^\Sigma + R_{\Lambda\Sigma} F^\Lambda \wedge F^\Sigma. \tag{1.9}$$

The aforementioned black hole effective potential V_{BH} [3], governing the radial evolution of the scalar fields in the black hole background (1.4) enjoys a very simple expression in terms of the matrix $\mathcal{M}$, namely,

$$V_{\mathrm{BH}} = -\frac{1}{2} Q^T \mathcal{M} Q\,. \tag{1.10}$$

As pioneered in [4], such a function arises upon reducing the general $D = 4$ Lagrangian (1.9) in the background (1.4) to the $D = 1$ almost geodesic action describing the radial evolution of the n_V scalar fields $\{U(\tau), \phi^i(\tau)\}$:

$$S_{D=1} = \int (U' + g_{ij}\phi'^i \phi'^j + \mathrm{e}^{2U} V_{\mathrm{BH}}(\phi(\tau), p, q))\mathrm{d}\tau, \tag{1.11}$$

where τ is the $D = 1$ affine evolution parameter in the extremal black hole background (1.4), and prime stands for differentiation with respect to it. In order to have the same equations of motion of the original theory, the action must be complemented with the Hamiltonian constraint (in the extremal case) [4]

$$(U')^2 + g_{ij}\phi'^i \phi'^j - \mathrm{e}^{2U} V_{\mathrm{BH}}(\phi(\tau), p, q) = 0. \tag{1.12}$$

The black hole effective potential can be written in terms of the superpotential W as

$$V_{\mathrm{BH}} = W^2 + 2g^{ij}\partial_i W \partial_j W. \tag{1.13}$$

This formula can be viewed as a differential equation defining W for a given black hole effective potential V_{BH}, and it can lead to multiple choices: only one of those will correspond to BPS solutions (i.e. to the usual superpotential), while others will be associated with non-BPS ones. In both cases, W allows to rewrite the ordinary second-order supergravity equations of motion

$$\frac{\mathrm{d}^2 U}{\mathrm{d}\tau^2} = \mathrm{e}^{2U} V_{\mathrm{BH}} \tag{1.14}$$

$$\frac{\mathrm{d}^2 \phi^i}{\mathrm{d}\tau^2} = g^{i\bar{j}} \frac{\partial V_{\mathrm{BH}}}{\partial \phi_{\bar{j}}} \mathrm{e}^{2U}, \tag{1.15}$$

as first-order flow equations, defining the radial evolution of the scalar fields ϕ^i and the warp factor U from asymptotic infinity towards the black hole horizon [11]:

$$U' = -\mathrm{e}^U W\,, \qquad \phi'^i = -2\mathrm{e}^U g^{ij}\partial_j W. \tag{1.16}$$

Besides the horizon entropy $S_{\mathrm{BH}} = \pi W_{\mathrm{H}}^2$ and the first-order flows, the value at radial infinity of the superpotential W also encodes other basic properties of the extremal black hole, which are its ADM mass [1], given by

$$M_{\mathrm{ADM}}(\phi_0^i, Q) \equiv W(\phi_\infty, Q), \tag{1.17}$$

and the scalar charges

$$\Sigma^i \equiv \phi'^i_\infty = -2g^{ij}(\phi_\infty)\frac{\partial W}{\partial \phi^i}(\phi_\infty, Q). \tag{1.18}$$

For $N \geq 2$, the fake superpotential for the non-BPS branch [14,15] has been computed for wide classes of models [11,16–23], based on symmetric geometries of moduli spaces, using as a tool the U-duality symmetry of the underlying supergravity. A universal procedure for its construction in $N = 2$ special geometries has been established [19,20], which generalizes the results obtained for the so-called $N = 2$ STU model [24].

2. ...to Groups of type E_7

As yielded by the treatment above, the black hole entropy S is invariant under the electric-magnetic duality, within the framework first defined in [8], in which the non-compact U-duality group has a symplectic action both on the charge vector Q (1.7) and on the scalar fields (through the definition of a flat symplectic bundle [25] over the scalar manifold itself) (see [26] for a review). The latter property makes relevant the mathematical notion of groups of type E_7.

