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A highly accurate method to solve Fisher's equation

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Abstract. In this study, we present a new and very accurate numerical method to approximate the Fisher's-type equations. Firstly, the spatial derivative in the proposed equation is approximated by a sixth-order compact finite difference (CFD6) scheme. Secondly, we solve the obtained system of differential equations using a third-order total variation diminishing Runge–Kutta (TVD-RK3) scheme. Numerical examples are given to illustrate the efficiency of the proposed method.

Keywords. Fisher's equation; compact finite difference; Taylor expansion series; total variation diminishing Runge–Kutta; numerical solutions.

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1. Introduction

In 1937, Fisher proposed a nonlinear reaction–diffusion equation to describe the propagation of a viral mutant in an infinitely long habitat [1]. This equation is encountered in various applications such as gene propagation [1,2], tissue engineering [3], autocatalytic chemical reactions [4], combustion [5], and neurophysiology [6]. The proposed nonlinear reaction–diffusion equation is defined by

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2} + \alpha u(1-u), \qquad x \in (-\infty, \infty), \quad t \geq 0, \tag{1}$$

where β is the diffusion coefficient, α is the reactive factor, t is the time, x is the distance and $u(x,t)$ is the population density. The analytical properties and subsequent computation for minimum wave speed have been easily interpreted by removing the explicit dependence on coefficients α and β in (1) by a suitable rescaling of x and t. After rescaling the time $t^* = \beta t$ and space $x^* = (\beta/\alpha)^{1/2} x$, and dropping the asterisk notation, eq. (1) becomes [7,8]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u). \tag{2}$$

Equation (2) may be transformed into an ordinary differential equation by substituting $u = u(z) = u(x - ct)$. Kolmogorov *et al* [9] showed that with appropriate initial and boundary conditions, there exists a travelling wave solution to eq. (2) of wave speed c for every $c \geq 2$.

In the past several decades, there has been great activity in developing numerical and analytical methods for the Fisher's equation. Ablowitz and Zeppetella [10] found explicit solutions of the Fisher's equation for a special wave speed. Asymptotic solutions have been found for the d-dimensional Fisher's equation in [11]. Singular perturbation method has been applied to solve eq. (2) by Puri *et al* in [12,13]. Carey and Shen [7] solved the Fisher's reaction–diffusion equation by a least-squares finite element approximation. In [14], a nonlinear transformation was introduced to solve the Fisher's equation. Adomian's decomposition method has been applied to approximate the solutions of the proposed equation in [15,16]. Qiu and Sloan in [17] used a moving mesh method to simulate travelling wave solutions of the proposed equation. Al-Khaled in [18] has presented the Sinc collocation method to find the solutions of Fisher's equation. In [19], the authors investigated the solution of the Fisher's equation by the pseudospectral method. Recently, Mittal and Jiwari [20] have applied the differential quadrature method to approximate the solution of the Fisher's equation. A numerical scheme for solving the Fisher's equation, which permits the usage of very large discretization mesh sizes in space and time, has been proposed in [21,22].

As we know, the compact finite difference method [23,24] is a powerful mathematical device for finding approximate solutions of various kinds of equations. Wirz *et al* presented a compact finite difference method to approximate the hyperbolic equations [25]. In [26], a compact finite difference scheme has been applied to solve Euler and Navier–Stokes equations. Dehghan and Taleei used a compact split-step finite difference method to solve the Schrödinger equations [27]. A high-order compact finite difference method was applied for systems of reaction–diffusion equations in [28]. The solution of the Helmholtz equation was approximated by a sixth-order compact finite difference (CFD6) method in [29]. In [30], a CFD6 scheme has been presented to approximate the integro-differential equations. Sari and Gürarslan in [31] had combined a CFD6 scheme for first derivative in space and a third-order total variation diminishing Runge–Kutta (TVD-RK3) scheme in time to approximate the Burgers' equation. In this paper, we present a CFD6 scheme to approximate the second-order spatial derivative in the Fisher's equation. A TVD-RK3 [32,33] method has been applied to solve the obtained system. We shall see from the numerical results that the proposed method is more accurate than the method presented in [20].

