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Octonionic Lorenz-like condition

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Abstract. In this study, the octonion algebra and its general properties are defined by the Cayley–Dickson's multiplication rules for octonion units. The field equations, potential equations and Maxwell equations for electromagnetism are investigated with the octonionic equations and these equations can be compared with their vectorial representations. The potential and wave equations for fields with sources are also provided. By using Maxwell equations, a Lorenz-like condition is newly suggested for electromagnetism. The existing equations including the photon mass provide the most acknowledged Lorenz condition for the magnetic monopole and the source.

Keywords. Lorenz condition; octonion; Maxwell equations; Klein–Gordon equation; magnetic monopole.

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1. Introduction

The quaternions, octonions and Clifford algebra, which are especially used in quantum theories and symmetries of elementary particles in high-energy physics, are based on four and eight components, respectively. There is a lot of studies in physics with quaternions and octonions. Majerník [1–3], for example, studied the general field, potential and wave equations in the quaternionic and complex quaternionic forms. Gamba represented Maxwell equations using a different octonionic form with Modul 7, which is a multiplication method for products of octonion's units [4]. Tanışlı and Özgür wrote the Dirac equation and angular momentum with complex quaternions [5]. The gauge transformation and the electromagnetic energy conservation with biquaternions were investigated by Tanışlı [6]. Tolan *et al* reformulated the electromagnetism with octonions [7] and Candemir *et al* rewrote the Proca–Maxwell equations with hyperbolic octonions for magnetic monopoles [8]. Quaternionic and octonionic spinors and the Dirac equation were investigated by Toppan [9]. Also, Bernevig *et al* defined the generalization of quantum Hall phenomenon in eight-dimensional space with octonions [10]. Vlaenderen and Waser [11] and Negi *et al* [12] investigated the electromagnetism and electrodynamics equations with quaternions and complex quaternions. Jancewicz investigated the electrodynamics with Clifford algebra and multivectors [13]. Adler [14] described the quaternionic quantum mechanics and quantum fields.

There are several products for octonion units. In this study, the Cayley–Dickson notation is used. The general octonionic field, potential equation for field with source can be represented by this method in the octonionic form. In general, the electromagnetic fields are the solutions of Maxwell equations, which can be defined by four vectorial equations, two quaternionic equations or a single complex quaternionic equation (equivalent to a Clifford algebra equation). They are represented by an octonionic equation in the present paper.

Organization of the paper is as follows: Section 2 introduces the octonion algebra and its properties. The general octonionic field equations, octonionic potential equation for fields with sources and the Maxwell equations are presented in §3. Moreover, the octonionic Maxwell equations for the magnetic monopole and the source are given. By using the Maxwell equations, a Lorenz-like condition is also derived from the magnetic and electric fields in §4. In §5, the duality symmetry in octonionic form is briefly given. Conclusions are drawn in §6.

2. Octonion algebra

The octonion algebra is nonassociative and noncommutative. In fact, octonions form an alternative division algebra. A real octonion **P**, which possesses eight octonion units denoted by $\boldsymbol{e}_\mu,\ \mu \in \{0, 1, ..., 7\}$, is [7,15,16]

$$\mathbf{P} = \sum_{\mu=0}^{7} p_\mu \mathbf{e}_\mu,$$

where p_μs are real numbers and $\mathbf{e}_0$ is a unit of the algebra. An octonion is also transcribed as a linear combination of two quaternions as follows:

$$\mathbf{P} = P' + P''\mathbf{e}_4,$$

where P' and P'' are quaternions. The subtraction and addition of two octonions **P** and **Q** are defined as

$$\mathbf{P} \mp \mathbf{Q} = \sum_{\mu=0}^{7} (p_\mu \mp q_\mu)\mathbf{e}_\mu.$$

