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The role of the Jacobi last multiplier and isochronous systems

PARTHA GUHA[1,*] and ANINDYA GHOSE CHOUDHURY[2]
[1]S.N. Bose National Centre for Basic Sciences, JD Block, Sector III, Salt Lake, Kolkata 700 098, India
[2]Department of Physics, Surendranath College, 24/2 Mahatma Gandhi Road, Kolkata 700 009, India
*Corresponding author. E-mail: partha@bose.res.in

Abstract. We employ Jacobi's last multiplier (JLM) to study planar differential systems. In particular, we examine its role in the transformation of the temporal variable for a system of ODEs originally analysed by Calogero–Leyvraz in course of their identification of isochronous systems. We also show that JLM simplifies to a great extent the proofs of isochronicity for the Liénard-type equations.

Keywords. Jacobi last multiplier; isochronous systems; Liénard equation; commuting vector fields.

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1. Introduction

This method of the last multiplier was first described by Jacobi [1,2] in the course of his lectures given during the 1842–43 academic year at the University of Königsberg. Since then, it has found a number of applications in dynamics. The method essentially seeks to construct a first integral for an n-dimensional system when $(n-2)$ conserved densities are known. In other words, if a Jacobi multiplier is known along with $(n-2)$ first integrals, we can reduce locally the n-dimensional system to a two-dimensional vector field on the intersection of the $(n-2)$ level sets formed by the first integrals; the technique of Jacobi's last multiplier then yields an extra first integral.

The relationship between the Jacobi last multiplier M and the Lagrangian L for any second-order equation is determined by a formula, derived by Madhava Rao [3] in the 1940s. It appears that the relation between the Jacobi last multiplier and the existence of Lagrangian functions were the subjects of investigation by only a few people in the early 1900s [4]. However, the precise nature of their inter-relation was clarified by Rao. Thereafter this did not attract much attention among researchers in the field of differential equations. Recently, Nucci and Leach derived Lagrangians for many second-order differential equations using Jacobi's last multiplier [5].

1.1 *Jacobi's construction of the last multiplier*

Let $M = M(x_1, \ldots, x_n)$ be a non-negative C^1 function non-identically vanishing on any open subset of $\mathbf{R}^n$. Consider a set of first-order equations

$$\frac{\mathrm{d}x_r}{\mathrm{d}t} = W_r(x_1, \ldots, x_n), \quad r = 1, \ldots, n, \tag{1.1}$$

where the vector fields $(W_1, W_2, \ldots, W_n)$ are functions of $(x_1, \ldots, x_n, t)$. Let $(c_1, \ldots, c_k)$ be a set of constant of motions of these sets of equations. The Jacobi last multiplier may be regarded as the density associated with the invariant measure $\int_\Omega M\,\mathrm{d}x$, where Ω is any open subset of $\mathbf{R}^n$. Thus the invariance of flux implies

$$\int_\Omega M\delta x_1 \cdots \delta x_k = \int_{\phi_t(\Omega)} M \frac{\partial(x_1, \ldots, x_k)}{\partial(c_1, \ldots, c_k)} \delta c_1 \cdots \delta c_k, \tag{1.2}$$

where $\phi_t(\cdot)$ is the flow generated by the solutions $\mathbf{x} = \mathbf{x}(t)$ of $\dot{\mathbf{x}} = \mathbf{W}(\mathbf{x})$. In other words, $\phi_t(\Omega)$ is the transformation of the domain Ω under the flow generated by the solution [6]. This invariant condition yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{M \frac{\partial(x_1, \ldots, x_k)}{\partial(c_1, \ldots, c_k)}\right\} = 0$$

or

$$\frac{\mathrm{d}M}{\mathrm{d}t} \frac{\partial(x_1, \ldots, x_k)}{\partial(c_1, \ldots, c_k)} + M \sum_{p=1}^{k} \frac{\partial(x_1, \ldots, x_{p-1}, W_p, \ldots, x_k)}{\partial(c_1, \ldots, c_k)} = 0 \tag{1.3}$$

so that

$$\frac{\mathrm{d}M}{\mathrm{d}t} \frac{\partial(x_1, \ldots, x_k)}{\partial(c_1, \ldots, c_k)} + M \sum_{p=1}^{k} \frac{\partial W_p}{\partial x_p} \cdot \frac{\partial(x_1, \ldots, x_k)}{\partial(c_1, \ldots, c_k)} = 0,$$

and leads to the following equation:

$$\frac{\mathrm{d}M}{\mathrm{d}t} + M \sum_{i=1}^{n} \frac{\partial W_i}{\partial x_i} = 0. \tag{1.4}$$

2. Isochronous systems

A (classical) dynamical system is said to be isochronous if it displays an open region in its phase space in which all its solutions are completely periodic, i.e., periodic in all degrees of freedom, with the same fixed period. The linear harmonic oscillator is the prototype of an isochronous system and all other isochronous systems are isoperiodic with the harmonic oscillator. Research on isochronous system has taken a new turn in recent times. A large number of articles (for example, [7–9]) have appeared in the last few years inspired by the novel ideas developed by Francesco Calogero in this regard [10].

