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Linearization of systems of four second-order ordinary differential equations

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Abstract. In this paper we provide invariant linearizability criteria for a class of systems of four second-order ordinary differential equations in terms of a set of 30 constraint equations on the coefficients of all derivative terms. The linearization criteria are derived by the analytic continuation of the geometric approach of projection of two-dimensional systems of cubically semi-linear second-order differential equations. Furthermore, the canonical form of such systems is also established. Numerous examples are presented that show how to linearize nonlinear systems to the free particle Newtonian systems with a maximally symmetric Lie algebra relative to $sl(6, \Re)$ of dimension 35.

Keywords. Linearization; geometric projections; maximally symmetric; complex Newtonian systems.

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1. Introduction

The linearization, that is, mapping a nonlinear differential equation into a linear differential equation, is an important tool in the theory of differential equations. To differentiate such nonlinear differential equations from the linear equations, it is crucial to develop invariant linearizability criteria. It started with the pioneering work of Sophus Lie (see e.g. [1]) who proved that to be linearizable, a second-order ordinary differential equation (ODE) must be at most cubically semi-linear and an overdetermined system of conditions on the coefficients in it must be satisfied. These conditions were later reduced to a set of two conditions by Tressé (see e.g. [1]). Over the past few years a considerable amount of research was done in this area. It was proved in [2] that there exists five equivalence classes of linearizable systems of two second-order ODEs. The most general form of systems of second-order nonlinear ODEs that are transformable to the free particle equations was

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found and proved to be cubically semi-linear in the dependent variables [3]. Separately, the geometric approach of projection was exploited to find a maximally symmetric class of square systems of two linearizable second-order cubically semi-linear ODEs [4]. It was done by projecting down a system of three geodesic equations to one lower dimension. Systems of two quadratically semi-linear ODEs were also investigated on the basis of their geometry and algebras to obtain linearization criteria [5,6].

Recently, complex symmetry analysis (CSA) was used [7,8] to study the algebraic properties of two-dimensional systems with the help of the symmetry structures of the corresponding scalar complex ODEs. Linearizable systems that can be obtained by CSA [8] were proved not to be equivalent [9] to those found by geometric methods [4]. Here we use the equivalence of scalar second-order ODEs (see [1]) to find a canonical form of a class of systems of four second-order ODEs. This class will be the main topic of discussion in this paper. The equivalence of linear two-dimensional systems under point transformations [2] is used to extract the simplest linear forms for this class. We call these equivalence classes of linear systems 'special classes' as the coefficients give rise to a symmetry algebra for these systems which are different from their classical analogues. The main aim of the paper is to employ CSA to obtain a class of systems of four cubically semi-linear ODEs subject to linearization if the coefficients in that class satisfy 30 constraint equations. This class of linearizable systems is transformable to a system of Newtonian free particle systems with a maximally symmetry algebra identical to $sl(6, \Re)$.

The outline of the paper is as follows. The preliminaries are in §1. Section 2 deals with the derivation of the canonical forms of systems of four second-order linear ODEs. Section 3 deals with linearizability criteria for systems of four ODEs. In §5 we mention computer packages and algorithms to check linearizability of such systems. Applications are presented in §6. Section 7 provides a summary and discussion of the results.

2. Preliminaries

The general form of linear systems to which a nonlinear system can be mapped via point transformations is crucial to develop linearization criteria. For example, two-dimensional linearizable systems are characterized into five equivalence classes. Any system of n second-order nonhomogeneous linear ODEs,

$$\ddot{\mathbf{u}} = \kappa\dot{\mathbf{u}} + \lambda\mathbf{u} + \mathbf{c}, \tag{1}$$

where the dot denotes the differentiation with respect to t, can be mapped invertibly [6] by point transformations to one of the following forms (called counterparts of the Laguerre–Forsyth canonical forms):

$$\ddot{\mathbf{v}} = \mu\mathbf{v}, \quad \ddot{\mathbf{w}} = \nu\dot{\mathbf{w}}, \tag{2}$$

where κ, λ, μ, ν are $n \times n$ matrices and $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{c}$ are vectors. The number of arbitrary coefficients for a system of n ODEs in eq. (1) is $2n^2 + n$. The first of the equations in (2) gives a general form of systems of two linear ODEs,

$$\ddot{v}_1 = a(t)v_1 + b(t)v_2, \qquad \ddot{v}_2 = c(t)v_1 + d(t)v_2, \tag{3}$$

with four arbitrary coefficients. It can be further reduced to

$$u_1'' = \tilde{a}(x)u_1 + \tilde{b}(x)u_2, \quad u_2'' = \tilde{c}(x)u_1 - \tilde{a}(x)u_2, \tag{4}$$

where prime denotes differentiation with respect to x, with only three arbitrary coefficients [2].

A system of two second-order ODEs,

$$\begin{aligned} u_1'' &= \omega_1(x, u_1, u_2, u_1', u_2'), \\ u_2'' &= \omega_2(x, u_1, u_2, u_1', u_2'), \end{aligned} \tag{5}$$

corresponds to a complex differential equation

$$u'' = \omega(x, u, u'), \tag{6}$$

where $u(x) = u_1(x) + iu_2(x)$ is an analytic function, if the coefficients in (1) satisfy a set of conditions [10]. There exists only one class of linear scalar ODEs that have a maximal Lie algebra relative to $sl(3, \Re)$. A linear system of two ODEs that arises from CSA is of the form

$$\ddot{v}_1 = \beta_1(t)v_1 - \beta_2(t)v_2, \quad \ddot{v}_2 = \beta_2(t)v_1 + \beta_1(t)v_2, \tag{7}$$

with two arbitrary coefficients. But unlike the previous case (3), it can be reduced to a linear form with only one arbitrary coefficient,

$$u_1'' = -\beta_3(x)u_2, \quad u_2'' = \beta_3(x)u_1, \tag{8}$$

by invoking the transformations given in [10]. This is the optimal canonical form for linear two-dimensional systems.