The first axiomatic characterization of groups of type E_7 through a module (irrep.) was given in 1967 by Brown [27]. A group G of type E_7 is a Lie group endowed with a representation $\mathbf{R}$ such that:

(1) $\mathbf{R}$ is symplectic, i.e.,

$$\exists!\mathbb{C}_{[MN]} \equiv \mathbf{1} \in \mathbf{R}\times_a\mathbf{R}; \tag{2.1}$$

(the subscript a stands for symmetric and skew-symmetric throughout) in turn, $\mathbb{C}_{[MN]}$ defines a non-degenerate skew-symmetric bilinear form (symplectic product); given two different charge vectors Q_1 and Q_2 in $\mathbf{R}$, such a bilinear form is defined as

$$\langle Q_1, Q_2\rangle \equiv Q_1^M Q_2^N \mathbb{C}_{MN} = -\langle Q_2, Q_1\rangle. \tag{2.2}$$

(2) $\mathbf{R}$ admits a unique rank-4 completely symmetric primitive G-invariant structure, usually named K-tensor

$$\exists!\mathbb{K}_{(MNPQ)} \equiv \mathbf{1} \in [\mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}]_s; \tag{2.3}$$

thus, by contracting the K-tensor with the same charge vector Q in $\mathbf{R}$, one can construct a rank-4 homogeneous G-invariant polynomial, named $\mathcal{I}_4$:

$$\mathcal{I}_4(Q) \equiv \frac{1}{2}\mathbb{K}_{MNPQ}Q^M Q^N Q^P Q^Q, \tag{2.4}$$

which corresponds to the evaluation of the rank-4 symmetric form $\mathbf{q}$ induced by the K-tensor on four identical modules $\mathbf{R}$:

$$\begin{aligned}\mathcal{I}_4(Q) &= \frac{1}{2}\,\mathbf{q}(Q_1, Q_2, Q_3, Q_4)|_{Q_1=Q_2=Q_3=Q_4\equiv Q} \\ &\equiv \frac{1}{2}\left[\mathbb{K}_{MNPQ}Q_1^M Q_2^N Q_3^P Q_4^Q\right]_{Q_1=Q_2=Q_3=Q_4\equiv Q}. \end{aligned} \tag{2.5}$$

A famous example of quartic invariant in $G = E_7$ is the Cartan–Cremmer–Julia invariant [28], constructed out of the fundamental irrep. $\mathbf{R} = \mathbf{56}$.

(3) If a trilinear map $T\colon \mathbf{R}\times\mathbf{R}\times\mathbf{R}\to\mathbf{R}$ is defined such that

$$\langle T(Q_1, Q_2, Q_3), Q_4\rangle = \mathbf{q}(Q_1, Q_2, Q_3, Q_4), \tag{2.6}$$

then it holds that

$$\langle T(Q_1, Q_1, Q_2), T(Q_2, Q_2, Q_2)\rangle = \langle Q_1, Q_2\rangle\,\mathbf{q}(Q_1, Q_2, Q_2, Q_2). \tag{2.7}$$

This last property makes the group of type E_7 amenable to a treatment in terms of (rank-3) Jordan algebras and related Freudenthal triple systems.

Remarkably, groups of type E_7, appearing in $D = 4$ supergravity as U-duality groups, admit a $D = 5$ uplift to groups of type E_6, as well as a $D = 3$ downlift to groups of type E_8; see [29]. It should also be recalled that split form of exceptional Lie groups appear in the exceptional Cremmer–Julia [30] sequence $E_{D(D)}$ of U-duality groups of M-theory compactified on a D-dimensional torus, in $D = 3, 4, 5$.

