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The exact solutions for the interaction $V(r) = \alpha r^{2d-2} - \beta r^{d-2}$ by Nikiforov–Uvarov method

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Abstract. The exact solutions for the two- and N-dimensional Schrödinger equation have been rederived for the potential $V(r) = \alpha r^{2d-2} - \beta r^{d-2}$ by Nikiforov–Uvarov method. Specific results are presented for (i) the hydrogen atom and (ii) an isotropic harmonic oscillator. The dimensionality of the problem is seen to enter into these relations in such a way that one can immediately verify the corresponding three-dimensional results. The local accidental degeneracies are also explained for the two- and N-dimensional problems.

Keywords. Nikiforov–Uvarov (NU) method; exact solutions for two- and N-dimensional problem; separate case study; local accidental degeneracies.

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1. Introduction

Many of the physical phenomena of nature are characterized by some basic differential equations. For example, quantum mechanical phenomena are described by Schrödinger's equation, which dictates the dynamics of some quantum systems represented by a Hamiltonian operator. One is primarily interested in finding all eigenvalues and eigenstates of such Hamiltonians. As a consequence, finding a large class of analytically exactly solvable quantum systems is an important goal and this search has already been initiated by Schrödinger using the factorization method [1,2].

Recently, there has been renewed interest in solving simple quantum mechanical systems within the framework of the Nikiforov–Uvarov (NU) method [3]. This algebraic technique is based on solving the second-order linear differential equation which has been used successfully to solve Schrödinger, Dirac, Klein–Gordon and Duffin–Kemmer–Petiau wave equation in the presence of some well-known central and non-central potentials [4–13].

In this paper, we focus our attentions to deal with the Hamiltonians

$$H = \frac{1}{2M} P^2 + \alpha r^{2d-2} - \beta r^{d-2}, \qquad \alpha, \beta > 0, \tag{1}$$

where $P^2 = -\Delta^2$, $r = |\mathbf{r}|$ and d is a positive rational number, have been shown [14,15] to admit local accidental degeneracies in all dimensions: For various choices of the parameters, there are more bound states with energy $E = 0$ than expected from rotational invariance. Mayrand and Vinet [16] have provided a complete group-theoretic analysis of the quantum dynamics in two dimensions and, in particular, have observed that the manifest $O(2)$ rotational invariance of the eigenvalue equation $H\psi = E\psi$, is enlarged to an $SU(2)$ symmetry on the null space of H. They also have shown that the $E = 0$ degenerate solutions describing bound states are connected to each other through the action of $SU(2)$ generators and parity transformation thereby explaining the local accidental degeneracy.

In the present work, our main objective is to solve Schrödinger equation by NU [3] method which is an alternative treatment [17–19] of the Schrödinger equation for the potential $V(r) = \alpha r^{2d-2} - \beta r^{d-2}$ in two and also N dimensions at $E = 0$ eigenvectors. We have also studied the accidental degeneracies of the zero energy levels for this potential in two and N dimensions.

This paper is organized as follows: After a brief introductory discussion of the NU method in §2, we obtain the eigenvalues and eigenfunctions for the potential $V(r) = \alpha r^{2d-2} - \beta r^{d-2}$ for two dimensions in §3 and that for N dimensions in §4 and finally conclusions have been drawn in §5.

2. Basic equations of Nikiforov–Uvarov method

The NU [3] method is based on reducing the second-order differential equation to a generalized equation of hypergeometric type. In this sense, the Schrödinger equation, after employing an appropriate coordinate transformation $s = s(r)$, transforms to the following form:

$$\psi_n''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi_n'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s) = 0, \tag{2}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most second degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. To find the particular solution of eq. (2), one can use the following transformation as

$$\psi_n(s) = \phi_n(s) y_n(s) \tag{3}$$

leading to a hypergeometric-type equation such as

$$\sigma(s) y_n''(s) + \tau(s) y_n'(s) + \lambda y_n(s) = 0, \tag{4}$$

where

$$\sigma(s) = \pi(s) \frac{\phi_n(s)}{\phi_n'(s)}, \tag{5}$$

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s), \qquad \tau'(s) < 0. \tag{6}$$

Here, the most significant point at this stage is that prime factor of $\tau(s)$ shows the differentials at first degree and must be negative to reproduce physically acceptable λ-values which is defined as

$$\lambda = \lambda_n = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s), \quad n = 0, 1, 2, 3, ... \tag{7}$$