A system of two cubically semi-linear ODEs of the form

$$\begin{aligned} &u_1'' + \alpha_1 u_1'^3 + 2\alpha_2 u_1'^2 u_2' + \alpha_3 u_1' u_2'^2 + \beta_1 u_1'^2 + 2\beta_2 u_1' u_2' + \beta_3 u_2'^2 \\ &\quad + \gamma_1 u_1' + \gamma_2 u_2' + \delta_1 = 0, \\ &u_2'' + \alpha_1 u_1'^2 u_2' + 2\alpha_2 u_1' u_2'^2 + \alpha_3 u_2'^3 + \beta_4 u_1'^2 + 2\beta_5 u_1' u_2' + \beta_6 u_2'^2 \\ &\quad + \gamma_3 u_1' + \gamma_4 u_2' + \delta_2 = 0, \end{aligned} \tag{9}$$

was obtained by projecting down a system of geodesic equations from a 3×3 system to a 2×2 system [4]. The necessary and sufficient conditions to linearize such nonlinear systems (9) were given in the form of 15 constraint equations on their coefficients which comes from the condition that the underlying manifold for the projected geodesic equations be *flat*. Moreover, these results include the quadratic and linear (in the first derivative) cases as special subcases.

3. Canonical forms for systems of four ODEs

The general four-dimensional system of linear ODEs is given by

$$f_i'' = \kappa_{ij} f_j' + \lambda_{ij} f_j + c_i, \quad i, j = 1, ..., 4, \tag{10}$$

where summation over repeated indices is assumed and the functions κ_{ij}, λ_{ij} and c_i are arbitrary functions of x. The above system has 36 coefficients. The algebraic classification of different symmetry algebras and the number of linearizable classes of such systems are not known except for the maximally symmetric case $sl(6, \Re)$ of the free particle Newtonian system. Here our interest lies in extracting those linearizable classes in the canonical forms which arise from CSA.

There are two possible ways to extract canonical forms for systems of four linear second-order ODEs. First we use the equivalence of the complex linear scalar differential equations. In [9] it was shown that the optimal canonical form for a special class of systems of two ODEs with Lie algebras of dimensions 6 and 7 contains only one coefficient. We complexify this special class of system corresponding to a linear scalar complex equation and obtain a unique class of those four-dimensional systems that involve only two arbitrary coefficients. The procedure whereby we complexify the two-dimensional system obtained from a scalar complex equation is called nested complexification; for example,

$$\begin{aligned} f_1'' &= -\alpha_1(x) f_3 + \alpha_2(x) f_4, \\ f_2'' &= -\alpha_1(x) f_4 - \alpha_2(x) f_3, \\ f_3'' &= \alpha_1(x) f_1 - \alpha_2(x) f_2, \\ f_4'' &= \alpha_1(x) f_2 + \alpha_2(x) f_1, \end{aligned} \tag{11}$$

corresponds to a linear two-dimensional complex system (8) with the identification $u_1 = f_1 + i f_2,\ u_2 = f_3 + i f_4$ and $\beta_3(x) = \alpha_1(x) + i\alpha_2(x)$. Furthermore, eq. (8) can be mapped to a scalar complex equation via point transformations

$$\nu_1 = \rho u_1, \quad \nu_2 = \rho u_2, \quad x = \int^t \rho^{-2}(s) \mathrm{d}s, \tag{12}$$

where $\rho'' = \beta_1 \rho$ and $\beta_3 = \rho^3 \beta_2$. Therefore we state the following theorem.

Theorem 1. *There exists an optimal canonical form of four-dimensional systems of linear ODEs arising from the nested complexification that contains only two arbitrary coefficients.*

Secondly a canonical form of linearizable systems can also be derived by utilizing the equivalence of systems of two complex ODEs. For example, the following system,

$$\begin{aligned} f_1'' &= \alpha_1(x) f_1 - \alpha_2(x) f_2 + \alpha_3(x) f_3 - \alpha_4(x) f_4, \\ f_2'' &= \alpha_2(x) f_1 + \alpha_1(x) f_2 + \alpha_4(x) f_3 + \alpha_3(x) f_4, \\ f_3'' &= \alpha_5(x) f_1 - \alpha_6(x) f_2 - \alpha_1(x) f_3 + \alpha_2(x) f_4, \\ f_4'' &= \alpha_6(x) f_1 + \alpha_5(x) f_2 - \alpha_2(x) f_3 - \alpha_1(x) f_4, \end{aligned} \tag{13}$$

with six arbitrary coefficients can be reduced to (4), using $u_1 = f_1 + i f_2,\ u_2 = f_3 + i f_4$ and all three parameters in (4) are complex functions of the real independent variable. Thus we obtain another canonical form which contains six arbitrary coefficients. These are the two linear forms for systems of four second-order ODEs with fewer arbitrary coefficients obtainable in the manner described. Note that these forms cannot be obtained using real symmetry analysis because the canonical form (2) requires at least $n^2 = 16$ arbitrary coefficients. We state these results in the form of the following theorem.

Theorem 2. *Any linearizable system of four second-order nonlinear ODEs obtainable from a complex scalar ODE or a system of two second-order complex ODEs is transformable to one of the forms* (11) *or* (13).

4. Linearization for systems of four ODEs

To establish an invariant linearizability criteria for a class of systems of four differential equations, we apply analytical continuation. The extension of the geometric approach of projection used in developing the criteria for systems of two semi-linear ODEs in the complex domain enables us to write down explicit criteria for those four-dimensional systems that are maximally symmetric, that is, which have a maximal Lie algebra $sl(6, \Re)$. This provides us a set of 30 equations to be satisfied by the coefficients that arise in this class of systems. In this regard we state a theorem which can easily be proved following the same lines developed for two-dimensional systems in [9].