A formal definition of an isochronous vector is as follows:

DEFINITION 2.1.

A vector $\mathbf{z}(t;\alpha)$ is isochronous with a period T if all its components are periodic with the same period T independent of the set of parameters $\alpha = \{\alpha_k\}$, i.e.,

$$z_j(t+T;\alpha) = z_j(t;\alpha). \tag{2.1}$$

Note that in this definition the number of parameters need not be equal to the number of components of $\mathbf{z}$.

2.1 *Isochronous dynamical systems*

Consider a system of three first-order ODEs:

$$\dot{x} = \phi(x,y,z)h(x), \tag{2.2}$$

$$\dot{y} = U(x,y,z), \tag{2.3}$$

$$\dot{z} = V(x,y,z). \tag{2.4}$$

The explicit forms of U and V are deduced below. The function $h(x) \neq 0$, but is otherwise arbitrary. Here ϕ plays a crucial role and we deduce its form rather than stipulate it. Following Calogero and Leyvraz [11] we demand that there exists a fictitious time τ such that

$$\dot{\tau} = \phi(x,y,z). \tag{2.5}$$

Then it follows that (2.2) can be written as $\dot{x} = \dot{\tau}h(x)$ and may therefore be separated to yield

$$\tau = \int^x \frac{\mathrm{d}u}{h(u)}, \tag{2.6}$$

and is independent of y and z. To determine the functions U and V such that the system of eqs (2.2)–(2.4) is isochronous, Calogero and Leyvraz used an ingenious trick, wherein they exploited the solutions of the simple harmonic oscillator. The equations of motion of the latter when written as a first-order system are

$$\dot{f}_1 = \Omega f_2 \quad \text{and} \quad \dot{f}_2 = -\Omega f_1, \tag{2.7}$$

and have the following solutions, namely

$$f_1(t) = C\sin(\Omega t + \Lambda) \quad \text{and} \quad f_2(t) = C\cos(\Omega t + \Lambda). \tag{2.8}$$

Here C is the amplitude and Λ represents the phase difference.

The functional forms of U and V appearing in (2.3) and (2.4) are determined in the following manner. Introduce two functions $F(x)$ and $G(x)$ such that

$$yF(x) = f_1 \quad \text{and} \quad zG(x) = f_2. \tag{2.9}$$

The time derivatives of the right-hand sides lead to the following equations for y and z, viz.,

$$\dot{y} = \Omega z\frac{G}{F} - y\phi\frac{\partial \log F(x)}{\partial x}h(x), \tag{2.10}$$

$$\dot{z} = -\Omega y \frac{F}{G} - z\phi \frac{\partial \log G(x)}{\partial x} h(x), \tag{2.11}$$

where we have used (2.7) and (2.2). Equations (2.10) and (2.11) fix the forms of the functions U and V appearing in (2.3) and (2.4) once ϕ, $F(x)$ and $G(x)$ are known. In other words, the systems of equations under investigation are given by (2.2), (2.10) and (2.11). We shall determine ϕ in terms of the Jacobi last multiplier (described below) of our system of equations.

However, prior to that we wish to note two features. First, if x is known as an explicit function of the time t, then y and z can be determined algebraically from (2.9). Secondly, since f_1 and f_2 are both periodic functions with period $T = 2\pi/\Omega$, if $x(t+T) = x(t)$, it follows that y and z are also periodic and have the same period as x. Therefore, as all the dynamical variables have the same period, the system is naturally isochronous.