This paper is organized as follows. In §2, a sixth-order compact finite difference scheme for the second-order spatial derivative in conjunction with a TVD-RK3 method in time is presented to solve the Fisher's equation. Numerical results that illustrate the efficiency of the proposed method are reported in §3. Finally, a conclusion is given in §4.

2. Method and discussion

In this section, we introduce a CFD6 scheme for the second-order spatial derivative and implement it to solve the Fisher's equation. Compact finite differencing is a means for achieving high-order discretization of differential equations without enlarging the bandwidth of the resulting set of discrete equations. This method uses the values of the function and its derivatives at consecutive points. In order to find numerical solution of the Fisher's-type equations, it should be discretized in both space and time. The spatial region $[a, b]$ is discretized by N equidistant points with space step $h = x_{i+1} - x_i$, $i = 1, \ldots, N-1$, where

$$a = x_1 < x_2 < \cdots < x_N = b. \tag{3}$$

Lele in [24], introduced a fourth-order approximation for the second-order derivative at interior points by

$$\alpha u''_{i-1} + u''_i + \alpha u''_{i+1} = b\frac{u_{i+2} - 2u_i + u_{i-2}}{4h^2} + a\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}, \tag{4}$$

where

$$a = \frac{4}{3}(1-\alpha), \quad b = \frac{1}{3}(-1+10\alpha). \tag{5}$$

The relations between the coefficients α, a and b are derived by matching the Taylor series coefficients of various orders. The sixth-order tridiagonal scheme is obtained by substituting $\alpha = 1/10$ in eq. (4).

$$u''_{i-1} + 10u''_i + u''_{i+1} = \frac{12}{h^2}(u_{i-1} - 2u_i + u_{i+1}). \tag{6}$$

Compact schemes are introduced for the nodes near the boundary by Taylor series expansion in [30]. Similarly, for the new compact schemes in boundary points, we use Taylor series expansion. Consider the following scheme at first point:

$$10u''_1 + u''_2 = \frac{12}{h^2}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \alpha_4 u_4 + \alpha_5 u_5 + \alpha_6 u_6), \tag{7}$$

where $\alpha_i, i = 1, \ldots, 6$, are the parameters to be decided. To obtain a sixth-order scheme, Taylor series expansion about point x_1 are inserted in eq. (7) and the terms by the order of h are collected. Then terms of various orders are placed equal to zero. This gives the following linear system of equations:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1/2 & 2 & 9/2 & 8 & 25/2 \\ 0 & 1/6 & 4/3 & 9/2 & 32/3 & 125/6 \\ 0 & 1/24 & 2/3 & 27/8 & 32/3 & 625/24 \\ 0 & 1/120 & 4/15 & 81/40 & 128/15 & 625/24 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 11/12 \\ 1/12 \\ 1/24 \\ 1/72 \end{pmatrix}. \tag{8}$$

Finally, the unknown parameters are obtained by getting solution of the system of eq. (8). In a similar manner we obtain a six-order scheme at the end point. Therefore, the sixth-order schemes for boundary points x_1 and x_N can be written as

$$\begin{cases} 10u''_1 + u''_2 = (12/h^2)((115/36)u_1 - (1555/144)u_2 + (89/6)u_3 - (773/72)u_4 \\ \qquad + (151/36)u_5 - (11/16)u_6), \\ u''_{N-1} + 10u''_N = (12/h^2)((-11/16)u_{N-5} + (151/36)u_{N-4} - (773/72)u_{N-3} \\ \qquad + (89/6)u_{N-2} - (1555/144)u_{N-1} + (115/36)u_N). \end{cases} \tag{9}$$

Equation (6) together with eq. (9) may be written in the matrix form

$$BU'' = AU, \tag{10}$$

where

$$B = \begin{pmatrix} 10 & 1 & 0 & \cdots & 0 \\ 1 & 10 & 1 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 1 & 10 & 1 \\ 0 & \cdots & 0 & 1 & 10 \end{pmatrix}_{N\times N},$$

$$A = \frac{12}{h^2}\begin{pmatrix} 115/36 & -1555/144 & 89/6 & -773/72 & 151/36 & -11/16 & 0 \\ 1 & -2 & 1 & 0 & 0 & & \vdots \\ & \ddots & \ddots & \ddots & & & \\ \vdots & & 0 & 0 & 1 & -2 & 1 \\ 0 & -11/16 & 151/36 & -773/72 & 89/6 & -1555/144 & 115/36 \end{pmatrix}_{N\times N}.$$