It is intriguing to notice that the first paper on groups of type E_7 was written about a decade before the discovery of extended ($N = 2$) supergravity [31], in which electromagnetic duality symmetry was observed [32]. The connection of groups of type E_7 to supergravity can be summarized by stating that all $2 \leq N \leq 8$-extended supergravities in $D = 4$ with symmetric scalar manifolds G/H have G of type E_7 [33,34], with the exception of $N = 2$ group $G = U(1, n)$ and $N = 3$ group $G = U(3, n)$. These latter in fact have a quadratic invariant Hermitian form $\left(Q_1, \bar{Q}_2\right)$, whose imaginary part is the symplectic (skew-symmetric) product and whose real part is the symmetric quadratic invariant $\mathcal{I}_2(Q)$ defined as follows:

$$\mathcal{I}_2(Q) \equiv [\mathrm{Re}(Q_1, \bar{Q}_2)]_{Q_1=Q_2}; \tag{2.8}$$

$$\langle Q_1, \bar{Q}_2\rangle = -\mathrm{Im}(Q_1, \bar{Q}_2). \tag{2.9}$$

Thus, the fundamental representations of pseudounitary groups $U(p, n)$, which have a Hermitian quadratic invariant form, do not strictly qualify for groups of type E_7.

In theories with groups of type E_7, the Bekenstein–Hawking black hole entropy is given by

$$S = \pi\sqrt{|\mathcal{I}_4(Q)|}, \tag{2.10}$$

as it was proved for the case of $G = E_{7(7)}$ (corresponding to $N = 8$ supergravity) in [35]. For $N = 2$ group $G = U(1, n)$ and $N = 3$ group $G = U(3, n)$ the analogue of (2.10) reads as

$$S = \pi\,|\mathcal{I}_2(Q)|\,. \tag{2.11}$$

Table 1. $N \geq 3$ supergravity sequence of groups G of the corresponding G/H symmetric spaces, and their symplectic representations $\mathbf{R}$.

N	G	$\mathbf{R}$
3	$U(3, n)$	$(\mathbf{3}+\mathbf{n})$
4	$SL(2, \mathbb{R}) \times \mathrm{SO}(6, n)$	$(\mathbf{2}, \mathbf{6}+\mathbf{n})$
5	$SU(1,5)$	$\mathbf{20}$
6	$SO^*(12)$	$\mathbf{32}$
8	$E_{7(7)}$	$\mathbf{56}$

Table 2. $N = 2$ choices of groups G of the G/H symmetric spaces and their symplectic representations $\mathbf{R}$. The last four lines refer to 'magic' $N = 2$ supergravities.

G	$\mathbf{R}$
$U(1, n)$	$(\mathbf{1}+\mathbf{n})$
$SL(2, \mathbb{R}) \times \mathrm{SO}(2, n)$	$(\mathbf{2}, \mathbf{2}+\mathbf{n})$
$SL(2, \mathbb{R})$	$\mathbf{4}$
$Sp(6, \mathbb{R})$	$\mathbf{14}'$
$SU(3,3)$	$\mathbf{20}$
$SO^*(12)$	$\mathbf{32}$
$E_{7(-25)}$	$\mathbf{56}$

For $3 < N \leq 8$ the following groups of type E_7 are relevant: $E_{7(7)}$, $SO^*(12)$, $SU(1,5)$, $SL(2,\mathbb{R}) \times SO(6,n)$ (see table 1). In $N = 2$ cases of symmetric vector multiplets' scalar manifolds, there are six groups of type E_7 [36]: $E_{7(-25)}$, $SO^*(12)$, $SU(3,3)$, $Sp(6,\mathbb{R})$, $SL(2,\mathbb{R})$, and $SL(2,\mathbb{R}) \times SO(2,n)$ (see table 2). Here n is the integer describing the number of matter (vector) multiplets for $N = 4, 3, 2$.

3. Orbits

Here we report some results on the stratification of the $\mathbf{R}$ irrep. space of simple groups G of type E_7 (for a recent account, with a detailed list of references, see [37]).