2.2 *Explicit determination of isochronous systems and the Jacobi last multiplier*

In this section we return to the explicit determination of the relationship between the JLM and the first-order system of ODEs given by (2.2), (2.10) and (2.11). The determining equation of the JLM for the system is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\big(\log M\big) &= -h(x)\phi_x + y\frac{F'(x)}{F}h(x)\phi_y + z\frac{G'(x)}{G}h(x)\phi_z \\ &\quad -\phi\left(h_x + \frac{F'(x)}{F}h + \frac{G'(x)}{G}h\right). \end{aligned} \tag{2.12}$$

Here the subscripts denote the usual partial derivatives. We demand that the r.h.s. of (2.12) be proportional to ϕ, i.e., we set $\mathrm{d}\log M/\mathrm{d}t = \alpha\phi$, where α is a constant. This leads to the following first-order partial differential equation determining the unknown function ϕ:

$$-\phi_x + y\frac{F'(x)}{F}\phi_y + z\frac{G'(x)}{G}\phi_z = \phi\left(\frac{\alpha}{h} + \frac{h'(x)}{h} - \frac{F'(x)}{F} - \frac{G'(x)}{G}\right). \tag{2.13}$$

A particular solution of this equation is given by

$$\phi(x,y,z) = \frac{K}{yzh(x)}\exp\left(-\alpha\int^x \frac{\mathrm{d}u}{h(u)}\right), \tag{2.14}$$

where K is a constant. Consequently the system of equations assumes the simplified form

$$\begin{aligned} \dot{x} &= \frac{K}{yz}\exp\left(-\alpha\int^x \frac{\mathrm{d}u}{h(u)}\right) \\ \dot{y} &= \Omega z\frac{G}{F} - \frac{K}{z}\exp\left(-\alpha\int^x \frac{\mathrm{d}u}{h(u)}\right)\frac{\partial \log F(x)}{\partial x} \\ \dot{z} &= -\Omega y\frac{F}{G} - \frac{K}{y}\exp\left(-\alpha\int^x \frac{\mathrm{d}u}{h(u)}\right)\frac{\partial \log G(x)}{\partial x}. \end{aligned} \tag{2.15}$$

It is now observed that, if we define

$$\dot{\tau} = \phi,$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}\log M = \alpha\dot{\tau} \quad \text{or} \quad \tau = \log M^{1/\alpha}, \tag{2.16}$$

then from (2.2) we have

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = h(x) \quad \text{or} \quad \tau = \int^x \frac{\mathrm{d}u}{h(u)}, \tag{2.17}$$

from which we can in principle obtain $x = x(\tau)$ by inversion. Hence it remains to find τ as an explicit function of the time t. This is relatively easy in view of the expression for ϕ given by (2.14). Since $\dot{\tau} = \phi$, using (2.17) and (2.9) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\alpha\tau} = \frac{\alpha K}{f_1 f_2}\frac{1}{S(x)}, \tag{2.18}$$

where

$$S(x) = \frac{h(x)}{F(x)G(x)}.$$

However, assuming we have already determined $x = x(\tau)$ from (2.17), it follows that $S(\tau) = S(x(\tau))$. Consequently, from (2.18) we have, using the solutions f_1 and f_2 given by (2.7),

$$N(\tau) := \int S(\tau)\mathrm{e}^{\alpha\tau}\mathrm{d}\tau = \frac{K}{C^2\Omega}\log|\tan(\Omega t + \Lambda)|. \tag{2.19}$$

Note that the explicit form of $S(x)$ depends upon our choice of the as yet unspecified functions $F(x)$ and $G(x)$ respectively. Thus, for a suitable choice of the function $S(\tau)$, if we can obtain an invertible function $N(\tau)$, then τ becomes an explicit function of the time t, i.e.,

$$\tau = N^{-1}\left[\frac{K}{C^2\Omega}\log|\tan(\Omega t + \Lambda)|\right], \tag{2.20}$$

and it is obvious that it is periodic with period $T = 2\pi/\Omega$. It follows from (2.16) that the JLM, M, is also periodic. In the following examples we illustrate this formalism, setting $\alpha = 1$ without loss of generality.

Note that $\tau(t)$ is singular, hence the corresponding dynamical systems that we have constructed below are not, strictly speaking, periodic, and their time evolution should be considered in a loose sense, as soon as it hits a singularity. The system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ exhibits a finite time blow-up if there exist both $t_1 \in \mathbf{R}^n$ such that for all $m \in \mathbf{R}$, there exists an $\epsilon > 0$ such that for $|t - t_1| < \epsilon$, we have $||\mathbf{x}(t;\mathbf{x}_0)|| > m$, where $||\cdot||$ is any L^p-norm. The blow-up is forward in time if $t_1 > t_0$ and backward in time if $t_1 < t_0$. Generally, we denote $\lim_{t\to t_1}||\mathbf{x}(t;\mathbf{x}_0)|| \to \infty$ to denote such a blow-up.