Now, we review the TVD-RK3 method to approximate the solution of ordinary differential equation of the form

$$u_t = \mathcal{L}(u), \tag{11}$$

where $\mathcal{L}$ is a linear/nonlinear operator. The time interval $[0, T]$ is divided into M small cells equally and let $k = T/M$ (time mesh size). The TVD-RK3 method to solve the proposed system is given by (see [33])

$$\begin{aligned} u^{(1)} &= u^n + k\mathcal{L}(u^n), \\ u^{(2)} &= \frac{3}{4}u^n + \frac{1}{4}u^{(1)} + \frac{1}{4}k\mathcal{L}(u^{(1)}), \\ u^{n+1} &= \frac{1}{3}u^n + \frac{2}{3}u^{(2)} + \frac{2}{3}k\mathcal{L}(u^{(2)}), \end{aligned} \tag{12}$$

where n is the step of the method.

Now, we briefly describe our method to solve the Fisher's-type equations. The second-order spatial derivative in the proposed equations is obtained via eq. (10). Then the obtained system of ordinary differential equations is solved by the TVD-RK3 method. In the next section, some numerical examples are studied to demonstrate the accuracy and efficiency of the proposed method.

3. Illustrative examples

In this section, four examples are provided to illustrate the validity and effectiveness of the proposed method. In all the examples, the initial and boundary conditions are directly obtained from analytical solutions. The computations associated with the examples in this paper are performed using MATLAB 7. Although, we have considered problem (2) for our discussion, one can apply the proposed method to solve the more general problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(u),$$

where F is a real function. Therefore, we consider this problem for our numerical examples.

Example 1. Consider the Fisher's equation given in [34]:

$$u_t = u_{xx} + u^2(1-u), \qquad 0 < x < 1, \tag{13}$$

with the initial condition

$$u(x,0) = \frac{1}{1+\mathrm{e}^{x/\sqrt{2}}}. \tag{14}$$

In this case the exact solution is given by

$$u(x,t) = \frac{1}{1+\mathrm{e}^{(x-vt)/\sqrt{2}}}. \tag{15}$$

Comparisons are made with analytical solution and differential quadrature method (DQM) [20] for $N = 13$ and $k = 0.00005$ in table 1. It shows that the numerical solutions are in good agreement with analytical solutions and it is observed that the new CFD6 method is more accurate than the DQM. Absolute error between the numerical and analytical solution is also depicted at different time levels for $N = 20$, $k = 0.0001$ and $N = 80$, $k = 0.00005$ in figure 1.

Example 2. Consider the following generalized Fisher's equation in domain [0, 1]:

$$u_t = u_{xx} + u(1-u^{\alpha}) \tag{16}$$

with the initial condition

$$u(x,0) = \left\{\frac{1}{2}\tanh\left(-\frac{\alpha}{2\sqrt{2\alpha+4}}x\right) + \frac{1}{2}\right\}^{2/\alpha}. \tag{17}$$

The exact solution is presented in [20] by

$$u(x,t) = \left\{\frac{1}{2}\tanh\left\{-\frac{\alpha}{2\sqrt{2\alpha+4}}\left(x - \frac{\alpha+4}{\sqrt{2\alpha+4}}t\right)\right\} + \frac{1}{2}\right\}^{2/\alpha}. \tag{18}$$

In table 2, the obtained results for $N = 13$, $k = 0.00005$ and $\alpha = 1$ are compared with the exact solution and the solution of the DQM for $t = 0.5$ and 1.0. Figures 2 and 3

Table 1. Comparison of results for Example 1 with $N = 13$ and $k = 0.00005$.

t	x	DQM [20]	New CFD6	Exact	Absolute error
0.5	0.25	0.51831	0.518298	0.518298	$1.41e-010$
	0.75	0.43038	0.430373	0.430373	$3.96e-010$
1.0	0.25	0.58012	0.580110	0.580110	$3.59e-010$
	0.75	0.49243	0.492418	0.492418	$1.71e-010$