It is necessary to define the products of octonion basic elements before the product of two octonions. According to Cayley–Dickson method, the product rules for octonion units can be defined as [7]

$$-\mathbf{e}_4\mathbf{e}_i = \mathbf{e}_i\mathbf{e}_4 = \hat{\mathbf{e}}_i, \tag{1}$$

$$\mathbf{e}_4\hat{\mathbf{e}}_i = -\hat{\mathbf{e}}_i\mathbf{e}_4 = \mathbf{e}_i, \quad \mathbf{e}_4\mathbf{e}_4 = -\mathbf{e}_0, \tag{2}$$

$$\mathbf{e}_i\mathbf{e}_j = -\delta_{ij}\mathbf{e}_0 + \epsilon_{ijk}\mathbf{e}_k, \quad \hat{\mathbf{e}}_i\hat{\mathbf{e}}_j = -\delta_{ij}\mathbf{e}_0 - \epsilon_{ijk}\mathbf{e}_k, \tag{3}$$

$$-\hat{\mathbf{e}}_j\mathbf{e}_i = \mathbf{e}_i\hat{\mathbf{e}}_j = -\delta_{ij}\mathbf{e}_4 - \epsilon_{ijk}\hat{\mathbf{e}}_k, \tag{4}$$

where $\hat{\mathbf{e}}_k \equiv \mathbf{e}_{4+k}$, $i, j, k \in \{1, 2, 3\}$. The multiplication of octonionic basic elements can be represented by a multiplication table given in table 1.

As mentioned, the multiplication of octonions is nonassociative, but in special cases, when two basic elements are the same, the product is associative (no summation over repeated indices):

$$(\mathbf{e}_\mu\mathbf{e}_\mu)\,\mathbf{e}_\nu = \mathbf{e}_\mu(\mathbf{e}_\mu\mathbf{e}_\nu), \tag{5}$$

$$(\mathbf{e}_\mu\mathbf{e}_\nu)\,\mathbf{e}_\mu = \mathbf{e}_\mu(\mathbf{e}_\nu\mathbf{e}_\mu). \tag{6}$$

According to the above rules, the octonionic product of two octonions $\mathbf{P}$ and $\mathbf{Q}$ is transcribed as

$$\mathbf{PQ} = P_0Q_0 + P_0\mathbf{Q} + \mathbf{P}Q_0 - \vec{\mathbf{P}}\cdot\vec{\mathbf{Q}} + \vec{\mathbf{P}}\times\vec{\mathbf{Q}},$$

where P_0, Q_0 and $\vec{\mathbf{P}}$, $\vec{\mathbf{Q}}$ are the scalar and vectorial parts of octonions $\mathbf{P}$ and $\mathbf{Q}$ respectively. If $\mathbf{P}$ and $\mathbf{Q}$ possess only imaginary parts, the scalar and vectorial products of $\mathbf{P}$ and $\mathbf{Q}$ can be written as

$$\vec{\mathbf{P}}\cdot\vec{\mathbf{Q}} = -\frac{1}{2}[\mathbf{PQ} + (\mathbf{PQ})^*]$$

and

$$\vec{\mathbf{P}}\times\vec{\mathbf{Q}} = \frac{1}{2}[\mathbf{PQ} - (\mathbf{PQ})^*],$$

where $\mathbf{P}^* = P_0 - \vec{\mathbf{P}}$. In other words, the conjugate of an octonion is defined as changing of the sign of imaginary parts of octonion [15–18]. Hence, the conjugate of the octonions possesses some properties as follows:

$$(\mathbf{P}^*)^* = \mathbf{P}, \quad (\mathbf{P}+\mathbf{Q})^* = \mathbf{P}^* + \mathbf{Q}^*, \quad (\mathbf{PQ})^* = \mathbf{Q}^*\mathbf{P}^*.$$

Table 1. The product diagram of octonion units in Cayley–Dickson notation.

	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$
$\mathbf{e}_0$	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$
$\mathbf{e}_1$	$\mathbf{e}_1$	$-\mathbf{e}_0$	$\mathbf{e}_3$	$-\mathbf{e}_2$	$\mathbf{e}_5$	$-\mathbf{e}_4$	$-\mathbf{e}_7$	$\mathbf{e}_6$
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_3$	$-\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_6$	$\mathbf{e}_7$	$-\mathbf{e}_4$	$-\mathbf{e}_5$
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_2$	$-\mathbf{e}_1$	$-\mathbf{e}_0$	$\mathbf{e}_7$	$-\mathbf{e}_6$	$\mathbf{e}_5$	$-\mathbf{e}_4$
$\mathbf{e}_4$	$\mathbf{e}_4$	$-\mathbf{e}_5$	$-\mathbf{e}_6$	$-\mathbf{e}_7$	$-\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
$\mathbf{e}_5$	$\mathbf{e}_5$	$\mathbf{e}_4$	$-\mathbf{e}_7$	$\mathbf{e}_6$	$-\mathbf{e}_1$	$-\mathbf{e}_0$	$-\mathbf{e}_3$	$\mathbf{e}_2$
$\mathbf{e}_6$	$\mathbf{e}_6$	$\mathbf{e}_7$	$\mathbf{e}_4$	$-\mathbf{e}_5$	$-\mathbf{e}_2$	$\mathbf{e}_3$	$-\mathbf{e}_0$	$-\mathbf{e}_1$
$\mathbf{e}_7$	$\mathbf{e}_7$	$-\mathbf{e}_6$	$\mathbf{e}_5$	$\mathbf{e}_4$	$-\mathbf{e}_3$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$-\mathbf{e}_0$