In supergravity, this corresponds to U-duality invariant constraints defining the charge orbits of a single-centred extremal black hole, namely of the G-invariant conditions defining the 'rank' of the dyonic charge vector Q (1.7) in $\mathbf{R}$ as an element of the corresponding Freudenthal triple system (FTS) (see [38,39], and references therein). The symplectic indices $M = 1, \ldots, \mathbf{f}$ ($\mathbf{f} \equiv \dim_{\mathbb{R}} \mathbf{R}\,(G)$) are raised and lowered with the symplectic metric $\mathbb{C}_{MN}$ defined by (2.1). By recalling the definition (2.4) of the unique primitive rank-4 G-invariant polynomial constructed with Q in $\mathbf{R}$, the 'rank' of a non-null Q as an element of FTS(G) ranges from four to one, and it is manifestly G-invariantly characterized as follows:

(1) rank $(Q) = 4$. This corresponds to 'large' extremal black holes, with non-vanishing area of the event horizon (exhibiting attractor mechanism [3,4]):

$$\mathcal{I}_4(Q) < 0 \qquad \text{or} \qquad \mathcal{I}_4(Q) > 0. \tag{3.1}$$

(2) rank $(Q) = 3$. This corresponds to 'small' light-like extremal black holes, with vanishing area of the event horizon:

$$\mathcal{I}_4(Q) = 0, \qquad T(Q, Q, Q) \neq 0. \tag{3.2}$$

(3) rank $(Q) = 2$. This corresponds to 'small' critical extremal black holes:

$$T(Q, Q, Q) = 0, \qquad 3T(Q, Q, P) + \langle Q, P\rangle\, Q \neq 0. \tag{3.3}$$

(4) rank $(Q) = 1$. This corresponds to 'small' doubly-critical extremal BHs [40–42]:

$$3T(Q, Q, P) + \langle Q, P\rangle Q = 0,\ \forall P \in \mathbf{R}. \tag{3.4}$$

Let us consider the doubly-critical condition (3.4) more in detail. At least for simple groups of type E_7, the following holds:

$$\mathbf{R} \times_s \mathbf{R} = \mathbf{Adj} + \mathbf{S}, \tag{3.5}$$

$$\mathbf{R} \times_a \mathbf{R} = \mathbf{1} + \mathbf{A}, \tag{3.6}$$

where $\mathbf{S}$ and $\mathbf{A}$ are suitable irreps. For example, for $G = E_7$ ($\mathbf{R} = \mathbf{56}$, $\mathbf{Adj} = \mathbf{133}$) one gets (see [43])

$$(\mathbf{56} \times \mathbf{56})_s = \mathbf{133} + \mathbf{1463}; \tag{3.7}$$

$$(\mathbf{56} \times \mathbf{56})_a = \mathbf{1} + \mathbf{1539}. \tag{3.8}$$

For such groups, one can construct the projection operator on $\mathbf{Adj}(G)$:

$$\mathcal{P}_{AB}{}^{CD} = \mathcal{P}_{(AB)}{}^{(CD)}; \tag{3.9}$$

$$\mathcal{P}_{AB}{}^{CD}\frac{\partial^2\mathcal{I}_4}{\partial Q^C\partial Q^D} = \left.\frac{\partial^2\mathcal{I}_4}{\partial Q^A\partial Q^B}\right|_{\mathbf{Adj}(G)}; \tag{3.10}$$

$$\mathcal{P}_{AB}{}^{CD}\mathcal{P}_{CD}{}^{EF}\frac{\partial^2\mathcal{I}_4}{\partial Q^E\partial Q^F} = \mathcal{P}_{AB}{}^{EF}\frac{\partial^2\mathcal{I}_4}{\partial Q^E\partial Q^F}, \tag{3.11}$$

where (recall (3.5))