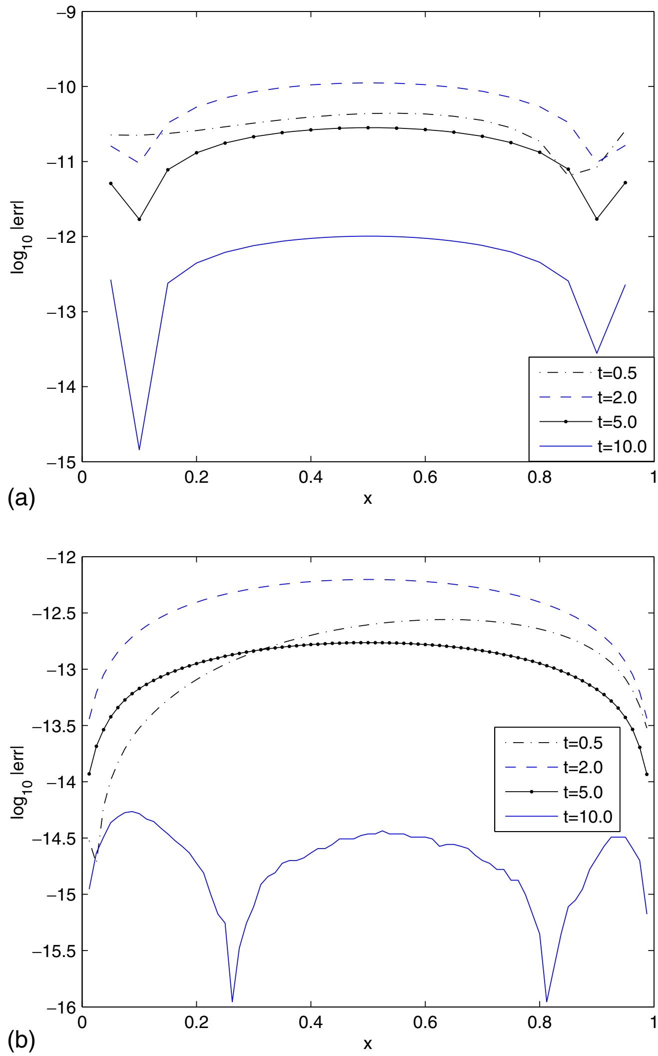


Figure 1. Log 10 of the absolute error in Example 1 at different time levels. (**a**) $k = 0.0001$ and $N = 20$, (**b**) $k = 0.00005$ and $N = 80$.

Table 2. Comparison of results for Example 2 with $\alpha = 1$, $N = 13$ and $k = 0.00005$.

t	x	DQM [20]	New CFD6	Exact	Absolute error
0.5	0.25	0.33412	0.334094	0.334094	$2.27e-011$
	0.75	0.27838	0.278353	0.278353	$5.00e-011$
1.0	0.25	0.45576	0.455739	0.455739	$2.30e-011$
	0.75	0.39544	0.395411	0.395411	$4.80e-012$