The square of the norm for an octonion $\mathbf{P}$:

$$N(\mathbf{P}) = \mathbf{P}\mathbf{P}^* = \sum_{\mu=0}^{7} (p_\mu)^2$$

is a real number and the identity

$$N(\mathbf{PQ}) = N(\mathbf{P})N(\mathbf{Q})$$

is valid for it. The inverse of an octonion $\mathbf{P}$ is

$$\mathbf{P}^{-1} = \frac{\mathbf{P}^*}{N(\mathbf{P})}.$$

For complex z_μs, the complexified octonion Z is defined as

$$Z = \sum_{\mu=0}^{7} z_\mu \mathbf{e}_\mu.$$

The octonionic conjugate Z^* of Z is $Z^* = z_0 - \sum_{i=1}^{7} z_i \mathbf{e}_i$. It follows that the product ZZ^* is equal to the following equation:

$$N(Z) = ZZ^* = Z^*Z = \sum_{\mu=0}^{7} z_\mu^2.$$

According to Cayley–Dickson method, the octonionic differential operator D can be defined as

$$D = \frac{i}{c}\frac{\partial}{\partial t} + \mathbf{e}_5 \frac{\partial}{\partial x} + \mathbf{e}_6 \frac{\partial}{\partial y} + \mathbf{e}_7 \frac{\partial}{\partial z}. \tag{7}$$

Its octonionic conjugate is

$$D^* = \frac{i}{c}\frac{\partial}{\partial t} - \mathbf{e}_5 \frac{\partial}{\partial x} - \mathbf{e}_6 \frac{\partial}{\partial y} - \mathbf{e}_7 \frac{\partial}{\partial z}. \tag{8}$$

The product DD^* will be

$$DD^* = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}, \tag{9}$$

where Δ is the Laplacian

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{10}$$

3. Maxwell equations for magnetic sources and massive photon

To consider a more general case, we transcribe the Maxwell equations in the presence of electric and magnetic sources. The possible existence of magnetic charges has fascinated physicists since Dirac's classic papers [20]. In addition to the electric charge density ρ and the electric current density $\vec{j}$ one introduces the magnetic charge density ρ' and the magnetic current density $\vec{j}\,'$. In CGS system of units, the Maxwell equations assume more symmetric form (comp. [7] and [19], eq. (6.150)):

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \tag{11}$$

$$\nabla \cdot \mathbf{B} = 4\pi\rho', \tag{12}$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = -\frac{4\pi}{c}\mathbf{j}\,', \tag{13}$$

$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{j}, \tag{14}$$

where $\mathbf{E}$ is the electric field vector and $\mathbf{B}$ is the magnetic induction vector.

In electromagnetic theory, without magnetic charges, the very existence of the vector potential $\mathbf{A}$ is assured from the equation $\nabla \cdot \mathbf{B} = 0$ by the Poincaré lemma. Equation (12) shows that if magnetic charges are present this lemma cannot be applied. Let us quote the paper by Kyriakopoulos [21]: "The theories of electrodynamics involving magnetic monopoles can be classified into two groups depending upon the number of electromagnetic potentials employed in the formalism. Dirac [20] in his original paper formulated the theory in terms of a single vector potential A^μ, introducing the so-called Dirac string, at the end of which a magnetic monopole is attached, and along which the vector potential is singular. This is the approach followed by most people. Another approach was originated by Cabibbo and Ferrari [22] who introduced two vector potentials A^μ and ${}^*A^\mu$."