Theorem 3. *A system of four second-order cubically semi-linear ODEs,*

$$\begin{aligned}
f_1'' &+ (\alpha_{11}f_1' - 3\alpha_{12}f_2' + 2\alpha_{21}f_3' - 2\alpha_{22}f_4' + \beta_{11}^1)f_1'^2 \\
&- (3\alpha_{11}f_1' - \alpha_{12}f_2' + 2\alpha_{21}f_3' - 2\alpha_{22}f_4' - \beta_{11}^1)f_2'^2 \\
&+ (\alpha_{31}f_1' - \alpha_{32}f_2' + \beta_{31}^1)f_3'^2 + (\alpha_{31}f_1' + \alpha_{32}f_2' - \beta_{31}^1)f_4'^2 \\
&+ 2(2\alpha_{22}f_1'f_2'f_3' + 2\alpha_{21}f_1'f_2'f_4' + \alpha_{32}f_1'f_3'f_4' - \alpha_{31}f_2'f_3'f_4') \\
&- 2(\beta_{12}^1f_1'f_2' - \beta_{21}^1f_1'f_3' + \beta_{22}^1f_1'f_4' + \beta_{22}^1f_2'f_3' + \beta_{21}^1f_2'f_4' + \beta_{32}^1f_3'f_4') \\
&+ \gamma_{11}^1f_1' - \gamma_{12}^1f_2' + \gamma_{21}^1f_3' - \gamma_{22}^1f_4' + \delta_{11} = 0, \qquad (14)\\
f_2'' &+ (\alpha_{12}f_1' + 3\alpha_{11}f_2' + 2\alpha_{22}f_3' + 2\alpha_{21}f_4' + \beta_{12}^1)f_1'^2 \\
&- (3\alpha_{12}f_1' + \alpha_{11}f_2' + 2\alpha_{22}f_3' + 2\alpha_{21}f_4' - \beta_{12}^1)f_2'^2 \\
&+ (\alpha_{32}f_1' + \alpha_{31}f_2' + \beta_{32}^1)f_3'^2 - (\alpha_{32}f_1' + \alpha_{31}f_2' - \beta_{32}^1)f_4'^2 \\
&+ 2(2\alpha_{21}f_1'f_2'f_3' + 2\alpha_{22}f_1'f_2'f_4' - \alpha_{31}f_1'f_3'f_4' - \alpha_{32}f_2'f_3'f_4') \\
&+ 2(\beta_{11}^1f_1'f_2' + \beta_{22}^1f_1'f_3' + \beta_{21}^1f_1'f_4' + \beta_{21}^1f_2'f_3' - \beta_{22}^1f_2'f_4' + \beta_{31}^1f_3'f_4') \\
&+ \gamma_{12}^1f_1' + \gamma_{11}^1f_2' + \gamma_{22}^1f_3' + \gamma_{21}^1f_4' + \delta_{12} = 0, \qquad (15)\\
f_3'' &+ (\alpha_{11}f_3' - \alpha_{12}f_4' + \beta_{11}^2)f_1'^2 - (\alpha_{11}f_3' - \alpha_{12}f_4' - \beta_{11}^2)f_2'^2 \\
&+ (2\alpha_{21}f_1' - 2\alpha_{22}f_2' + \alpha_{31}f_3' - 3\alpha_{32}f_4' + \beta_{31}^2)f_3'^2 \\
&- (2\alpha_{21}f_1' - 2\alpha_{22}f_2' + 3\alpha_{31}f_3' - \alpha_{32}f_4' - \beta_{31}^2)f_4'^2 \\
&- 2(\alpha_{12}f_1'f_2'f_3' + \alpha_{11}f_1'f_2'f_4' + 2\alpha_{22}f_1'f_3'f_4' + 2\alpha_{21}f_2'f_3'f_4') \\
&- 2(\beta_{12}^2f_1'f_2' - \beta_{21}^2f_1'f_3' + \beta_{22}^2f_1'f_4' + \beta_{22}^2f_2'f_3' + \beta_{21}^2f_2'f_4' + \beta_{32}^2f_3'f_4') \\
&+ \gamma_{11}^2f_1' - \gamma_{12}^2f_2' + \gamma_{21}^2f_3' - \gamma_{22}^2f_4' + \delta_{21} = 0, \qquad (16)\\
f_4'' &+ (\alpha_{12}f_3' + \alpha_{11}f_4' + \beta_{21}^2)f_1'^2 - (\alpha_{12}f_3' + \alpha_{11}f_4' - \beta_{12}^2)f_2'^2 \\
&+ (2\alpha_{22}f_1' + 2\alpha_{21}f_2' + \alpha_{32}f_3' + 3\alpha_{31}f_4' + \beta_{32}^2)f_3'^2 \\
&- (2\alpha_{22}f_1' + 2\alpha_{21}f_2' + 3\alpha_{32}f_3' + \alpha_{31}f_4' - \beta_{32}^2)f_4'^2 \\
&+ 2(\alpha_{11}f_1'f_2'f_3' - \alpha_{12}f_1'f_2'f_4' + 2\alpha_{21}f_1'f_3'f_4' - 2\alpha_{22}f_2'f_3'f_4') \\
&+ 2(\beta_{11}^2f_1'f_2' - \beta_{22}^2f_1'f_3' + \beta_{21}^2f_1'f_4' + \beta_{21}^2f_2'f_3' - \beta_{22}^2f_2'f_4' + \beta_{31}^2f_3'f_4') \\
&+ \gamma_{12}^2f_1' + \gamma_{11}^2f_2' + \gamma_{22}^2f_3' + \gamma_{21}^2f_4' + \delta_{22} = 0, \qquad (17)
\end{aligned}$$

where

$$\alpha_{ij} \neq \alpha_{ji}, \quad \beta^k_{ij} \neq \beta^k_{ji}, \quad \gamma^k_{ij} \neq \gamma^k_{ji}, \quad \delta_{ij} \neq \delta_{ji}, \tag{18}$$

is linearizable with respect to the linearizability of the system of two equations if and only if the coefficients satisfy the 30 *constraint equations given in Appendix.*