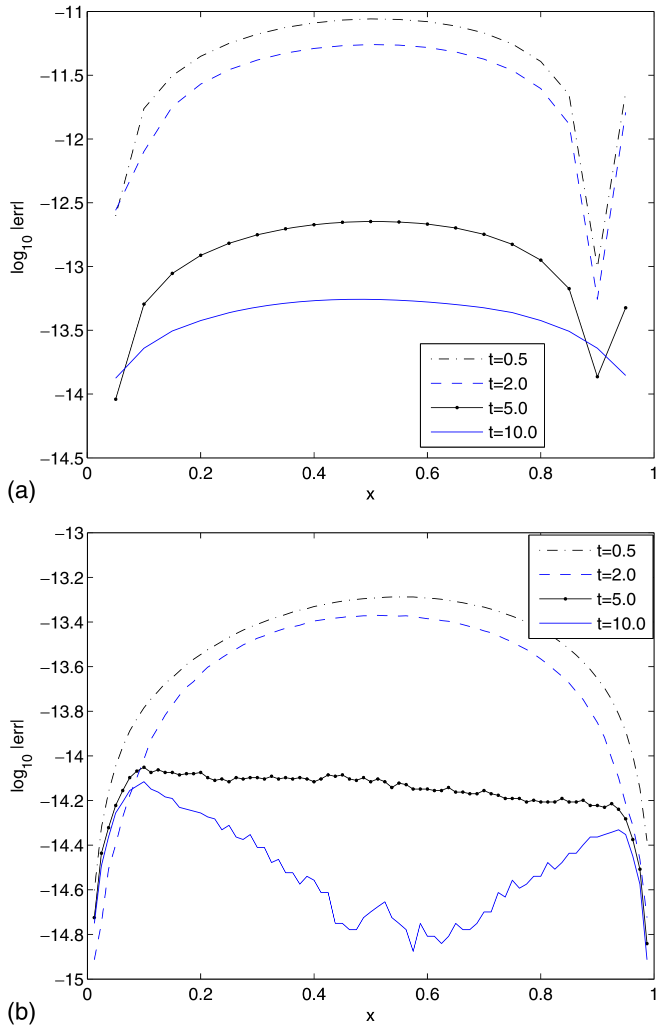


Figure 2. Log 10 of the absolute error in Example 2 with $\alpha = 1$ at different time levels. (**a**) $k = 0.0001$ and $N = 20$, (**b**) $k = 0.00005$ and $N = 80$.

illustrate the graph of the absolute error of the numerical solutions with $\alpha = 1$ and 6 at different time levels for $N = 20$, $k = 0.0001$ and $N = 80$, $k = 0.00005$. As can be seen from the figures, the proposed method gives highly accurate results.

Example 3. We now consider the Fisher's equation given in [16]:

$$u_t = u_{xx} + \alpha u(1 - u), \tag{19}$$

subject to the initial condition

$$u(x, 0) = \frac{1}{(1 + \mathrm{e}^{\sqrt{\alpha/6}x})^2}, \tag{20}$$

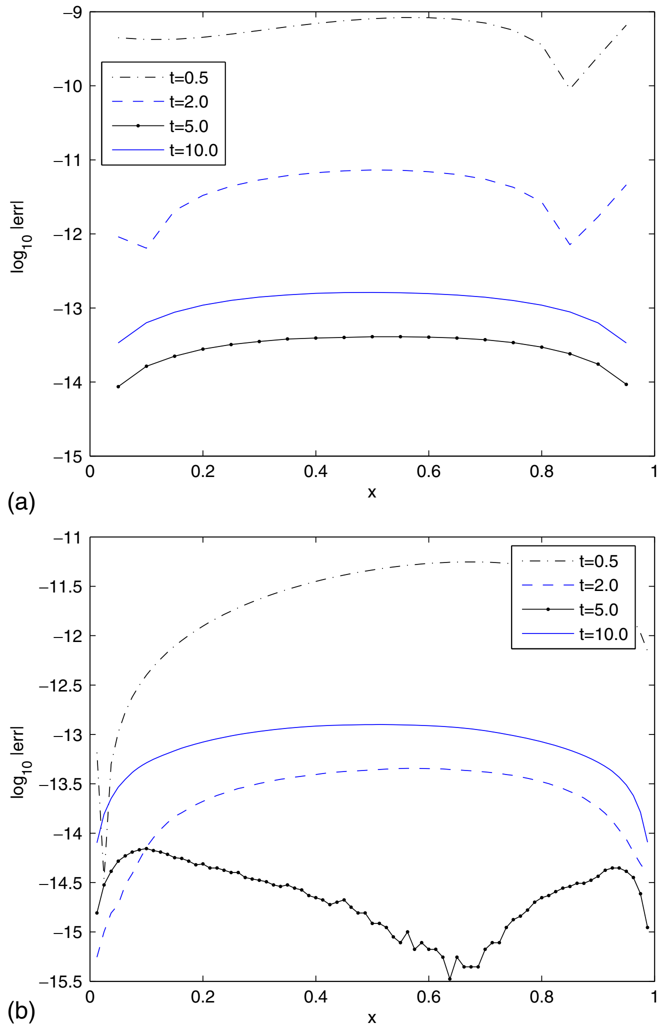


Figure 3. Log 10 of the absolute error in Example 2 with $\alpha = 6$ at different time levels. (**a**) $k = 0.0001$ and $N = 20$, (**b**) $k = 0.00005$ and $N = 80$.