Our article is in the same avenue. In addition to the traditional scalar potential φ and vector potential $\mathbf{A}$ one can introduce extra potentials φ' and $\mathbf{A}'$ related to the magnetic sources. The electromagnetic field quantities are expressed by them in the following relations:

$$\mathbf{E} \; = \; -\nabla\varphi - \frac{1}{c}\,\frac{\partial \mathbf{A}}{\partial t} + \nabla \times \mathbf{A}', \tag{15}$$

$$\mathbf{B} \; = \; -\nabla\varphi' + \frac{1}{c}\,\frac{\partial \mathbf{A}'}{\partial t} + \nabla \times \mathbf{A}. \tag{16}$$

Substitution of (15) in eq. (11) yields

$$-\Delta\varphi - \frac{1}{c}\,\frac{\partial}{\partial t}\,\nabla \cdot \mathbf{A} = 4\pi\rho. \tag{17}$$

The Lorenz condition

$$\nabla \cdot \mathbf{A} + \frac{1}{c}\,\frac{\partial\varphi}{\partial t} = 0 \tag{18}$$

allows to replace the second term at the left-hand side of (17) by $(1/c^2)\,(\partial^2\varphi/\partial t^2)$ to obtain the following equation:

$$\Delta\varphi - \frac{1}{c^2}\,\frac{\partial^2\varphi}{\partial t^2} = -4\pi\rho, \tag{19}$$

which is the inhomogeneous d'Alembert equation. Similarly, the substitution of (16) in eq. (12) provides

$$-\Delta\varphi' + \frac{1}{c}\frac{\partial}{\partial t}\nabla\cdot\mathbf{A}' = 4\pi\rho'. \tag{20}$$

The Lorenz condition for potentials φ' and $\mathbf{A}'$ should have a different form, namely

$$\nabla\cdot\mathbf{A}' - \frac{1}{c}\frac{\partial\varphi'}{\partial t} = 0 \tag{21}$$

to replace the second term in (20) by $(1/c^2)\,(\partial^2\varphi'/\partial t^2)$ and to obtain again the inhomogeneous d'Alembert equation

$$\Delta\varphi' - \frac{1}{c^2}\frac{\partial^2\varphi'}{\partial t^2} = -4\pi\rho'. \tag{22}$$

Inserting (15) and (16) into eq. (13) and using the identity $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$ yields

$$\nabla\left(\nabla\cdot\mathbf{A}' - \frac{1}{c}\frac{\partial\varphi'}{\partial t}\right) - \Delta\mathbf{A}' + \frac{1}{c^2}\frac{\partial\mathbf{A}'}{\partial t^2} = -\frac{4\pi}{c}\mathbf{j}'.$$

Due to the Lorenz condition (21), this equation reduces to the inhomogeneous d'Alembert equation:

$$\Delta\mathbf{A}' - \frac{1}{c^2}\frac{\partial^2\mathbf{A}'}{\partial t^2} = \frac{4\pi}{c}\mathbf{j}'. \tag{23}$$

Similarly, insertion of (15) and (16) into eq. (14) and using the Lorenz condition (18) leads to the inhomogeneous d'Alembert equation

$$\Delta\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{j}. \tag{24}$$

The fact that the substitutions of relations (15) and (16) into the Maxwell equations lead to the inhomogeneous d'Alembert equation for all four potentials, is a proof that these relations are proper ones. Equations (22) and (23) demonstrate that the additional potentials φ' and A' are related to the magnetic sources in analogy to the relation of φ and A to the electric sources [22a].

The next step of generalization is an assumption that photons possess a rest mass. Let it be equal to m. The d'Alembert operator $DD^* = \Delta - (1/c^2)\,(\partial/\partial t^2)$ should be replaced by the Klein–Gordon operator

$$\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - k_0^2 = DD^* - k_0^2, \tag{25}$$

where $k_0 = mc/\hbar$. Accordingly, the Maxwell equations (5)–(8) have to be modified in order to imply, by relations (9) and (10), the inhomogeneous Klein–Gordon equations.

The Maxwell equations for massive photons possess the following forms:

$$\nabla \cdot \mathbf{E} + k_0^2 \varphi = 4\pi\rho, \tag{26}$$

$$\nabla \cdot \mathbf{B} + k_0^2 \varphi' = 4\pi\rho', \tag{27}$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} + k_0^2 \mathbf{A}' = -\frac{4\pi}{c}\mathbf{j}', \tag{28}$$

$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} - k_0^2 \mathbf{A} = \frac{4\pi}{c}\mathbf{j}. \tag{29}$$

(The introduction of the term $k_0^2\varphi$ in the photon wave equation yields the Klein–Gordon equation [23]).