We illustrate this issue in our next example.

Example 1: When $h(x) = x$ and $F(x) = G(x) = x$.

It follows from (2.14) that $\phi(x, y, z) = K/x^2yz$ and $S(x) = x^{-1}$. Therefore the system (2.15) is given by

$$\dot{x} = \frac{K}{xyz}, \quad \dot{y} = \Omega z - \frac{K}{x^2 z}, \quad \dot{z} = -\Omega y - \frac{K}{x^2 y}. \tag{2.21}$$

While from (2.17) we have $x(\tau) = \mathrm{e}^{\tau}$ and hence $S(\tau) = \mathrm{e}^{-\tau}$ leading to

$$N(\tau) = \tau = \frac{K}{C^2\Omega}\log|\tan(\Omega t + \Lambda)|$$

so that τ is determined as a function of time t and is periodic. Choosing $K = C^2\Omega$ we obtain the solution of the above system as

$$x(t) = |\tan(\Omega t + \Lambda)|, \quad y(t) = \frac{C\sin(\Omega t + \Lambda)}{|\tan(\Omega t + \Lambda)|}, \quad z(t) = \frac{C\cos(\Omega t + \Lambda)}{|\tan(\Omega t + \Lambda)|}. \tag{2.22}$$

Since (2.21) is singular at $x = y = z = 0$, $(\Omega t + \Lambda) \neq 0, \pm\frac{\pi}{2}, \pi$ as otherwise $\sin(\Omega t + \Lambda)$ and $\cos(\Omega t + \Lambda)$ would vanish. Regrading the singularity of $\tan(\Omega t + \Lambda)$ at $(\Omega t + \Lambda) = \pm\pi/2$ one will notice that this corresponds to the limit $\tau \to \infty$. It is usual to fix Λ by the requirement $\tau(0) = 0$. Thus the dynamical system constructed here is isochronous in a loose sense owing to the existence of a finite-time singularity.

Example 2: When $h(x) = x = F(x)$ and $F(x)G(x) = 1$.

In this case the system of equations assumes the form

$$\dot{x} = \frac{K}{xyz}, \quad \dot{y} = \Omega\frac{z}{x^2} - \frac{K}{x^2 z}, \quad \dot{z} = -\Omega x^2 y + \frac{K}{x^2 y}. \tag{2.23}$$

As in the previous example we have $x(\tau) = \mathrm{e}^{\tau}$. However, $S(x) = x$ so that $S(\tau) = \mathrm{e}^{\tau}$ and from (2.19) we obtain

$$\tau(t) = \frac{1}{2}\log\left|\frac{2K}{C^2\Omega}\log|\tan(\Omega t + \Lambda)|\right|.$$

In terms of $\tau(t)$ given above we arrive at the following solution, namely,

$$x(t) = \mathrm{e}^{\tau(t)}, \quad y(t) = C\sin(\Omega t + \Lambda)\mathrm{e}^{-\tau(t)}, \quad z(t) = C\cos(\Omega t + \Lambda)\mathrm{e}^{\tau(t)}. \tag{2.24}$$

Example 3: In this example we investigate the applicability of the technique described above to ordinary differential equations of the following class:

$$\ddot{x} + F(x)\dot{x}^2 + G(x) = 0. \tag{2.25}$$

Consider the following system of first-order ordinary differential equations:

$$\dot{x} = f(x)y, \quad \dot{y} = -\frac{f'(x)}{2}y^2 + \Omega^2 h(x). \tag{2.26}$$

This is equivalent to the following second-order equation:

$$\ddot{x} = \frac{1}{2}\frac{f'(x)}{f(x)}\dot{x}^2 + \Omega^2 f(x)h(x). \tag{2.27}$$

So comparison with (2.25) shows that

$$F(x) = -\frac{1}{2}\frac{f'(x)}{f(x)} \quad \text{and} \quad G(x) = -\Omega^2 f(x)h(x).$$

Let $f(x)$ be defined as

$$f(x) = \frac{\int^x -2h(\bar{x})\mathrm{d}\bar{x}}{h^2(x)}, \tag{2.28}$$

where $h(x)$ is any integrable real valued function, such that $f(x) > 0$.