It is to be noted that λ or λ_n is obtained from a particular solution of the form $y_n(s)$ which is a polynomial of degree n and satisfies the Rodrigues relation [20]

$$y_n(s) = \frac{B_n}{\rho(s)} \frac{\mathrm{d}^n}{\mathrm{d}s^n} \left(\sigma^n(s)\rho(s)\right). \tag{8}$$

In this equation, B_n is the normalization constant and the weight function $\rho(s)$ satisfies the condition

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\sigma(s)\rho(s)\right) = \tau(s)\rho(s). \tag{9}$$

Here, the function $\pi(s)$ and the parameter λ are defined as

$$\pi(s) = \frac{\sigma' - \tilde{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2}\right)^2 - \tilde{\sigma} + k\sigma}, \tag{10}$$

$$\lambda = \lambda_n = k + \pi'(s), \tag{11}$$

where $\pi(s)$ obviously is a polynomial depending on the transformation function $s(r)$ and the determination of k is the essential point in the calculation of $\pi(s)$, for which the discriminant of the square root in eq. (10) is set to zero so that, the expression of $\pi(s)$ becomes the square of a polynomial of first degree. In this case, an equation for k is obtained. After solving this equation of k, the obtained values of k are substituted in eq. (11) to find the values of λ. Then, by comparing eqs (7) and (11), one can obtain the values of λ_n.

It is well known that many special functions of mathematics represent solutions to differential equations of the form in eq. (2) where the function $\tilde{\tau}/\sigma$ and $\tilde{\sigma}/\sigma^2$ are well defined for any particular function [21]. Bearing this in mind, we proceed first with a transformation of Schrödinger equation to the one similar to eq. (2).

3. Eigenfunctions and eigenvalues in two dimensions

In two dimensions, the Hamiltonian given in eq. (1) reads in polar coordinates as

$$H = -\frac{1}{2}\left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2\right) + V(r), \tag{12}$$

where

$$V(r) = \alpha r^{2d-2} - \beta r^{d-2}. \tag{13}$$

(We have set the mass $\hbar = M = 1$). We shall be concerned with the zero-energy eigenspace of this operator, that is with the set of the solution to

$$H|\psi\rangle = 0. \tag{14}$$

Let us point out immediately that eq. (14) encompasses two familiar problems, i.e. the Schrödinger equations for particles in a Coulomb or harmonic oscillator potential. Indeed for $d = 1$, eq. (14) reduces to

$$\left(-\frac{1}{2}\Delta^2 - \frac{\beta}{r} \right)|\psi\rangle = -\alpha|\psi\rangle, \tag{15}$$

while for $d = 2$, it becomes

$$\left(-\frac{1}{2}\Delta^2 + \alpha r^2 \right)|\psi\rangle = \beta|\psi\rangle. \tag{16}$$

In each of these cases, one of the coupling constants plays the role of energy eigenvalue. Separation of variables is achieved by supplementing eq. (14) with

$$L|\psi\rangle = m|\psi\rangle, \qquad m \in Z, \tag{17}$$

where $L = -i\partial_\theta$ is the angular momentum operator. The single valuedness of $|\psi\rangle$ requires m to be an integer.

In order to apply the NU method, we rewrite eq. (14) using

$$|n, m\rangle = \psi_{n,m} = R_n(r)\mathrm{e}^{im\theta} \tag{18a}$$

and a new variable of the form $s = -\sqrt{(2\alpha/d)}r^d$, like

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}R_n(s) + \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}s}R_n(s) + \frac{1}{\mathrm{d}^2 s^2}\left(\frac{2\beta d}{\sqrt{2\alpha}}s - \mathrm{d}^2 s^2 - m^2\right)R_n(s) = 0 \tag{18b}$$

which leads to a hypergeometric-type equation. After comparing of eq. (18b) with eqs (2) and (3), we obtain the following definitions:

$$R_n(s) = \phi_n(s)y_n(s), \tag{19a}$$

and

$$\tilde{\tau}(s) = d, \quad \sigma(s) = sd, \quad \tilde{\sigma}(s) = \frac{2\beta d}{\sqrt{2\alpha}}s - \mathrm{d}^2 s^2 - m^2. \tag{19b}$$

