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Application of the trial equation method for solving some nonlinear evolution equations arising in mathematical physics

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Abstract. In this paper some exact solutions including soliton solutions for the KdV equation with dual power law nonlinearity and the $K(m, n)$ equation with generalized evolution are obtained using the trial equation method. Also a more general trial equation method is proposed.

Keywords. Trial equation method; KdV equation; $K(m, n)$ equation; dual-power law; soliton solution.

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1. Introduction

Nonlinear phenomena exist in all the fields such as fluid mechanics, plasma physics, optical fibres, biology, solid-state physics, chemical kinematics, chemical physics, and so on. The investigation of exact solutions of nonlinear evolution equations plays an important role in the analysis of these physical phenomena and gradually becomes one of the most important and significant tasks. To obtain the travelling wave solutions, many methods were attempted, such as the inverse scattering method [1], Hirotas bilinear transformation [2,3], the tanh method [4–7], sine–cosine method [8,9], homogeneous balance method [10], exp-function method [11–17], (G'/G)-expansion method [18–20], and so on. These methods derived many solutions to most nonlinear evolution equations. Then Ma and Fuchssteiner proposed a powerful method for finding exact solutions to nonlinear differential equations [21]. Their key idea is to expand solutions of given differential equations as functions of solutions of solvable differential equations, in particular, polynomial and rational functions. This idea is so good that many types of nonlinear differential equations can be solved by it. Also, Liu proposed the trial equation method and applied this method to some nonlinear evolution equations [22–26]. For example, consider a differential equation of u. We always assume that its exact solution satisfies a solvable equation $(u')^2 = F(u)$.

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Therefore, our aim is just to find the function F. Liu has obtained a number of exact solutions to many nonlinear differential equations when $F(u)$ is a polynomial or a rational function. However, for some nonlinear ordinary differential equations with rank inhomogeneous, we cannot find a polynomial $F(u)$ or a rational function $F(u)$. Therefore, Du took F as an irrational function, and hence proposed a new trial equation method to solve these kinds of equations [27]. Also, Liu proposed a new version of the trial equation method which is suitable for nonlinear partial differential equations with variable coefficients [28].

In this study, we apply the trial equation method to seek exact solutions of the generalized partial differential equation with higher-order nonlinearity. When the equations of this form are transformed to the nonlinear ordinary differential equations, the term $(u')^2$ has been obtained. This is an inconvenient term for these equations. So, we adopted Liu's approach and proposed an irrational version of that approach. Using this method, we obtain some new travelling wave solutions to the KdV equation with dual power law nonlinearity [29,30]:

$$u_t + au^p u_x + bu^{2p} u_x + \delta u_{xxx} = 0, \tag{1}$$

and the $K(m, n)$ equation with generalized evolution [31–33]

$$(u^n)_t + au^m u_x + b(u^n)_{xxx} = 0. \tag{2}$$

2. Trial equation method

According to Ma–Fuchsseiter's idea and Liu's trial equation method to nonlinear evolution equations, we consider a trial equation method which can be suitable to the nonlinear partial differential equations with higher order nonlinearity.

Step 1. We consider the following nonlinear partial differential equation for a function u of two real variables, space x and time t:

$$P(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \dots) = 0. \tag{3}$$

Under the travelling wave transformation $u(x, t) = u(\eta), \eta = x - ct$, where c is a free constant, we reduce eq. (3) to a nonlinear ordinary differential equation of the form:

$$N(t, x, u, u', u'', \dots) = 0, \tag{4}$$

where the prime denotes differentiation with respect to η. Sometimes, the order of eq. (4) can be reduced.

Step 2. Take the trial equation

$$(u')^2 = F(u) = \sum_{i=0}^{s} a_i u^i, \tag{5}$$

where s and a_i are constants to be determined. Substituting eq. (5) and other derivative terms such as u'' or u''' and so on into eq. (4) yields a polynomial $G(u)$ of u. According to the balance principle we can determine the value of s. Setting the coefficients of $G(u)$ to zero, we get a system of algebraic equations. Solving this system, we shall determine c and values of $a_0,\ a_1, \dots,\ a_s$.

Step 3. Rewrite eq. (5) by the integral form

$$\pm(\eta - \eta_0) = \int \frac{1}{\sqrt{F(u)}}\mathrm{d}u. \tag{6}$$

According to the complete discrimination system of the polynomial, we classify the roots of $F(u)$, and solve the integral equation (6). Thus we obtain the exact solutions to eq. (3).