Define the pair of conjugate variables H and Θ in the spirit of Calogero and Levyraz by

$$H := \sqrt{f(x)}y, \quad \Theta := \left(\int^x -2h(\bar{x})\mathrm{d}\bar{x}\right)^{1/2}, \tag{2.29}$$

where x and y are assumed to be canonical variables, with Poisson brackets $\{x, y\} = 1$. It is straightforward to verify that $\{\Theta, H\} = 1$, and we define an Ω modified Hamiltonian by

$$\widetilde{H} = \frac{1}{2}(H^2 + \Omega^2\Theta^2). \tag{2.30}$$

This leads to the following equations $\dot{\Theta} = H$ and $\dot{H} = -\Omega^2\Theta$ with solutions

$$H(t) = H(0)\cos(\Omega t) - \Theta(0)\Omega\sin(\Omega t) \tag{2.31}$$

$$\Theta(t) = \Theta(0)\cos(\Omega t) + \frac{H(0)}{\Omega}\sin(\Omega t). \tag{2.32}$$

On the other hand, the evolution equations for x and y as determined by the Hamiltonian $\widetilde{H}$ are

$$\dot{x} = \frac{\partial \widetilde{H}}{\partial y} = f(x)y, \quad \dot{y} = -\frac{\partial \widetilde{H}}{\partial x} = -\frac{f'(x)}{2}y^2 + \Omega^2 h(x), \tag{2.33}$$

where we have made explicit use of (2.30) and the definitions of H and Θ given in (2.29).

Given a suitable function $h(x)$, one can in principle solve for x from $\Theta^2 = \int^x -2h(\bar{x})\mathrm{d}\bar{x}$, say $x(t) = K[\Theta^2(t)]$ and obtain $y(t) = \frac{H(t)}{[f(K(\Theta^2(t)))]^{1/2}}$ from the first equation in (2.29). But, as $H(t)$ and $\Theta(t)$ vary periodically with time period $T = 2\pi/\Omega$, it follows that x and y also evolve with the same period. The system of equations in (2.33) is equivalent to (2.27) by construction.

3. Isochronicity and commuting vector fields from a Lagrangian perspective

We now demonstrate another aspect of isochronicity using JLM in connection with Lagrangian formalism. Of late it has come to light that a system of ODEs of the form

$$\dot{x} = p_1(x)y \tag{3.1}$$

$$\dot{y} = q_0(x) + q_2(x)y^2 \tag{3.2}$$

may admit isochronous motion for a fairly large class of functions p_1, q_0 and q_2 of the polynomial type.

Such a system may be conveniently recast as a single second-order ODE, namely,

$$\ddot{x} + f(x)\dot{x}^2 + g(x) = 0, \tag{3.3}$$

where

$$f(x) = -\frac{p_{1x} + q_2}{p_1}, \quad g(x) = -p_1 q_0.$$

As shown in [12] (and references therein) this second-order equation admits a Lagrangian description via the existence of a Jacobi last multiplier (JLM) $M(x)$, of the form

$$M(x) = \mathrm{e}^{2F(x)} \quad \text{with} \quad F(x) = \int^x f(s)\mathrm{d}s, \tag{3.4}$$

with

$$L(x, \dot{x}) = \frac{1}{2}M(x)\dot{x}^2 - V(x), \tag{3.5}$$

where the potential

$$V(x) = \int^x M(s)g(s)\mathrm{d}s.$$

The following proposition is well-known (see [13,14]).

Proposition 3.1. *The necessary condition for mapping of the potential $V(x)$ to that of the linear harmonic oscillator is*

$$g'(x) + f(x)g(x) = 1.$$

Proof. Suppose there exists a transformation $x \to Q$ such that $V(x) = \frac{1}{2}Q^2$ where $Q = \int^x \mathrm{e}^{F(s)}\mathrm{d}s$. Then

$$V'(x) = QQ_x \Rightarrow M(x)g(x) = Q\sqrt{M(x)} \Rightarrow \sqrt{M(x)}g(x) = Q.$$

Differentiation again with respect to x yields

$$g'(x) + \frac{1}{2}\frac{M'}{M}g = 1,$$

which implies from (3.4) that

$$g'(x) + f(x)g(x) = 1.$$

It is also well known that isochronous motion can also arise from the isotonic potential. The following proposition gives the necessary condition for this.

Proposition 3.2. *The necessary condition for mapping of $V(x)$ to the isotonic potential $V(x) = \frac{1}{8}Q^2 + \frac{c^2}{Q^2}$ is given by the condition*

$$f(x) = -\frac{5}{4}\left[\frac{g''(x) + (f(x)g(x))'}{g'(x) + f(x)g(x) - \frac{1}{4}}\right] + \frac{g'''(x) + (f(x)g(x))''}{g''(x) + (f(x)g(x))'}$$

where

$$Q = \int^x \sqrt{M(s)}\mathrm{d}s.$$

Proof. The proof is along similar lines as indicated above.