where the exact solution is given by

$$u(x,t) = \frac{1}{\left(1 + e^{\sqrt{\frac{\alpha}{6}}x - \frac{5}{6}\alpha t}\right)^2}. \tag{21}$$

In table 3, we give the absolute errors between the exact and numerical results obtained by the new CFD6 for $N = 13$, $k = 0.00005$ and $\alpha = 6$. A comparison with the results given in table 3 shows that the proposed method is more accurate than the DQM. In order to see the error distributions in this example with $\alpha = 6$, figure 4 is plotted for $N = 20$, $k = 0.0001$ and $N = 80$, $k = 0.00005$.

Table 3. Comparison of results for Example 3 with $\alpha = 6$, $N = 13$ and $k = 0.00005$.

t	x	DQM [20]	New CFD6	Exact	Absolute error
0.5	0.25	0.81847	0.818393	0.818393	$4.26e-010$
	0.75	0.72592	0.725824	0.725824	$4.43e-009$
1.0	0.25	0.98293	0.982919	0.982919	$3.18e-011$
	0.75	0.97208	0.972071	0.972071	$1.62e-010$

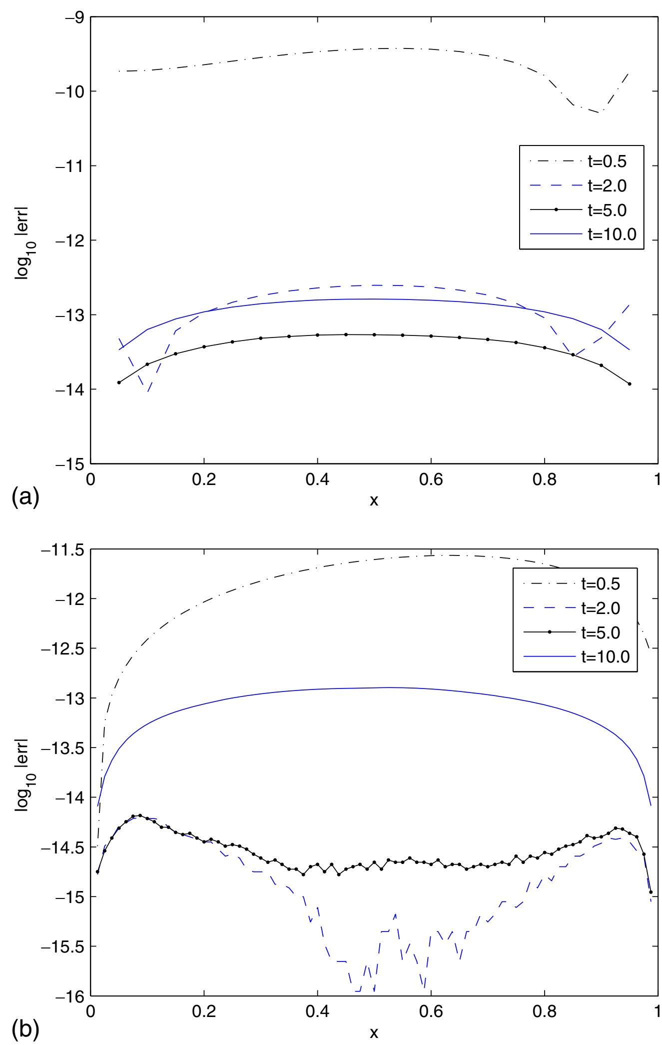
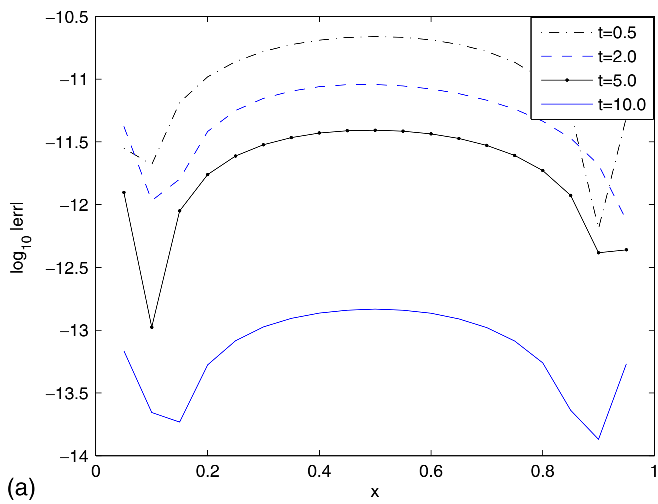