Once the linearizability criteria are satisfied by a system, the next step involves the construction of invertible linearizing transformations which can often be difficult. In the subsequent section we show how a complex linearizing transformation of the form

$$X = X(x), \quad U = U(x, u, v), \quad V = V(x, u, v) \tag{19}$$

plays the part of generating invertible transformations to linearize the corresponding nonlinear systems of four ODEs. Notice that the independent variable X depends only on x. The above transformation is a special case of the general invertible complex point transformation

$$X = X(x, u, v), \quad U = U(x, u, v), \quad V = V(x, u, v), \tag{20}$$

for two-dimensional systems. The dependence of X on the complex variables (x, u, v) makes it a complex independent variable. This may seem strange as the decomposition of the linearized system would yield a system of partial differential equations instead of ODEs. However, the fact is that the resultant system is equipped with Cauchy–Riemann equations which makes the system integrable and the solution is obtained in the original variables upon using invertible transformations. This line of approach is followed in [10,11] for two-dimensional nonlinear systems corresponding to nonlinear complex scalar equations and it was found that there exist two classes of such systems, namely, linearizable and nonlinearizable systems. It is shown that there exist nonlinearizable systems with Lie algebras of dimensions $\{1, 2, 3, 4\}$ which can be complex-linearized using complex transformations (20). In this paper we restrict our investigation to the transformations (19) that yield a set of real invertible transformations for systems of four ODEs. Furthermore, the solution of a nonlinear system can be obtained from the linearized system by utilizing such real transformations. The complete characterization of all four-dimensional nonlinear systems and their Lie algebras subject to the complex transformations (20) will be the subject of a forthcoming paper.

5. Computer algorithm

The linearizability conditions include 30 partial differential equations of the first order in the coefficients of a nonlinear system of four ODEs, relative to the dependent and independent variables. To avoid cumbersome calculations, we use a computer algebra system, for example, MAPLE or CRACK, to check the linearizability of a system. For this purpose we have translated the 30 constraint equations in the form of a computer code in MAPLE to test linearizability for a nonlinear system of the form (14)−(17) whose coefficients are used as inputs. The code also determines the symmetry algebra of a system.

6. Applications

We consider many examples to verify the theory developed for the linearization of systems of four second-order ODEs. It is important to note that nonlinearity depends on the coupling of both derivative and nonderivative terms in such systems for which there could be many choices. Furthermore, there can be $3! = 6$ types of coupling, for example, the first equation can be coupled with the other three in the system. The first example illustrates how the complex variable approach reveals the complete integrability of a system that has the maximal algebra $sl(6, \Re)$. Example 3 is of great interest for two main reasons. First, it is a geodesic type system of differential equations in which the nonlinearity arises due to the quadratic terms in first-order derivatives [5]. Such equations appear frequently in relativity in the study of shortest paths on curved manifold. Secondly, this system can be extended to higher dimensions to yield the linearization of a system of $4n$ ODEs which has the Lie algebra $sl(2(2n + 1), \Re)$ where $n \geq 1$.

Example 1. Consider a quadratically semi-linear system of four second-order ODEs,

$$\begin{aligned}
&f_1'' - f_1' + f_1'^2 - f_2'^2 = 0,\\
&f_2'' - f_2' + 2f_1'f_2' = 0,\\
&f_3'' - f_3' + f_3'^2 - f_4'^2 = 0,\\
&f_4'' - f_4' + 2f_3'f_4' = 0,
\end{aligned} \tag{21}$$

where the first two and last two equations are coupled in the first derivatives of the dependent variables f_1, f_2 and f_3, f_4 respectively. If we substitute all the coefficients $\beta_{11}^1 = \beta_{31}^2 = 1$ and $\gamma_{11}^1 = \gamma_{21}^2 = -1$ in the linearizability conditions, then it can be verified that the above system is linearizable. To find the linearizing transformations to map system (21) into a linear target system $F_\alpha'' = 0,\ \alpha = 1, ..., 4$, we use

$$\begin{aligned}
&X = \exp(x), \quad F_1 = \exp(f_1)\cos(f_2), \quad F_2 = \exp(f_1)\sin(f_2),\\
&F_3 = \exp(f_3)\cos(f_4), \quad F_4 = \exp(f_3)\sin(f_4).
\end{aligned} \tag{22}$$

Example 2. We now take an example of a quadratically semi-linear system with the same coupling as above but whose coefficients involve the dependent and independent variables,

$$\begin{aligned}
&f_1'' - \frac{f_1(f_1'^2 - f_2'^2) + 2f_2f_1'f_2'}{f_1^2 + f_2^2} - xf_1 = 0,\\
&f_2'' - \frac{2f_1f_1'f_2' - f_2(f_1'^2 - f_2'^2)}{f_1^2 + f_2^2} - xf_2 = 0,\\
&xf_3'' + xf_3'^2 - xf_4'^2 + 2f_3' = 0,\\
&xf_4'' + 2xf_3'f_4' + 2f_4' = 0.
\end{aligned} \tag{23}$$

Linearizability of this system is ensured by Theorem 3 since the coefficients

$$\begin{aligned}
&\beta_{11}^1 = \frac{-f_1}{f_1^2 + f_2^2}, \quad \beta_{12}^1 = \frac{f_2}{f_1^2 + f_2^2}, \quad \beta_{31}^2 = 1,\\
&\gamma_{21}^2 = \frac{2}{x}, \quad \delta_{11} = -xf_1, \quad \delta_{12} = -xf_2,
\end{aligned} \tag{24}$$

satisfy the 30 constraint equations given for linearizability. The linear target system in this case is

$$F_1'' = X^{-4}, \quad F_2'' = F_3'' = F_4'' = 0, \tag{25}$$

upon utilizing the transformations,

$$X = \frac{1}{x}, \;\; F_1 = \frac{1}{2x}\ln(f_1^2 + f_2^2), \;\; F_2 = \frac{1}{x}\arctan\left(\frac{f_2}{f_1}\right),$$
$$F_3 = \exp(f_3)\cos(f_4), \;\; F_4 = \exp(f_3)\sin(f_4). \tag{26}$$

Notice that the above linearizing transformations came from an invertible complex transformation,

$$X = \frac{1}{x}, \quad U = \frac{1}{x}\ln u, \quad V = \exp v. \tag{27}$$

We used CRACK to verify that the system (23) has 35 symmetries. In all other examples we used MAPLE to verify the number of symmetries.