Substituting these values into eq. (10), we obtain $\pi(s)$ function as

$$\pi(s) = \pm\sqrt{\mathrm{d}^2 s^2 + \left(kd - \frac{2\beta d}{\sqrt{2\alpha}}\right)s + m^2}. \tag{20}$$

To find the value of k, the discriminant of eq. (20) under the square root has to be zero, so that the expression becomes the square of a polynomial of first degree. For this, we put

$$\mathrm{d}^2 s^2 + \left(kd - \frac{2\beta d}{\sqrt{2\alpha}}\right)s + m^2 = 0. \tag{21}$$

Now, solving eq. (21) we obtain the values of s as

$$s = \frac{1}{2d^2}\left(-\left(kd - \frac{2\beta d}{\sqrt{2\alpha}}\right) \pm \sqrt{\left(kd - \frac{2\beta d}{\sqrt{2\alpha}}\right)^2 - 4m^2 d^2} \right). \tag{22}$$

Again, for our purpose the value of k will be obtained by solving

$$\left(kd - \frac{2\beta d}{\sqrt{2\alpha}}\right)^2 - 4m^2d^2 = 0. \tag{23}$$

and we get the double roots of k from eq. (23) as

$$k_{\pm} = \frac{2\beta d}{\sqrt{2\alpha}} \pm 2|m|. \tag{24}$$

Substituting the acceptable value k_- into eq. (20), the acceptable value of $\pi(s)$ is obtained as

$$\pi_-(s) = -sd + |m|. \tag{25}$$

Here, we select $\pi_-(s)$ for which the function $\tau(s)$ in eq. (6) has a negative derivative. Therefore, the function $\tau(s)$ satisfies these requirements, with

$$\tau_-(s) = d - 2sd + 2|m|, \qquad \tau'(s) = -2d. \tag{26}$$

By putting the values of $\tau'_-(s), \sigma''(s), \pi'_-(s)$ and k_- into eqs (7) and (11), we get the values of parameter λ

$$\lambda_n = 2nd \tag{27}$$

and

$$\lambda = \frac{2\beta d}{\sqrt{2\alpha}} - 2|m| - d. \tag{28}$$

From eqs (27) and (28), we get a quantization condition like

$$\frac{\beta}{d\sqrt{2\alpha}} - \frac{d+2|m|}{2d} = n, \qquad n = 0, 1, 2, 3, ...; m = 0, \pm1, \pm2, \pm3, ... \tag{29}$$

Let us now find the corresponding eigenfunctions for this potential. Due to the NU method, the polynomial solutions of the hypergeometric function $y_n(s)$ depends on the determination of the weight function $\rho(s)$ which satisfies the differential equation (9). Putting the values of $\tau(s)$ and $\sigma(s)$ in eq. (9) and solving it, we obtain the weight function $\rho(s)$ as

$$\rho(s) = \frac{1}{d} s^{2|m|/d} \mathrm{e}^{-2s}. \tag{30}$$

Substituting the values of $\sigma(s)$ and $\rho(s)$ into the Rodrigues relation given in eq. (8), the polynomial solution $y_n(s)$ of eq. (4) is obtained in the following form:

$$y_n(s) = B_{nl} \mathrm{d} s^{-2|m|/d} \mathrm{e}^{2s} \frac{\mathrm{d}^n}{\mathrm{d}s^n}\left(s^{n+(2|m|/d)}\mathrm{e}^{-2s}\right), \tag{31}$$

where B_n is the normalization constant. Again, the polynomial solution $y_n(s)$ given in eq. (31) can also be expressed in terms of the associated Laguerre polynomials as

$$y_n(s) = B\mathrm{d}^{n+1} n! L_n^{2|m|/d}(2s). \tag{32}$$

By substituting $\pi_-(s)$ and $\sigma(s)$ in eq. (5) and solving it, we get $\phi_n(s)$ as

$$\phi_n(s) = s^{|m|/d} \mathrm{e}^{-s}. \tag{33}$$

Again substituting the values of the polynomial solution y_n and $\phi_n(s)$ in eq. (19a), we get the radial solution of eq.(18b) as

$$R_n(s) = A_n s^{|m|/d} \mathrm{e}^{-s} L_n^{2|m|/d}(2s), \tag{34}$$

where A_n is the normalization constant. An orthonormalized basis of wave functions for the bound states with energy $E_{n,m} = 0$ can be obtained by substituting the values of $R_n(s)$ in eq. (18a) as