3. Applications

Example 1. *The KdV equation with dual power law nonlinearity*

In order to look for travelling wave solutions of eq. (1), we make the transformation $u(x, t) = u(\eta)$, $\eta = x - ct$, where c is an arbitrary constant. Then, integrating this equation twice and setting the integration constant to zero, we have

$$-cu + \frac{a}{p+1}u^{p+1} + \frac{b}{2p+1}u^{2p+1} + \delta u'' = 0, \tag{7}$$

where p is a positive integer and a, b, δ are free parameters.

Equation (7), using the transformation $u = v^{1/p}$, reduces to

$$Mvv'' + N(v')^2 - cPv^2 + Rv^3 + Tv^4 = 0, \tag{8}$$

where

$$\begin{aligned} M &= \delta p(p+1)(2p+1), \quad N = \delta(1-p^2)(2p+1), \\ P &= p^2(p+1)(2p+1), \quad R = ap^2(2p+1) \end{aligned}$$

and

$$T = bp^2(p+1).$$

Substituting trial equation (5) into eq. (8) and using the balance principle we get $s = 4$. Using the solution procedure of the trial equation method, we obtain the system of algebraic equations as follows:

$$\begin{aligned} &a_0 N = 0, \\ &a_1 M + 2a_1 N = 0, \\ &a_2 M + a_2 N - cP = 0, \\ &3a_3 M + 2a_3 N + 2R = 0, \\ &2a_4 M + a_4 N + T = 0. \end{aligned}$$

Solving the above system of algebraic equations, we obtain the following results:

$$a_0 = 0, \quad a_1 = 0, \quad a_2 = a_2, \quad a_3 = -\frac{2R}{3M+2N}, \quad a_4 = a_4, \quad c = \frac{a_2(M+N)}{P}, \tag{9}$$

where a_2 and a_4 are free parameters. Substituting these results into eqs. (5) and (6), we get

$$\pm(\eta-\eta_0)=\int\frac{1}{\sqrt{a_2v^2-\frac{2R}{3M+2N}v^3+a_4v^4}}\mathrm{d}v. \tag{10}$$

Integrating eq. (10), we get the exact solution of eq. (1) as follows:

$$v(\eta)=\left[-\frac{a_3}{2a_2}-\frac{(4a_2a_4-a_3^2)\exp(-\sqrt{a_2}\eta)+a_2\exp(\sqrt{a_2}\eta)}{4a_2\sqrt{a_2}}\right]^{-1} \tag{11}$$

and

$$v(\eta)=\left[-\frac{a_3}{2a_2}+\frac{(a_3^2-4a_2a_4)\exp(\sqrt{a_2}\eta)+a_2\exp(-\sqrt{a_2}\eta)}{4a_2\sqrt{a_2}}\right]^{-1}. \tag{12}$$

Using the properties

$$\sinh(\eta)=\frac{\exp(\eta)-\exp(-\eta)}{2},\qquad \cosh(\eta)=\frac{\exp(\eta)+\exp(-\eta)}{2}, \tag{13}$$

when $a_3^2-4a_2a_4=\pm a_2$ and $u=v^{1/p}$, it is easy to see that the solutions (11) and (12) can reduce to the soliton solutions

$$u(x,t)=\frac{A}{(D-\cosh[B(x-ct)])^{1/p}}, \tag{14}$$

$$u(x,t)=\frac{A}{(D-\sinh[B(x-ct)])^{1/p}}, \tag{15}$$

$$u(x,t)=\frac{A}{(D+\cosh[B(x-ct)])^{1/p}}, \tag{16}$$

where $c=\delta a_2/p^2$, $A=(2\sqrt{a_2})^{1/p}$, $B=\sqrt{a_2}$ and $D=2ap^2/(\delta\sqrt{a_2}(p+1)(p+2))$. Also, eq. (15) represents a singular soliton solution for eq. (1). Here A is the amplitude of the soliton, c is the velocity and B is the inverse width of the soliton. From this point of view, we can say that the solitons exist for $a\delta>0$ as long as $B>0$, which is guaranteed from $a_2>0$. Also, it is seen that eq. (16) is the solution obtained by the solitary wave ansatz method in [34].

Example 2. *The* $K(m,n)$ *equation with generalized evolution*

Using a variation η defined as $\eta=x-ct$, integrating and equating the integration constant to zero, we can convert eq. (2) into ordinary differential equation which reads as

$$-cu^n+\frac{a}{m+1}u^{m+1}+b(u^n)''=0, \tag{17}$$

where $m,\ n$ are positive integers and $a,\ b$ are free constants.