Example 3. In this case we consider a coupled system of four real ODEs of geodesic type given by

$$f_1'' + f_1'^2 - f_2'^2 = 0, \;\; f_2'' + 2f_1'f_2' = 0,$$
$$f_3'' + f_3'^2 - f_4'^2 + 2(f_1'f_3' - f_2'f_4') = 0,$$
$$f_4'' + 2(f_3'f_4' + f_1'f_4' + f_3'f_2') = 0, \tag{28}$$

where the first two equations are coupled in the first derivatives of f_1 and f_2 and the last two equations contain the coupling relative to the first derivatives of all the dependent variables. It can be easily checked that the above system is linearizable by substituting the coefficients $\beta_{11}^1 = \beta_{21}^2 = \beta_{31}^2 = 1$ into the linearizability conditions. The system (28) can be mapped to $F_\alpha'' = 0$ by the real linearizing transformations,

$$F_1 = \exp(f_1)\cos(f_2), \quad F_2 = \exp(f_1)\sin(f_2),$$
$$F_3 = \exp(f_1 + f_3)\cos(f_2 + f_4), \quad F_4 = \exp(f_1 + f_3)\sin(f_2 + f_4). \tag{29}$$

We now present the generalization of the system (28) in a $(4n+1)$-dimensional space. We first observe that in system (28) the first two equations are coupled with each other and the last two equations contains full coupling. We take a system of $2n$ such equations of the form

$$f_j'' + f_j'^2 - g_j'^2 = 0,$$
$$g_j'' + 2f_j'g_j' = 0, \tag{30}$$

where $j = 1, ..., n$, and another system of $2n$ ODEs that contains full coupling, that is,

$$h_j'' + h_j'^2 - k_j'^2 + 2(f_j'h_j' - g_j'k_j') = 0,$$
$$k_j'' + 2(h_j'k_j' + f_j'k_j' + h_j'g_j') = 0, \tag{31}$$

where **f**, **g**, **h**, **k**, are n-dimensional vectors with $4n$ dependent functions of x. The systems (30) and (31) can also be regarded as a surface in a $(12n + 1)$-dimensional space $(x, \mathbf{f}, \mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{f}', \mathbf{g}', \mathbf{h}', \mathbf{k}', \mathbf{f}'', \mathbf{g}'', \mathbf{h}'', \mathbf{k}'')$. We now introduce a complex structure on the $12n$-dimensional space with the identification

$$u_j = u_j(x) := f_j(x) + ig_j(x), \quad v_j = v_j(x) := h_j(x) + ik_j(x). \tag{32}$$

Therefore, the systems (30) and (31) are mapped to a system of $2n$ ODEs in a $2n$-complex dimensional space,

$$u_j'' + u_j'^2 = 0, \quad v_j'' + v_j'^2 + 2u_j'v_j' = 0, \quad j = 1, 2, ..., n, \tag{33}$$

which can be linearized to the complexified free particle system,

$$U_j'' = 0, \quad V_j'' = 0, \tag{34}$$

using the complex transformations,

$$U_j = \exp u_j, \quad V_j = \exp(u_j + v_j). \tag{35}$$

Consequently, systems (30) and (31) are linearized to the maximally symmetric system $sl(2(2n+1), \Re)$,

$$\begin{aligned} &F_j'' = 0, \quad G_j'' = 0, \\ &H_j'' = 0, \quad K_j'' = 0, \end{aligned} \tag{36}$$

using the real transformations obtained from (35). The next example is also of geodesic type with all equations coupled together.

Example 4. We now present a case where the nonlinear system is a fully coupled four-dimensional system,

$$\begin{aligned} &f_1'' - f_1'^2 + f_2'^2 - f_3'^2 + f_4'^2 = 0, \quad f_2'' - 2(f_1'f_2' + f_3'f_4') = 0, \\ &f_3'' - 2(f_1'f_3' - f_2'f_4') = 0, \quad f_4'' - 2(f_1'f_4' + f_2'f_3') = 0. \end{aligned} \tag{37}$$

This can be obtained from a system of two complex ODEs, that is, from the complexified Newtonian system,

$$u'' - u'^2 - v'^2 = 0, \quad v'' - 2u'v' = 0, \tag{38}$$

which has a four-dimensional Lie algebra with commutation relations

$$[X_\alpha, X_\beta] = 0, \quad [X_\alpha, X_4] = X_\alpha, \quad \alpha, \beta = 1, 2, 3, \tag{39}$$

identified as $L_{4,2}$ of symmetries. This is reducible to the free particle system as the coefficients $\beta^1_{11} = \beta^1_{31} = \beta^2_{21} = -1$ satisfy the linearizability conditions. The following transformations of the dependent variables,

$$\begin{aligned}
F_1 &= c_1 \exp(f_3 - f_1)\cos(f_4 - f_2) + c_2 \exp(-f_3 - f_1)\cos(f_4 + f_2),\\
F_2 &= c_1 \exp(f_3 - f_1)\sin(f_4 - f_2) + c_2 \exp(-f_3 - f_1)\sin(f_4 + f_2),\\
F_3 &= c_2 \exp(-f_3 - f_1)\cos(f_4 + f_2) - c_1 \exp(f_3 - f_1)\cos(f_4 - f_2),\\
F_4 &= -c_2 \exp(-f_3 - f_1)\sin(f_4 + f_2) - c_1 \exp(f_3 - f_1)\sin(f_4 - f_2),
\end{aligned} \tag{40}$$

maps the nonlinear system to $F''_\alpha = 0,\ \alpha = 1, ..., 4.$

Example 5. We conclude by presenting a maximally symmetric system with variable coefficients,

$$\begin{aligned}
&xf''_1 + f'_1 + xf'^2_1 - xf'^2_2 = 0,\quad xf''_2 + f'_2 + 2xf'_1 f'_2 = 0,\\
&xf''_3 + f'_3 + xf'^2_3 - xf'^2_4 + 2xf'_1 f'_3 - 2xf'_2 f'_4 = 0,\\
&xf''_4 + f'_4 + 2xf'_3 f'_4 + 2xf'_1 f'_4 + 2xf'_2 f'_3 = 0.
\end{aligned} \tag{41}$$