$$|n, m\rangle = \psi_{n,m} = C_{n,m} \mathrm{e}^{im\theta} r^{|m|} \mathrm{e}^{(-\sqrt{2\alpha}/d r^d)} L_n^{2|m|/d} \left(2\sqrt{\frac{2\alpha}{d}} r^d \right), \tag{35}$$

where $C_{n,m}$ is the normalization constant. When eq. (29) is satisfied, the degeneracies among the bound states are observed. Taking $d = d_1/d_2$ with d_1 and d_2 two relatively prime positive integers, eq. (29) can be written in the form $\beta = (1/d_2)\sqrt{2\alpha}(D + (d_1/2))$, with $D = d_1((\beta d_2/d_1\sqrt{2\alpha}) - \frac{1}{2})$ taken to be the non-negative integer $D = nd_1 + |m|d_2$. It is then easy to conclude that the states $|n, m\rangle$ and $|n', m'\rangle$ belong to the same zero-energy eigenspace provided $(n' - n)d_1 = (m - m')d_2$. Actually, the states $....|2n - n', 2m - m'\rangle, |n, m\rangle, |n', m'\rangle, |2n' - n, 2m' - m\rangle, ...$ (if admissible) will then all be degenerate. Restrictions arise from the fact that the quantum numbers n and m must satisfy $0 \le n \le D/d_1, 0 \le |m| \le D/d_2$. It should be stressed that these accidental degeneracies are local; they appear only for the level $E = 0$. In fact, with $d = d_1/d_2$ fixed, it is seen from $\beta = (1/d_2)\sqrt{2\alpha}(D + (d_1/2))$ that different coupling constants α and β, hence different potentials, correspond to different values of D. In the special cases where $d = 1$ and 2, we have already noted that one of the coupling constants can be taken to be an energy eigenvalue. In this alternative interpretation, different D correspond to different energies and accidental degeneracies are then present for every level. This is of course well known to happen for the Coulomb and harmonic oscillator problems; these are, however, the only cases within the class of potentials considered for which such a reinterpretation is possible and global accidental degeneracy seen.

3.1 *The special cases $d = 1$ and 2*

We briefly indicate in this section how the Coulomb and harmonic oscillator problems fit as special cases in our analysis.

3.1.1 *The coulomb potential* ($d = 1$).

For $d = 1$, the Hamiltonian given in eqs (12) and (13) becomes $H = H_{\mathrm{C}} + \alpha$ with

$$H_{\mathrm{C}} = -\frac{1}{2}\left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 \right) - \frac{\beta}{r}. \tag{36}$$

The equation $H|\psi\rangle = 0$ is identified as the Schrödinger equation $H_{\mathrm{C}}|\psi\rangle = E|\psi\rangle$ for a particle moving in a β/r potential in two dimensions [8]. The energy eigenvalue E is $-\alpha$. The spectrum

$$E = -\alpha = -\frac{\beta^2}{2\left(n + |m| + \frac{1}{2}\right)^2} \tag{37}$$

and the wave function

$$\psi_{n,m} = C_{n,m} \mathrm{e}^{im\theta} r^{|m|} \mathrm{e}^{-\sqrt{2\alpha} r} L_n^{2|m|}(2\sqrt{2\alpha} r), \tag{38}$$

where the normalization constant

$$C_{n,m} = (-1)^n \frac{1}{\sqrt{2\pi}} (2\sqrt{2\alpha})^{|m|+1} \sqrt{\frac{\Gamma(n+1)}{(2n+|m|+1)\Gamma(n+2|m|+1)}}. \tag{39}$$

3.1.2 *The harmonic oscillator potential* ($d = 2$). For $d = 2$, the Hamiltonian given in eqs (12) and (13) becomes $H = H_{\text{h.o.}} - \beta$ with

$$H_{\mathrm{C}} = -\frac{1}{2}\left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2\right) + \alpha r^2. \tag{40}$$