Figure 4. Log 10 of the absolute error in Example 3 with $\alpha = 6$ at different time levels. (**a**) $k = 0.0001$ and $N = 20$, (**b**) $k = 0.00005$ and $N = 80$.

Table 4. Comparison of results for Example 4 with $a = 0.2$, $N = 13$ and $k = 0.00005$.

t	x	DQM [20]	New CFD6	Exact	Absolute error
0.5	0.25	0.67492	0.675373	0.675373	$7.13e-011$
	0.75	0.72772	0.728174	0.728174	$9.74e-011$
1.0	0.25	0.72044	0.720433	0.720433	$1.02e-010$
	0.75	0.76946	0.769460	0.769460	$8.10e-011$



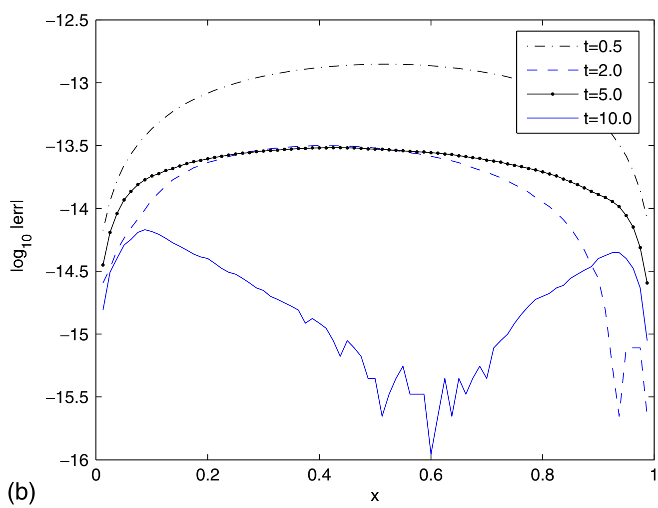


Figure 5. Log 10 of the absolute error in Example 4 with $a = 0.2$ at different time levels. (**a**) $k = 0.0001$ and $N = 20$, (**b**) $k = 0.00005$ and $N = 80$.

Example 4. In this example, we consider the nonlinear diffusion in Fisher's-type equation in domain [0, 1]:

$$u_t = u_{xx} + u(1-u)(u-a), \qquad 0 < a < 1, \tag{22}$$

subject to the initial condition

$$u(x,0) = \frac{1}{2}(1+a) + \frac{1}{2}(1-a)\tanh\left\{\sqrt{2}(1-a)\frac{x}{4}\right\}, \tag{23}$$

where the exact solution is given by (see [35])

$$u(x,t) = \frac{1}{2}(1+a) + \left(\frac{1}{2} - \frac{1}{2}a\right)\tanh\left\{\sqrt{2}(1-a)\frac{x}{4} + \frac{(1-a^2)}{4}t\right\}. \tag{24}$$

Table 4 presents a comparison between the new CFD6 method solutions and the DQM solutions with $a = 0.2$, $N = 13$ and $k = 0.00005$. As we see, our method is more effective than the DQM. For more investigation, the absolute error is plotted for this example with $a = 0.2$ in figure 5 for $N = 20$, $k = 0.0001$ and $N = 80$, $k = 0.00005$.

4. Conclusion

In this paper, the solution of the Fisher's equation is successfully approximated by a new high-order numerical method. A new CFD6 scheme for the second-order derivative in space combined with the TVD-RK3 method in time to solve the proposed equation has been presented. The obtained numerical results are compared with the exact solution and the earlier work in [20]. As the numerical results showed, performance of the method is in excellent agreement with exact solution. It may be concluded that the new CFD6 method is a very powerful and efficient technique for finding approximate solution for various kinds of linear/nonlinear problems.

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