$$\frac{\partial^2\mathcal{I}_4}{\partial Q^A\partial Q^B} = \left.\frac{\partial^2\mathcal{I}_4}{\partial Q^A\partial Q^B}\right|_{\mathbf{Adj}(G)} + \left.\frac{\partial^2\mathcal{I}_4}{\partial Q^A\partial Q^B}\right|_{\mathbf{S}(G)}; \tag{3.12}$$

$$\left.\frac{\partial^2\mathcal{I}_4}{\partial Q^A\partial Q^B}\right|_{\mathbf{Adj}(G)} = 2\,(1-\tau)\,(3\mathbb{K}_{ABCD} + \mathbb{C}_{AC}\mathbb{C}_{BD})Q^CQ^D; \tag{3.13}$$

$$\left.\frac{\partial^2\mathcal{I}_4}{\partial Q^A\partial Q^B}\right|_{\mathbf{S}(G)} = 2[3\tau\mathbb{K}_{ABCD} + (\tau-1)\,\mathbb{C}_{AC}\mathbb{C}_{BD}]Q^CQ^D, \tag{3.14}$$

where $\tau \equiv 2\mathbf{d}/\,[\mathbf{f}\,(\mathbf{f}+1)]$, $\mathbf{d} \equiv \dim_{\mathbb{R}}(\mathbf{Adj}(G))$. The explicit expression of $\mathcal{P}_{AB}{}^{CD}$ reads (for related results in terms of a map formulated in the '$4D/5D$ special coordinates' symplectic frame (and thus manifestly covariant under the $d = 5$ U-duality group G_5), see [44,45]) ($\alpha = 1, \ldots, \mathbf{d}$):

$$\mathcal{P}_{AB}{}^{CD} = \tau \left(3\mathbb{C}^{CE}\mathbb{C}^{DF}\mathbb{K}_{EFAB} + \delta^{C}_{(A}\delta^{D}_{B)}\right) = -t^{\alpha|CD}t_{\alpha|AB}, \tag{3.15}$$

where the relation [46] (see also [47])

$$\mathbb{K}_{MNPQ} = -\frac{1}{3\tau}t^{\alpha}_{(MN}t_{\alpha|PQ)} = -\frac{1}{3\tau}[t^{\alpha}_{MN}t_{\alpha|PQ} - \tau\mathbb{C}_{M(P}\mathbb{C}_{Q)N}], \tag{3.16}$$

where

$$t^{\alpha}_{MN} = t^{\alpha}_{(MN)}; \qquad t^{\alpha}_{MN}\mathbb{C}^{MN} = 0 \tag{3.17}$$

is the symplectic representation of the generators of the Lie algebra $\mathfrak{g}$ of G. Notice that $\tau < 1$ is nothing but the ratio of the dimensions of the adjoint **Adj** and rank-2 symmetric $\mathbf{R} \times_s \mathbf{R}$ (3.5) reps. of G, or equivalently the ratio of upper and lower indices of t^{α}_{MN}'s themselves.

4. Special Kähler geometry and *'generalization'* of groups of type E_7

Here we would like to discuss the characterization of special Kähler geometry (SKG) in terms of a suitable 'generalization' of the groups of type E_7, recently proposed in [48] (for some preliminary discussion, see also §4 of [49]).

As obtained in [50] (see eq. (5.36) therein), the following real function, which we dub 'entropy functional', can be defined on the vector multiplets' scalar manifold (note that the expression (4.1) is independent of the choice of the symplectic frame and manifestly invariant under diffeomorphisms in **M**) **M**:

$$\begin{aligned}\mathbb{I}_4 = {} & \left(|Z|^2 - Z_i\bar{Z}^i\right)^2 + \frac{2}{3}i(Z\bar{C}_{\bar{i}\bar{j}\bar{k}}Z^{\bar{i}}Z^{\bar{j}}Z^{\bar{k}} - \bar{Z}C_{ijk}\bar{Z}^i\bar{Z}^j\bar{Z}^k) \\ & -g^{i\bar{i}}C_{ijk}\bar{C}_{\bar{i}\bar{l}\bar{m}}\bar{Z}^j\bar{Z}^kZ^{\bar{l}}Z^{\bar{m}}.\end{aligned} \tag{4.1}$$