The substitution of (15) and (16) along with the Lorenz conditions (18) and (21) into eqs (26)–(29) lead to the following inhomogeneous Klein–Gordon equations:

$$DD^*\varphi - k_0^2\varphi = -4\pi\rho, \tag{30}$$

$$DD^*\varphi' - k_0^2\varphi' = -4\pi\rho', \tag{31}$$

$$DD^*\mathbf{A}' - k_0^2\mathbf{A}' = -\frac{4\pi}{c}\mathbf{j}', \tag{32}$$

$$DD^*\mathbf{A} - k_0^2\mathbf{A} = -\frac{4\pi}{c}\mathbf{j}. \tag{33}$$

4. Octonionic field equations

The Maxwell equations (11)–(14) constitute a collection of eight scalar equations. Therefore, octonions with their eight components are an appropriate tool for uniting the equations in a single formula. The octonionic electromagnetic field expressed in the Cayley–Dickson notation is

$$\mathbf{F} = iB_x\mathbf{e}_1 + iB_y\mathbf{e}_2 + iB_z\mathbf{e}_3 + E_x\mathbf{e}_5 + E_y\mathbf{e}_6 + E_z\mathbf{e}_7. \tag{34}$$

The current density expressed by octonions can be given as

$$\mathbf{J} = \frac{4\pi}{c}\left(-c\rho\mathbf{e}_0 + j_x'\mathbf{e}_1 + j_y'\mathbf{e}_2 + j_z'\mathbf{e}_3 + ic\rho'\mathbf{e}_4 - ij_x\mathbf{e}_5 - ij_y\mathbf{e}_6 - ij_z\mathbf{e}_7\right). \tag{35}$$

The potential in the octonion algebra will be chosen as follows:

$$\Phi = \varphi \mathbf{e}_0 + A'_x \mathbf{e}_1 + A'_y \mathbf{e}_2 + A'_z \mathbf{e}_3 + i\varphi' \mathbf{e}_4 - iA_x \mathbf{e}_5 - iA_y \mathbf{e}_6 - iA_z \mathbf{e}_7. \tag{36}$$

At this stage, the equation for the general octonionic field will be

$$D\mathbf{F} - k_0^2 \Phi = \mathbf{J}. \tag{37}$$

It is termed as the Proca–Maxwell equations for electromagnetism. Its two terms at the left-hand side written explicitly in components possess the form

$$\begin{aligned} D\mathbf{F} = &\left(-\frac{\partial E_x}{\partial x} - \frac{\partial E_y}{\partial y} - \frac{\partial E_z}{\partial z}\right)\mathbf{e}_0 + \left(-\frac{1}{c}\frac{\partial B_x}{\partial t} - \frac{\partial E_z}{\partial y} + \frac{\partial E_y}{\partial z}\right)\mathbf{e}_1 \\ &+ \left(-\frac{1}{c}\frac{\partial B_y}{\partial t} + \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right)\mathbf{e}_2 + \left(-\frac{1}{c}\frac{\partial B_z}{\partial t} - \frac{\partial E_y}{\partial x} + \frac{\partial E_x}{\partial y}\right)\mathbf{e}_3 \\ &+ i\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right)\mathbf{e}_4 + i\left(\frac{1}{c}\frac{\partial E_x}{\partial t} - \frac{\partial B_z}{\partial y} + \frac{\partial B_y}{\partial z}\right)\mathbf{e}_5 \\ &+ i\left(\frac{1}{c}\frac{\partial E_y}{\partial t} + \frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z}\right)\mathbf{e}_6 + i\left(\frac{1}{c}\frac{\partial E_z}{\partial t} - \frac{\partial B_y}{\partial x} + \frac{\partial B_x}{\partial y}\right)\mathbf{e}_7 \end{aligned}$$

and

$$-k_0^2 \Phi = -k_0^2\left(\varphi \mathbf{e}_0 + A'_x \mathbf{e}_1 + A'_y \mathbf{e}_2 + A'_z \mathbf{e}_3 + i\varphi' \mathbf{e}_4 - iA_x \mathbf{e}_5 - iA_y \mathbf{e}_6 - iA_z \mathbf{e}_7\right).$$