This system is linearizable and can be mapped to a system of the free particle system via the following change of the dependent and independent variables:

$$\begin{aligned}
&X = \ln x,\quad F_1 = \exp(f_1)\cos(f_2),\quad F_2 = \exp(f_1)\sin(f_2),\\
&F_3 = \exp(f_1 + f_3)\cos(f_2 + f_4),\quad F_4 = \exp(f_1 + f_3)\sin(f_2 + f_4).
\end{aligned} \tag{42}$$

These can be obtained by the complex linearizing transformations of the corresponding complex linearizable system. Linearizability is ensured since the coefficients $\beta^1_{11} = \beta^2_{21} = \beta^2_{31} = 1$ and $\gamma^1_{11} = \gamma^2_{21} = x^{-1}$ of the nonlinear system (39) satisfy the linearizability criteria.

7. Summary and discussion

We provided canonical forms for linear systems of four second-order ODEs to which a nonlinear system can be mapped by complex methods. We made use of the equivalence of linear scalar complex ODEs as well as the equivalence of systems of two complex ODEs of second order as the tool to establish canonical forms for systems of four ODEs. The next step was to construct linearizability criteria for such nonlinear systems. We established general forms of these systems which are transformable to a system of free particle equations, that is, it is at most cubically semi-linear in the first derivative. For this purpose we extended the geometric approach of projection developed by projecting down systems of geodesic equations by one dimension. These systems were in general algebraically inequivalent to those obtainable by CSA. By extending them to obtain a system of four second-order ODEs we provided the linearizability criteria for a system of four ODEs by using the corresponding linearizability criteria for systems of two complex ODEs. We found a class of four-dimensional systems which is maximally symmetric with regard to the base equations.

The linearizable systems that can be obtained by complex symmetry analysis are inequivalent to those obtained from geometric methods. Thus, complex symmetry analysis was proved to provide systems of ODEs with symmetry structures different from the systems that appear in classical symmetry analysis. Surprisingly there exist certain systems of two second-order ODEs that admit 4-, 3-, 2-, 1-dimensional algebra yet can be linearized using complex variables [10]. So it enables us to reveal complete integrability of a class of two-dimensional systems of nonmaximal symmetry algebras. Furthermore, the three equivalence classes of linearizable systems of ODEs that correspond to 6-, 7- and 15-dimensional algebra among five were also constructed. Therefore, the complex symmetry approach has an advantage over the standard Lie approach in linearizing systems of ODEs. Therefore, it would be very interesting to see the analogue of these results in the context of four-dimensional systems. The complex linearizability of such systems can be established.

Appendix

$$
\begin{aligned}
&2(\gamma^2_{11,x}-\delta_{21,f_1}-\delta_{22,f_2})+(\gamma^2_{21}\gamma^2_{11}-\gamma^2_{22}\gamma^2_{12}+\gamma^1_{11}\gamma^2_{11}-\gamma^1_{12}\gamma^2_{12})\\
&\quad-4(\beta^2_{11}\delta_{11}-\beta^2_{12}\delta_{12}+\beta^2_{21}\delta_{21}-\beta^2_{22}\delta_{22})=0,\\
&2(\gamma^2_{12,x}+\delta_{21,f_2}-\delta_{22,f_1})+(\gamma^2_{21}\gamma^2_{12}+\gamma^2_{11}\gamma^2_{22}+\gamma^1_{11}\gamma^2_{12}+\gamma^1_{12}\gamma^2_{11})\\
&\quad-4(\beta^2_{12}\delta_{11}+\beta^2_{11}\delta_{12}+\beta^2_{22}\delta_{21}+\beta^2_{21}\delta_{22})=0,\\
&4\beta^2_{11,x}-\gamma^2_{11,f_1}-\gamma^2_{12,f_2}-4(\alpha_{11}\delta_{21}-\alpha_{12}\delta_{22})+2(\beta^1_{11}\gamma^2_{11}-\beta^1_{12}\gamma^2_{12})\\
&\quad+2(\beta^2_{11}\gamma^2_{21}-\beta^2_{12}\gamma^2_{22})-2(\beta^2_{11}\gamma^1_{11}-\beta^2_{12}\gamma^1_{12})\\
&\quad-2(\beta^2_{21}\gamma^2_{11}-\beta^2_{22}\gamma^2_{12})=0,\\
&4\beta^2_{12,x}+\gamma^2_{11,f_2}-\gamma^2_{12,f_1}-4(\alpha_{11}\delta_{22}+\alpha_{12}\delta_{21})+2(\beta^1_{12}\gamma^2_{11}+\beta^1_{11}\gamma^2_{12})\\
&\quad+2(\beta^2_{12}\gamma^2_{21}+\beta^2_{11}\gamma^2_{22})-2(\beta^2_{12}\gamma^1_{11}+\beta^2_{11}\gamma^1_{12})\\
&\quad-2(\beta^2_{21}\gamma^2_{12}+\beta^2_{22}\gamma^2_{11})=0,\\
&12\beta^2_{21,x}-4\beta^1_{11,x}+\gamma^1_{11,f_1}+\gamma^1_{12,f_2}-16(\alpha_{21}\delta_{21}-\alpha_{22}\delta_{22})\\
&\quad-8(\beta^2_{11}\gamma^1_{21}-\beta^2_{12}\gamma^1_{22})+8(\beta^1_{21}\gamma^2_{11}-\beta^1_{22}\gamma^2_{12})\\
&\quad-3(\gamma^2_{21,f_1}+\gamma^2_{22,f_2})=0,\\
&12\beta^2_{22,x}-4\beta^1_{12,x}+\gamma^1_{12,f_1}-\gamma^1_{11,f_2}-16(\alpha_{22}\delta_{21}+\alpha_{21}\delta_{22})\\
&\quad-8(\beta^2_{11}\gamma^1_{22}+\beta^2_{12}\gamma^1_{21})+8(\beta^1_{21}\gamma^2_{12}+\beta^1_{22}\gamma^2_{11})\\
&\quad-3(\gamma^2_{22,f_1}+\gamma^2_{21,f_2})=0,\\
&2(\gamma^1_{21,x}-\delta_{11,f_3}-\delta_{12,f_4})+\gamma^1_{21}\gamma^2_{21}-\gamma^1_{22}\gamma^2_{22}+\gamma^1_{21}\gamma^1_{11}-\gamma^1_{22}\gamma^1_{12}\\
&\quad-4(\beta^1_{21}\delta_{11}-\beta^1_{22}\delta_{12}+\beta^1_{31}\delta_{21}-\beta^1_{32}\delta_{22})=0,\\
&2(\gamma^1_{22,x}-\delta_{12,f_3}+\delta_{11,f_4})+\gamma^1_{21}\gamma^2_{22}+\gamma^1_{22}\gamma^2_{21}+\gamma^1_{21}\gamma^1_{12}+\gamma^1_{22}\gamma^1_{11}\\
&\quad-4(\beta^1_{21}\delta_{12}+\beta^1_{22}\delta_{11}+\beta^1_{31}\delta_{22}+\beta^1_{32}\delta_{21})=0,\\
&4\beta^1_{31,x}-\gamma^1_{21,f_3}-\gamma^1_{22,f_4}-4(\alpha_{31}\delta_{11}-\alpha_{32}\delta_{12})\\
&\quad+2(\beta^2_{31}\gamma^1_{21}-\beta^2_{32}\gamma^1_{22}-\beta^1_{21}\gamma^1_{21}+\beta^1_{22}\gamma^1_{22}-\beta^1_{31}\gamma^2_{21}\\
&\quad+\beta^1_{32}\gamma^2_{22}+\beta^1_{31}\gamma^1_{11}-\beta^1_{32}\gamma^1_{12})=0,
\end{aligned}
$$