In this case, the equation $H|\psi\rangle = 0$ is identified as the Schrödinger equation $H_{\text{h.o.}}|\psi\rangle = E|\psi\rangle$ for a particle moving in a two-dimensional harmonic well [22]. The energy eigenvalue $E = \beta$. The spectrum

$$E = \beta = \sqrt{2\alpha}\left(n + |m| + \frac{1}{2}\right)^2 \tag{41}$$

and the wave function

$$\psi_{n,m} = C_{n,m} \mathrm{e}^{im\theta} r^{|m|} \mathrm{e}^{-\sqrt{2\alpha} r^2} L_n^{2|m|}(2\sqrt{2\alpha} r^2), \tag{42}$$

where the normalization constant

$$C_{n,m} = (-1)^n (2\alpha)^{|m|+1/4} \left(\frac{n!}{\pi(n+|m|)!}\right)^{1/2}. \tag{43}$$

4. Eigenvalues and eigenfunctions in N dimensions

In this article, we use NU method to deal with the N-dimensional ($N \geq 3$) Schrödinger equation and obtain the eigenstate and eigenvalue for the potential given in eq. (13). The extension sought by us, although straightforward, is quite instructive because laws of physics in N spatial dimensions may often lead to insights concerning laws of physics in lower dimensions [23–25].

Consider the motion of a particle of mass M in an N-dimensional Euclidian space. The time-independent Schrödinger equation for any integral dimension is given by [23–25]

$$\left(-\frac{\hbar^2}{2M}\Delta_N^2 + V_N\right)\psi = E\psi. \tag{44}$$

Here, the wavefunction ψ belongs to the energy eigenvalue E and Δ_N^2 and V_N stand for the N-dimensional Laplacian and potential respectively. Investigation of physical processes based on eq. (44) is a well-studied problem and many authors proceed by using the standard central potential $V(r) = \alpha r^{2d-2} - \beta r^{d-2}$ in place of V_N. Here r represents the N-dimensional radius $(\Sigma_i^N x_i^2)^{1/2}$. Going over to a spherical coordinate system with $N - 1$ angular variables and one radial coordinate we can write

$$\psi = R_{n,l}^{(N)}(r) Y_l^M(\theta_i), \tag{45}$$

where $Y_l^M(\cdot)$ represents contributions from the hyperspherical harmonics that arise in higher dimensions. The eigenvalues and eigenfunctions for generalized angular momentum operators in N-dimensional polar coordinates are determined [26] using results known from the factorization method [2]. However, from eqs (44) and (45) we have lth partial-wave radial Schrödinger equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{N-1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{l(l+N-2)}{r^2} - 2V(r)\right) R_{n,l}^{(N)}(r) = 2E_{n,l}^{(N)} R_{n,l}^{(N)}(r). \tag{46}$$

Here we have used $\hbar = M = 1$ and the superscript (N) on the radial function $R_{n,l}^{(N)}(r)$ and the energy eigenvalue $E_{n,l}^{(N)}$ merely stand for the dimensionality of the problem, the subscript n refers to a quantum number, the interpretation of which depends on the choice of $V(r)$.

In order to apply the NU method, we rewrite eq. (46) using a new variable of the form $s = -\sqrt{(2\alpha/d)}r^d$, like

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}s^2}R_{n,l}^{(N)}(s) &+ \frac{d+N-2}{sd}\frac{\mathrm{d}}{\mathrm{d}s}R_{n,l}^{(N)}(s)\\ &+ \frac{1}{\mathrm{d}^2 s^2}\left(\frac{2\beta d}{\sqrt{2\alpha}}s - \mathrm{d}^2 s^2 - l(l+N-2)\right)R_{n,l}^{(N)}(s) = 0\end{aligned} \tag{47}$$

which leads to a hypergeometric-type equation. After comparing eq. (47) with eq. (2), we obtain the following definitions as

$$R_{n,l}^{(N)}(s) = \phi_n(s) y_{n,l}(s), \tag{48a}$$

$$\tilde{\tau}(s) = d+N-2, \quad \sigma(s) = sd, \quad \tilde{\sigma}(s) = \frac{2\beta d}{\sqrt{2\alpha}}s - \mathrm{d}^2 s^2 - l(l+N-2). \tag{48b}$$

Substituting these values in eq. (10), we obtain the π function as

$$\pi(s) = -\frac{N-2}{2} \pm \sqrt{\mathrm{d}^2 s^2 + \left(kd - \frac{2\beta d}{\sqrt{2\alpha}}\right)s + \left(l + \frac{N-2}{2}\right)^2}. \tag{49}$$