Balancing $(u^n)''$ with u^{m+1} gives $s = (2m+4)/(n+1)$. To obtain closed form solutions, we use the transformation $u = v^{1/(m-n+1)}$ that will reduce eq. (17) into the ordinary differential equation

$$Mvv'' + N(v')^2 - cPv^2 + Rv^3 = 0, \tag{18}$$

where

$$M = bn(m+1)(m-n+1), \quad N = bn(m+1)(2n-m-1),$$

$$P = (m+1)(m-n+1)^2 \quad \text{and} \quad R = a(m-n+1)^2.$$

Substituting trial eq. (5) into eq. (18) and using the balance principle we get $s = 3$. Using the solution procedure of trial equation method, we obtain the system of algebraic equations as follows:

$$a_0 N = 0,$$

$$a_1 M + 2a_1 N = 0,$$

$$a_2 M + a_2 N - cP = 0,$$

$$3a_3 M + 2a_3 N + 2R = 0.$$

Solving the above system of algebraic equations, we obtain the following results:

$$a_0 = 0, \quad a_1 = 0, \quad a_2 = a_2, \quad c = \frac{a_2(M+N)}{P}, \quad a_3 = -\frac{2R}{3M+2N}. \tag{19}$$

Substituting these results into eqs (5) and (6), we get

$$\pm(\eta - \eta_0) = \int \frac{1}{\sqrt{a_2 v^2 - \frac{2R}{3M+2N} v^3}} \mathrm{d}v. \tag{20}$$

Integrating eq. (20) and using the transformation $u(\eta) = v^{1/(m-n+1)}(\eta)$ when $\eta = x - ct$, we obtain 1-soliton solution of eq. (2) as follows:

$$u(x,t) = \frac{A}{\cosh^{2/(m-n+1)}[B(x - ct - \eta_0)]}, \tag{21}$$

where

$$A = \left(\frac{a_2 bn(m+1)(m+n+1)}{2a(m-n+1)^2}\right)^{1/(m-n+1)}, \quad B = \frac{\sqrt{a_2}}{2}$$

and

$$c = a_2 b \left(\frac{n}{m-n+1}\right)^2.$$

Here, A is the amplitude of the soliton, c is the velocity and B is the inverse width of the soliton. From this point of view, we can say that the solitons exist for $ab > 0$ as long as $B > 0$, which is guaranteed from $a_2 > 0$.

4. Discussion

In our new trial equation method, the irrational trial equation (5) can be replaced by the following more general form:

$$(u')^2 = \sum_{i=0}^{k_1} a_i u^i + \left(\sum_{i=0}^{k_2} b_i u^i\right)\sqrt{\sum_{i=0}^{k_3} c_i u^i}, \tag{22}$$

where $a_0, \ldots, a_{k_1}, b_0, \ldots, b_{k_2}, c_0, \ldots, c_{k_3}$ are the constants to be determined. Therefore, we can give a more general trial equation method as follows:

Step 1. Take a trial equation (22). Correspondingly, we derive the following equation:

$$\begin{aligned} u'' =& \frac{1}{2}\sum_{i=1}^{k_1} i a_i u^{i-1} + \frac{1}{2}\left(\sum_{i=1}^{k_2} i b_i u^{i-1}\right)\sqrt{\sum_{i=0}^{k_3} c_i u^i} \\ &+ \frac{1}{4}\left(\sum_{i=0}^{k_2} b_i u^i\right)\left(\sum_{i=1}^{k_3} i c_i u^{i-1}\right)\left(\sum_{i=0}^{k_3} c_i u^i\right)^{-1/2} \end{aligned} \tag{23}$$

and other derivation terms such as u''', and so on.

Step 2. Substituting u', u'' and other derivation terms into eq. (4) yields the following equation:

$$G(u) + H(u)\sqrt{\sum_{i=0}^{k_3} c_i u^i} = 0, \tag{24}$$

where $G(u)$ and $H(u)$ are two polynomials of u. According to the balance principle, we can obtain the relation of k_1, k_2 and k_3 or their values.

Step 3. Taking concrete values of k_1, k_2 and k_3, and letting all coefficients of $G(u)$ and $H(u)$ to be zero yield a system of nonlinear algebraic equations. Solving the system, we get the values of $a_0, \ldots, a_{k_1}, b_0, \ldots, b_{k_2}, c_0, \ldots, c_{k_3}$.

Step 4. Substituting the results obtained in Step 3 into eq. (22) and integrating eq. (22), we can find the exact solutions of eq. (4).

5. Conclusion

In this paper, we used a trial equation method to obtain some soliton solutions to the KdV equation with dual power law nonlinearity and the $K(m, n)$ equation with generalized evolution. The obtained exact solutions are transformed into hyperbolic function solutions. We also proposed a more general trial equation method for solving the generalized nonlinear evolution equations. Consequently, we think that this method can also be applied to other nonlinear differential equations with higher-order nonlinearity.

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