The octonionic field equation (37) yields the following equalities for the coefficients in front of the octonionic units:

$$-\frac{\partial E_x}{\partial x} - \frac{\partial E_y}{\partial y} - \frac{\partial E_z}{\partial z} - k_0^2 \varphi = -4\pi\rho, \tag{38}$$

$$-\frac{1}{c}\frac{\partial B_x}{\partial t} + \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} - k_0^2 A'_x = \frac{4\pi}{c} j'_x, \tag{39}$$

$$-\frac{1}{c}\frac{\partial B_y}{\partial t} + \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} - k_0^2 A'_y = \frac{4\pi}{c} j'_y, \tag{40}$$

$$-\frac{1}{c}\frac{\partial B_z}{\partial t} - \frac{\partial E_y}{\partial x} + \frac{\partial E_x}{\partial y} - k_0^2 A'_z = \frac{4\pi}{c} j'_z, \tag{41}$$

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} - k_0^2 \varphi' = 4\pi\rho', \tag{42}$$

$$\frac{1}{c}\frac{\partial E_x}{\partial t} + \frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y} + k_0^2 A_x = -\frac{4\pi}{c} j_x, \tag{43}$$

$$\frac{1}{c}\frac{\partial E_y}{\partial t} + \frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z} + k_0^2 A_y = -\frac{4\pi}{c} j_y, \tag{44}$$

$$\frac{1}{c}\frac{\partial E_z}{\partial t} - \frac{\partial B_y}{\partial x} + \frac{\partial B_x}{\partial y} + k_0^2 A_z = -\frac{4\pi}{c} j_z. \tag{45}$$

In the above equations, eq. (38) is equivalent to eq. (26), eqs (39)–(41) correspond to three components of eq. (28), eq. (42) is the same as eq. (27) and eqs (43)–(45) constitute eq. (29).

5. Duality symmetry

By using identity (1) we can write (34) as

$$\mathbf{F} = i\mathbf{B} - \mathbf{e}_4\mathbf{E}. \tag{46}$$

6. Conclusions

Octonions are used for various subjects such as gravity, field theory and supersymmetry. Octonions, which are similar to the complex quaternions and dual quaternions from the point of view of components, are an eight-dimensional nonassociative algebra. On the other hand, the complex quaternions and dual quaternions include the definition of eight-dimensional component form of the four-dimensional numbers under the same rules. In this paper, the general field equation, the potential equation, for fields with sources and Maxwell equations have been transcribed by octonions. In the vectorial notations, the general field equation, potential equation and Maxwell equations are defined with two, four and four vectorial equations respectively. We represent all these equations as a single equation in octonionic form. These equations are arranged from general to special. The expanded Maxwell equations in the presence of k_0 are valid when the mass of photon is nonzero. In spite of experimental efforts, it has not been decided at present whether the photon has got a mass. As such, the expanded Maxwell equations only will stay as a theoretical study [3]. In a similar way, the existence of magnetic monopoles is also a theoretical study.

There are many definitions for the multiplications of octonion units. A usual method is used in this article. Because of the different products of octonion units in each of the method, the definitions for the differential operator, field, potential and source cause some differences. It is seen that the equations transcribed by the octonionic methods and their solutions are the same as vectorial, quaternionic, complex quaternionic and dual quaternionic definitions.

The study starts with the Maxwell equations for electric and magnetic sources. Hence an equation relating the field quantities to four electromagnetic potentials is proposed. In order to fulfill the inhomogeneous d'Alembert equation for them, a Lorenz-like condition for two new potentials is suggested. The fact that the substitution of relations (15) and (16) into the Maxwell equations lead to the inhomogeneous d'Alembert equation for all four potentials, is a proof that these relations are proper ones. Equations (19) and (23) demonstrate that the additional potentials φ' and A' are related to the magnetic sources in analogy to the relation of φ and A to the electric sources. Equations (26)–(29) and the derivation of eqs (30)–(33) in terms of the four potentials (φ, A), (φ', A') using Lorenz-like condition are new. The photon rest mass is also added to the computations, which implies the inhomogeneous Klein–Gordon equations for the potentials. The newly suggested Lorenz-like condition and the complex octonionic Maxwell equations with magnetic monopole and magnetic current density will contribute to the existing studies in literature. In other words, it is proved that the previous representations with quaternions, biquaternions and octonions for electromagnetism can be altered and rewritten using Lorenz-like condition, alternatively. The second-order differential equations (30)–(33) are expressed as a first-order equation (37) for an octonionic field.

At the end of these computations, it is shown that the octonions can be used for all these computations and also that we possess the compact, useful representations for the electromagnetism.

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