$Z_i \equiv D_iZ$ are the so-called 'matter charges' (D_i stands for the Kähler-covariant differential operator; see [51] and [52] for notation and further elucidation):

$$Z \equiv Q^MV^N\mathbb{C}_{MN}; \qquad Z_i \equiv Q^MV_i^N\mathbb{C}_{MN}, \tag{4.2}$$

with V^M denoting the vector of covariantly-holomorphic symplectic sections of SKG, and $V_i^M \equiv D_iV^M$. Furthermore, C_{ijk} is the rank-3, completely symmetric, covariantly holomorphic tensor of SKG (with Kähler weights $(2, -2)$) (see [53,54]):

$$\begin{aligned}& C_{ijk} \equiv \mathbb{C}_{MN}(D_iD_jV^M)D_kV^N = -ig_{i\bar{l}}\bar{f}^{\bar{l}}_{\Lambda}D_jD_kL^{\Lambda} = D_iD_jD_k\mathcal{S} = \mathrm{e}^KW_{ijk}; \\ & \bar{f}^{\bar{l}}_{\Lambda}\left(\bar{D}\bar{L}^{\Lambda}_{\bar{s}}\right) \equiv \delta^{\bar{l}}_{\bar{s}}, \qquad \mathcal{S} \equiv -iL^{\Lambda}L^{\Sigma}\mathrm{Im}(F_{\Lambda\Sigma}), \qquad \bar{\partial}_{\bar{l}}W_{ijk} = 0; \\ & \bar{D}_{\bar{i}}C_{jkl} = 0; \\ & D_{[i}C_{j]kl} = 0,\end{aligned} \tag{4.3}$$

the last property being a consequence, through the covariant holomorphicity of C_{ijk} and the SKG constraint on the Riemann tensor (see [53–55])

$$R_{j\bar{k}l\bar{m}} = -g_{j\bar{k}}g_{l\bar{m}} - g_{j\bar{m}}g_{l\bar{k}} + g^{i\bar{i}}C_{ijl}\bar{C}_{\bar{i}\bar{k}\bar{m}}, \tag{4.4}$$

of the Bianchi identities satisfied by the Riemann tensor $R_{i\bar{j}k\bar{l}}$.

Furthermore, $\mathbb{I}_4$ is an order-4 homogeneous polynomial in the fluxes $\mathcal{Q}$; this allows for the definition of the Q-independent rank-4 completely symmetric tensor Ω_{MNPQ} [49], whose general expression is explicitly computed here:

$$\Omega_{MNPQ} \equiv 2\frac{\partial^4\mathbb{I}_4}{\partial Q^{(M}\partial Q^N\partial Q^P\partial Q^{Q)}} \tag{4.5}$$

$$\begin{aligned} &= 2V_{(M}V_N\bar{V}_P\bar{V}_{Q)} + 2V_{i|(M}\bar{V}^i_N V_{j|P}\bar{V}^j_{Q)} - 4V_{(M}\bar{V}_N V_{i|P}\bar{V}^i_{Q)} \\ &\quad + \frac{2}{3}\left(V_{(M}V^{\bar{i}}_N V^{\bar{j}}_P \bar{D}_{\bar{i}}\bar{V}_{\bar{j}|Q} + \bar{V}_{(M}\bar{V}^i_N\bar{V}^j_P D_i V_{j|Q}\right) \\ &\quad -2g^{i\bar{i}}\bar{V}^j_{(M}V^{\bar{l}}_N D_i V_{j|N}\bar{D}_{\bar{i}}\bar{V}_{\bar{l}|Q)}, \end{aligned} \tag{4.6}$$

where the SKG defining relation (see [53–55])

$$D_i D_j V^M \equiv D_i V^M_j = iC_{ijk}\bar{V}^{k|M} \tag{4.7}$$

has been used in order to recast (4.5) in terms of V^M, V^M_i and $D_i V^M_j$ only.