As before, to find the values of k, the discriminant of eq. (49) under the square root has to be zero, so that the expression becomes the square of a polynomial of first degree. For this, we must put

$$\left(kd - \frac{2\beta d}{\sqrt{2\alpha}}\right)^2 - 4d^2\left(l + \frac{N-2}{2}\right)^2 = 0. \tag{50}$$

When the required arrangements are done with respect to the constant k, its double roots are derived as

$$k_{\pm} = \frac{2\beta}{\sqrt{2\alpha}} \pm 2\left(l + \frac{N-2}{2}\right). \tag{51}$$

Substituting the acceptable value of k_- into eq. (49), the acceptable value of $\pi(s)$ is obtained as

$$\pi_-(s) = -\frac{N-2}{2} - \left(\mathrm{d}s - \left(l + \frac{N-2}{2}\right)\right). \tag{52}$$

Here, we select the polynomial $\pi_-(s)$ for which the function $\tau(s)$ in eq. (6) has a negative derivative. Therefore, the function $\tau(s)$ satisfies these requirements, with

$$\tau_-(s) = d - 2ds + 2\left(l + \frac{N-2}{2}\right), \quad \tau'(s) = -2d. \tag{53}$$

From eqs (7) and (11), we get

$$\lambda_n = 2nd \tag{54}$$

and also

$$\lambda = \frac{2\beta d}{\sqrt{2\alpha}} - 2\left(l + \frac{N-2}{2}\right) - d. \tag{55}$$

From eqs (54) and (55), we get a quantization condition

$$\frac{\beta}{d\sqrt{2\alpha}} - \frac{d + 2\left(l + \frac{N-2}{2}\right)}{2d} = n, \quad n = 0, 1, 2, 3, \dots \ . \tag{56}$$

Let us now find the corresponding eigenfunctions for this potential. Due to the NU method, the polynomial solutions of the hypergeometric function $y_n(s)$ depend on the determination of the weight function $\rho(s)$ which satisfies the differential equation (9). As before, putting the values of $\tau(s)$ and $\sigma(s)$ in eq. (9) and solving it, we obtain the weight function $\rho(s)$ as

$$\rho(s) = \frac{1}{d} s^{\frac{2}{d}\left(l + \frac{N-2}{2}\right)} \mathrm{e}^{-2s}. \tag{57}$$

Substituting the values of $\sigma(s)$ and $\rho(s)$ into the Rodrigues relation given in eq. (8), the polynomial solution $y_n(s)$ of eq. (4) is obtained in the following form:

$$y_n(s) = B_{nl} s^{-\frac{2}{d}\left(l + \frac{N-2}{2}\right)} \mathrm{e}^{2s} \frac{\mathrm{d}^n}{\mathrm{d}s^n}\left(s^{n + \frac{2}{d}\left(l + \frac{N-2}{2}\right)} \mathrm{e}^{-2s}\right), \tag{58}$$

where B_n is the normalization constant. The polynomial solutions of $y_n(s)$ in eq. (58) can also be expressed in terms of the associated Laguerre polynomials as

$$y_n(s) = B_n d^{n+1} n! L_n^{\frac{2}{d}\left(l + \frac{N-2}{2}\right)}(2s). \tag{59}$$

By substituting $\pi_-(s)$ and $\sigma(s)$ in eq. (5) and solving it, we obtain $\phi_n(s)$ as

$$\phi_n(s) = s^{l/d} \mathrm{e}^{-s}. \tag{60}$$

Putting the values of $y_n(s)$ and $\phi_n(s)$ in eq. (48a), we get radial wave functions as

$$R_{n,l}^{(N)}(s) = A_n s^{l/d} \mathrm{e}^{-s} L_n^{\frac{2}{d}\left(l + \frac{N-2}{2}\right)}(2s), \tag{61}$$