Example 4: $\dot{x} = y(1+x)$, $\dot{y} = \frac{1}{4}(1-(1+x)^4) + y^2$

It is easy to verify that the condition stated in Proposition 3.2 is indeed satisfied by this system. The potential $V(x)$ may be expressed as $1/8(Q^2 + Q^{-2})$ with $Q = -(1+x)^{-1}$ [12].

It is obvious that (3.3) may be recast as the following system:

$$\dot{x} = z \tag{3.6}$$

$$\dot{z} = -f(x)z^2 - g(x). \tag{3.7}$$

On the other hand a Legendre transformation with $p := \partial L/\partial \dot{x} = M(x)\dot{x}$ enables us to define the Hamiltonian of the system by

$$H = \frac{1}{2}\left(\frac{p}{\sqrt{M(x)}}\right)^2 + V(x). \tag{3.8}$$

By defining

$$P := \frac{p}{\sqrt{M(x)}} \quad \text{and} \quad Q := \int^x \sqrt{M(s)}\mathrm{d}s, \tag{3.9}$$

we can rewrite the Hamiltonian H as

$$H = \frac{1}{2}P^2 + V(Q),$$

which leads to the standard Hamilton's equations of motion, namely

$$\dot{Q} = P, \quad \dot{P} = -V'(Q). \tag{3.10}$$

Equation (3.10) has the following associated vector field

$$X = P\frac{\partial}{\partial Q} - Q\frac{\partial}{\partial P}$$

and it is also well known that this commutes with the vector field Y given by

$$Y = \frac{1}{2}\left(Q\frac{\partial}{\partial Q} + P\frac{\partial}{\partial P}\right).$$

Consequently, the transversal commuting vector field associated with an isochronous system (3.6), (3.7) can easily be found out by simply transforming the vector field Y to the old variables z and x thereby identifying the corresponding system of ODEs which commutes with the given isochronous system. This procedure, based on the Lagrangian formalism, enables us to systematically determine the transversal commuting system of a large number of isochronous ODEs derived by different methods. In the sequel we illustrate its application by means of the following example from [15].

Example 5: $\dot{x} = y(-1 + ax^{n-1})$, $\dot{y} = x + ax^{n-2}y^2$.

This equation can be written in the form

$$\dot{x} = z \tag{3.11}$$

$$\dot{z} = \frac{nax^{n-2}}{(-1 + ax^{n-1})} z^2 + x(-1 + ax^{n-1}). \tag{3.12}$$

The JLM associated with this system is given by

$$M = (-1 + ax^{n-1})^{-2\left(1 + \frac{1}{n-1}\right)}, \tag{3.13}$$

while

$$Q = \int^x (-1 + as^{n-1})^{n/(n-1)} \mathrm{d}s = -\frac{x}{(-1 + ax^{n-1})^{u/(n-1)}} \tag{3.14}$$

$$P = \sqrt{M(x)} z = (-1 + ax^{n-1})^{n/(n-1)} z. \tag{3.15}$$

Equations (3.14) and (3.15) may be inverted to solve for x and z which are given by

$$x = \frac{Q}{[(-1)^n + aQ^{n-1}]^{1/n-1}}$$

and

$$z = \frac{(-1)^{n(n+1)/(n-1)} P}{[(-1)^n + aQ^{n-1}]^{n/n-1}}. \tag{3.16}$$

Since the vector field associated with (3.11) and (3.12) under the transformation given in (3.16) is transformed to the standard Hamiltonian vector field X as defined previously and whose corresponding commuting partner is given by Y above, all one needs to do is to write Y in terms of the variables x and z. A simple calculation yields the following result for the differential system commuting with the system of equations (3.11) and (3.12), viz.

$$\dot{x} = -\frac{x}{2}(-1 + ax^{n-1}) \quad \text{and} \quad \dot{z} = \frac{z}{2}(1 - anx^{n-1}), \tag{3.17}$$

or upon reverting back to the original variables x and y of our example the system

$$\dot{x} = -\frac{x}{2}(-1 + ax^{n-1}) \;\; \text{and} \;\; \dot{y} = \frac{y}{2}(1 - ax^{n-1}). \tag{3.18}$$

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