$$
\begin{aligned}
&4\beta^1_{32,x} - \gamma^1_{22,f_3} + \gamma^1_{21,f_4} - 4(\alpha_{32}\delta_{11} + \alpha_{31}\delta_{12})\\
&\quad + 2(\beta^2_{32}\gamma^1_{21} + \beta^2_{31}\gamma^1_{22} - \beta^1_{21}\gamma^1_{22} - \beta^1_{22}\gamma^1_{21} - \beta^1_{31}\gamma^2_{22}\\
&\quad - \beta^1_{32}\gamma^2_{21} + \beta^1_{31}\gamma^1_{12} + \beta^1_{32}\gamma^1_{11}) = 0,\\
&(\alpha_{21,f_1} + \alpha_{22,f_2} - \alpha_{11,f_3} - \alpha_{12,f_4}) + 2(\alpha_{11}\beta^1_{21} - \alpha_{12}\beta^1_{22} + \alpha_{21}\beta^2_{21}\\
&\quad - \alpha_{22}\beta^2_{22} - \alpha_{21}\beta^1_{11} + \alpha_{22}\beta^1_{12} - \alpha_{31}\beta^2_{11} + \alpha_{32}\beta^2_{12}) = 0,\\
&(\alpha_{22,f_1} - \alpha_{21,f_2} - \alpha_{12,f_3} + \alpha_{11,f_4}) + 2(\alpha_{11}\beta^1_{22} + \alpha_{12}\beta^1_{21} + \alpha_{21}\beta^2_{22} + \alpha_{22}\beta^2_{21}\\
&\quad - \alpha_{21}\beta^1_{12} - \alpha_{22}\beta^1_{11} - \alpha_{31}\beta^2_{12} - \alpha_{32}\beta^2_{11}) = 0,\\
&(\alpha_{31,f_1} + \alpha_{32,f_2} - \alpha_{21,f_3} - \alpha_{22,f_4}) + 2(\alpha_{11}\beta^1_{31} - \alpha_{12}\beta^1_{32} + \alpha_{21}\beta^2_{31}\\
&\quad - \alpha_{22}\beta^2_{32} - \alpha_{21}\beta^1_{21} + \alpha_{22}\beta^1_{22} - \alpha_{31}\beta^2_{21} + \alpha_{32}\beta^2_{22}) = 0,\\
&(\alpha_{32,f_1} - \alpha_{31,f_2} - \alpha_{22,f_3} + \alpha_{21,f_4}) + 2(\alpha_{11}\beta^1_{32} + \alpha_{12}\beta^1_{31} + \alpha_{21}\beta^2_{32}\\
&\quad + \alpha_{22}\beta^2_{31} - \alpha_{21}\beta^1_{22} - \alpha_{22}\beta^1_{21} - \alpha_{31}\beta^2_{22} - \alpha_{32}\beta^2_{21}) = 0,\\
&-6\alpha_{21,x} + 5(\alpha_{21}\gamma^1_{11} - \alpha_{22}\gamma^1_{12}) + 2(\alpha_{31}\gamma^2_{11} - \alpha_{32}\gamma^2_{12}) - 2(\beta^2_{21,f_3} + \beta^2_{22,f_4})\\
&\quad + 6(\beta^1_{31}\beta^2_{11} - \beta^1_{32}\beta^2_{12}) + (\alpha_{21}\gamma^2_{21} - \alpha_{22}\gamma^2_{22})\\
&\quad - 6(\beta^1_{21}\beta^2_{21} - \beta^1_{22}\beta^2_{22}) - 4(\beta^1_{21,f_1} + \beta^1_{22,f_2}) + 2(\beta^2_{31,f_1} + \beta^2_{32,f_2})\\
&\quad + 4(\beta^1_{11,f_3} + \beta^1_{12,f_4}) - 2(\alpha_{11}\gamma^1_{21} - \alpha_{12}\gamma^1_{22}) = 0,\\
&-6\alpha_{22,x} + 5(\alpha_{21}\gamma^1_{12} + \alpha_{22}\gamma^1_{11}) + 2(\alpha_{31}\gamma^2_{12} + \alpha_{32}\gamma^2_{11}) - 2(\beta^2_{22,f_3} + \beta^2_{21,f_4})\\
&\quad + 6(\beta^1_{31}\beta^2_{12} + \beta^1_{32}\beta^2_{11}) + (\alpha_{21}\gamma^2_{22} + \alpha_{22}\gamma^2_{21})\\
&\quad - 6(\beta^1_{21}\beta^2_{22} + \beta^1_{22}\beta^2_{21}) - 4(\beta^1_{22,f_1} - \beta^1_{21,f_2}) + 2(\beta^2_{32,f_1} - \beta^2_{31,f_2})\\