where A_n is the normalization constant. Taking as before $d = d_1/d_2$ with d_1 and d_2 two relatively prime positive integers and $\mathcal{L} = l + (N - 2/2)$, condition (54) can be written in the form $\beta = (1/d_2)\sqrt{2\alpha}(D + (d_1/2))$, with $D = d_1((\beta d_2/d_1\sqrt{2\alpha}) - \frac{1}{2})$ taken to be the non-negative integer $D = nd_1 + \mathcal{L}d_2$. Here, the states $|n, \mathcal{L}\rangle$ and $|n', \mathcal{L}'\rangle$ belong to the same zero-energy eigenspace provided $(n' - n)d_1 = (\mathcal{L} - \mathcal{L}')d_2$. Actually, the states $\dots|2n - n', 2\mathcal{L} - \mathcal{L}'\rangle, |n, \mathcal{L}\rangle, |n', \mathcal{L}'\rangle, |2n' - n, 2\mathcal{L}' - \mathcal{L}\rangle, \dots$(if admissible) will then all be degenerate. Restrictions arise from the fact that the quantum number n and $\mathcal{L}$ must satisfy $0 \le n \le D/d_1, 0 \le \mathcal{L} \le D/d_2$. It should be stressed that these accidental degeneracies are local and appear only for the level $E = 0$.

4.1 *The Coulomb potential* ($d = 1$)

For $d = 1$, the Hamiltonian given in eq. (1) becomes $H = H_{\mathrm{C}} + \alpha$ with the Schrödinger equation $H_{\mathrm{C}}|\psi\rangle = E|\psi\rangle$ for a particle moving in a β/r potential in N dimensions [27,28]. The energy eigenvalue E is $-\alpha$. The spectrum

$$E = -\alpha = -\frac{\beta^2}{2\left(n + l + \frac{N-1}{2}\right)^2} \tag{62}$$

and the wave function

$$R_{n,l}^{(3)}(r) = C_{n,m} r^l \mathrm{e}^{-\sqrt{2\alpha} r} L_n^{2\left(l+\frac{N-2}{2}\right)}\left(2\sqrt{2\alpha} r\right), \tag{63}$$

where $C_{n,m}$ is the normalization constant.

4.2 *The harmonic oscillator potential* ($d = 2$)

For $d = 2$, the Hamiltonian given in eq. (1) becomes $H = H_{\mathrm{h.o.}} - \beta$ with the Schrödinger equation $H_{\mathrm{h.o.}}|\psi\rangle = E|\psi\rangle$ for a particle moving in an N-dimensional harmonic well. The energy eigenvalue $E = \beta$. The spectrum

$$E = \beta = \sqrt{2\alpha}\left(2n + l + \frac{N}{2}\right) \tag{64}$$

and the radial wave function [28,29]

$$R_{n,l}^{(3)}(r) = \left(\frac{2n!}{\Gamma(n + l + N/2)}\right)^{1/2} r^l \mathrm{e}^{-\sqrt{2\alpha} r^2} L_n^{\left(l+\frac{N-2}{2}\right)}\left(2\sqrt{2\alpha} r^2\right). \tag{65}$$

5. Conclusion

We have been concerned with the Hamiltonians

$$H = \frac{1}{2m} P^2 + \alpha r^{2d-2} - \beta r^{d-2}, \quad \alpha, \beta > 0.$$

We have calculated the eigenvalues and eigenfunctions by NU method in two and N dimensions for the above Hamiltonians at $E = 0$ eigenstate and also obtained the results of the harmonic oscillator and Coulomb problem. The extension sought by us, although straightforward, is quite instructive because laws of physics in N spatial dimensions may often lead to insights concerning laws of physics in lower dimensions [23,24]. For example, Goodson *et al* [27] used the $1/N$ perturbation theory to calculate highly accurate energy eigenvalues for the ground and doubly excited states of He. The effect of electron correlation was included through dimensional scaling of the Scrhödinger equation. Energy was expanded in the parameter $\delta = 1/N$, N being treated as a continuous variable conversing to the physical situation $\delta = 1/3$. Thus, adaptation of the method described in ref. [28] for the N-dimensional Schrödinger equation is of considerable interest. Further, we note that a similar study is mathematically complicated [29] even for a method based on supersymmetric quantum mechanics [30–33]. We conclude by noting that the NU method is an elegant and powerful alternative technique. This technique removes the drawback in the original theory and bypass some difficulties in solving the Schrödinger equation. It also provides closed forms for the energy eigenvalues as well as the corresponding eigenfunctions. Here, the formalism systematically recovers known results in a natural way and allows one to extend certain results in particular cases. It is important to note that the local accidental degeneracies are signalled by the possibility of separating the variables in the Schrödinger equation in more than one coordinate system. This statement is also true in any dimensions.

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