Some further elaborations are possible. For e.g., by using (4.4), $\mathbb{I}_4$, (4.1) and Ω_{MNPQ}, eq. (4.6) can respectively be rewritten as

$$\begin{aligned} \mathbb{I}_4 = {} & |Z|^4 - (Z_i\bar{Z}^i)^2 - 2|Z|^2 Z_i\bar{Z}^i \\ & + \frac{2}{3}i\left(Z\bar{C}_{\bar{i}\bar{j}\bar{k}}Z^{\bar{i}}Z^{\bar{j}}Z^{\bar{k}} - \bar{Z}C_{ijk}\bar{Z}^i\bar{Z}^j\bar{Z}^k\right) - \mathcal{R}; \end{aligned} \tag{4.8}$$

$$\begin{aligned} \Omega_{MNPQ} = {} & 2V_{(M}V_N\bar{V}_P\bar{V}_{Q)} - 2V_{i|(M}\bar{V}^i_N V_{j|P}\bar{V}^j_{Q)} - 4V_{(M}\bar{V}_N V_{i|P}\bar{V}^i_{Q)} \\ & + \frac{2}{3}\left(V_{(M}V^{\bar{i}}_N V^{\bar{j}}_P \bar{D}_{\bar{i}}\bar{V}_{\bar{j}|Q} + \bar{V}_{(M}\bar{V}^i_N\bar{V}^j_P D_i V_{j|Q}\right) - R_{MNPQ}, \end{aligned} \tag{4.9}$$

where the sectional curvature of matter charges (eq. (5.3) of [56]; also note that (4.10) is different from the definition given by eq. (3.1.1.2.11) of [57]))

$$\mathcal{R} \equiv R_{i\bar{j}k\bar{l}}\bar{Z}^i Z^{\bar{j}}\bar{Z}^k Z^{\bar{l}}, \tag{4.10}$$

and the corresponding rank-4 completely symmetric tensor

$$R_{MNPQ} \equiv \frac{\partial^4\mathcal{R}}{\partial Q^{(M}\partial Q^N\partial Q^P\partial Q^{Q)}} = R_{i\bar{j}k\bar{l}}\bar{V}^i_{(M}V^{\bar{j}}_N\bar{V}^k_P V^{\bar{l}}_{Q)}, \tag{4.11}$$

has been introduced. Note that R_{MNPQ} can be regarded as the completely symmetric part of the 'symplectic pull-back' (through the symplectic sections V^M_i) of the Riemann tensor $R_{i\bar{j}k\bar{l}}$ of **M**.

Thus, SKG can be associated with a generalization of the class of groups of type E_7, based on $\mathbb{I}_4$ and the corresponding (generally field-dependent, non-constant) Ω-structure:

$$\text{SKG}\ :\ \begin{cases} \Omega_{MNPQ} : D_i\Omega_{MNPQ} = \partial_i\Omega_{MNPQ} \neq 0; \\ \\ \mathbb{I}_4 \equiv \frac{1}{2}\Omega_{MNPQ}Q^M Q^N Q^P Q^Q \Rightarrow D_i\mathbb{I}_4 = \partial_i\mathbb{I}_4 \neq 0. \end{cases} \tag{4.12}$$

Symmetric Kähler spaces have a covariantly constant Riemann tensor:

$$D_i R_{j\bar{k}l\bar{m}} = 0. \tag{4.13}$$

Within SKG, through the constraint (4.4), this implies the covariant constancy of the C-tensor (4.3):

$$D_{(i} C_{j)kl} = D_{(i} C_{jkl)} = 0, \tag{4.14}$$

which in turn yields the relation:

$$C_{p(kl} C_{ij)n} g^{n\bar{n}} g^{p\bar{p}} \bar{C}_{\bar{n}\bar{p}\bar{m}} = \frac{4}{3} g_{(l|\bar{m}} C_{|ijk)} \Leftrightarrow g^{n\bar{n}} R_{(i|\bar{m}|j|\bar{n}} C_{n|kl)} = -\frac{2}{3} g_{(i|\bar{m}} C_{|jkl)}. \tag{4.15}$$

Equivalently, symmetric SK manifolds can be characterized by stating that their Ω_{MNPQ} is independent of the scalar fields themselves, and it matches the $\mathbb{K}$-tensor $\mathbb{K}_{MNPQ}$ defining the rank-4 invariant $\mathbb{K}$-structure of the corresponding U-duality group of type E_7 [27] (see also [46], and references therein). Consequently, the corresponding 'entropy functional' $\mathbb{I}_4$ (4.1) is independent of the scalar fields themselves, and it is thus a constant function in $\mathbf{M}$, given by the unique algebraically-independent 1-centred U-duality invariant polynomial I_4:

$$\underset{(U\text{-duality group } G \text{ is of type } E_7)}{\text{symmetric SKG}} \Rightarrow \begin{cases} \Omega_{MNPQ} = \mathbb{K}_{MNPQ} \Rightarrow D_i \Omega_{MNPQ} = \partial_i \Omega_{MNPQ} = 0; \\ \\ \mathbb{I}_4 = \mathcal{I}_4 \equiv \frac{1}{2} \mathbb{K}_{MNPQ} Q^M Q^N Q^P Q^Q \Rightarrow D_i \mathbb{I}_4 = \partial_i \mathbb{I}_4 = 0. \end{cases} \tag{4.16}$$

In turn, within symmetric SKG, the pseudounitary U-duality group $U(1,s)$ (corresponding to $N = 2$ minimally coupled Maxwell–Einstein theory [58,59]) is 'degenerate', in the aforementioned sense that the corresponding $\mathcal{I}_4$ actually is the square of the order-2 $U(1,s)$-invariant polynomial $\mathcal{I}_2$. Indeed, $N = 2$ minimally coupled supergravity is characterized by $C_{ijk} = 0$, which plugged into (4.1) (by taking (4.16) into account) yields:

$$\underset{G=U(1,s)}{\text{symmetric SKG}} \Rightarrow \begin{cases} \Omega_{MNPQ} = \mathbb{K}_{MNPQ} \Rightarrow D_i \Omega_{MNPQ} = \partial_i \Omega_{MNPQ} = 0; \\ C_{ijk} = 0; \\ \mathbb{I}_4 = \mathcal{I}_4 = (|Z|^2 - Z_i \bar{Z}^i)^2 = \frac{1}{4} \mathcal{I}_2^2 \Rightarrow D_i \mathbb{I}_4 = \partial_i \mathbb{I}_4 = 0, \end{cases} \tag{4.17}$$

where the normalization of [60] (see eq. (2.15) therein) has been adopted.

We conclude by recalling that, as noticed in [50] and [49], the 'entropic functional' $\mathbb{I}_4$ (4.1) is related to the geodesic potential defined in the $D = 4 \to 3$ dimensional reduction of the considered $N = 2$ theory. Under such a reduction, the $D = 4$ vector multiplets' SK manifold $\mathbf{M}$ ($\dim_{\mathbb{C}} = n_V$) enlarges to a special quaternionic Kähler manifold $\mathfrak{M}$ ($\dim_{\mathbb{H}} = n_V + 1$) given by c-map [61,62] of $\mathbf{M}$ itself: $\mathfrak{M} = c(\mathbf{M})$. By specifying eq. (4.1) in the '$4D/5D$ special coordinates' symplectic frame, $\mathbb{I}_4$ matches the opposite of the function h defined by eq. (4.42) of [63], within the analysis of special quaternionic Kähler geometry. This relation can be strengthened by observing that the tensor Ω_{MNPQ} given by (4.5)–(4.6) is proportional to the Ω-tensor of quaternionic geometry, related to the quaternionic Riemann tensor by eq. (15) of [64] (for further comments, see [49]).

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