&\quad + 4(\beta^1_{12,f_3} - \beta^1_{11,f_4}) - 2(\alpha_{12}\gamma^1_{21} + \alpha_{11}\gamma^1_{22}) = 0,\\
&-2\alpha_{11,x} + (\alpha_{11}\gamma^1_{11} - \alpha_{12}\gamma^1_{12}) - 2(\beta^2_{11}\beta^2_{31} - \beta^2_{12}\beta^2_{32}) + \beta^2_{21,f_1}\\
&\quad + \beta^2_{22,f_2} - \beta^2_{11,f_3} - \beta^2_{12,f_4} + 2(\beta^2_{11}\beta^1_{21} - \beta^2_{12}\beta^1_{22} + \beta^2_{21}\beta^2_{21} - \beta^2_{22}\beta^2_{22})\\
&\quad + (\alpha_{11}\gamma^2_{21} - \alpha_{12}\gamma^2_{22}) - 2(\beta^2_{21}\beta^1_{11} - \beta^2_{22}\beta^1_{12}) = 0,\\
&-2\alpha_{12,x} + (\alpha_{11}\gamma^1_{12} + \alpha_{12}\gamma^1_{11}) - 2(\beta^2_{31}\beta^2_{12} + \beta^2_{11}\beta^2_{32})\\
&\quad + (\beta^2_{22,f_1} - \beta^2_{21,f_2} - \beta^2_{12,f_3} + \beta^2_{11,f_4}) + 2(\beta^2_{11}\beta^1_{22} + \beta^1_{21}\beta^2_{12} + 2\beta^2_{21}\beta^2_{22})\\
&\quad + (\alpha_{11}\gamma^2_{22} + \alpha_{12}\gamma^2_{21}) - 2(\beta^2_{21}\beta^1_{12} + \beta^2_{22}\beta^1_{11}) = 0,\\
&-2\alpha_{31,x} + (\alpha_{31}\gamma^1_{11} - \alpha_{32}\gamma^1_{12}) + (\alpha_{31}\gamma^2_{21} - \alpha_{32}\gamma^2_{22})\\
&\quad - (\beta^1_{31,f_1} + \beta^1_{32,f_2} - \beta^1_{21,f_3} - \beta^1_{22,f_4})\\
&\quad - 2(\beta^1_{11}\beta^1_{31} - \beta^1_{12}\beta^1_{32} - \beta^1_{21}\beta^1_{21} + \beta^1_{22}\beta^1_{22} + \beta^1_{21}\beta^2_{31}\\
&\quad - \beta^1_{22}\beta^2_{32} - \beta^1_{31}\beta^2_{21} + \beta^1_{32}\beta^2_{22}) = 0,\\
&-2\alpha_{32,x} + (\alpha_{32}\gamma^1_{11} + \alpha_{31}\gamma^1_{12}) + (\alpha_{31}\gamma^2_{22} + \alpha_{32}\gamma^2_{21})\\
&\quad - (\beta^1_{32,f_1} - \beta^1_{31,f_2} - \beta^1_{22,f_3} + \beta^1_{21,f_4})\\
&\quad - 2(\beta^1_{11}\beta^1_{32} + \beta^1_{12}\beta^1_{31} - 2\beta^1_{21}\beta^1_{22} + \beta^1_{21}\beta^2_{32} + \beta^1_{22}\beta^2_{31} - \beta^1_{31}\beta^2_{22} - \beta^1_{32}\beta^2_{21}) = 0,\\
&-8\beta^1_{11,x} + 2(\gamma^1_{11,f_1} + \gamma^1_{12,f_2}) - 6(\beta^2_{31}\gamma^2_{11} - \beta^2_{32}\gamma^2_{12}) + 12(\alpha_{11}\delta_{11} - \alpha_{12}\delta_{12})\\
&\quad - 8(\alpha_{21}\delta_{21} - \alpha_{22}\delta_{22}) - 4(\beta^2_{11}\gamma^1_{21} - \beta^2_{12}\gamma^1_{22})\\
&\quad + 10(\beta^1_{21}\gamma^2_{11} - \beta^1_{22}\gamma^2_{12}) + 12\beta^2_{21,x} - 3(\gamma^2_{11,f_3} + \gamma^2_{12,f_4})
\end{aligned}
$$

$$+6(\beta^2_{21}\gamma^2_{21} - \beta^2_{22}\gamma^2_{22} - \beta^2_{21}\gamma^1_{11} + \beta^2_{22}\gamma^1_{12}) = 0,$$

$$-8\beta^1_{12,x} + 2(\gamma^1_{12,f_1} - \gamma^1_{11,f_2}) - 6(\beta^2_{31}\gamma^2_{12} + \beta^2_{32}\gamma^2_{11}) + 12(\alpha_{12}\delta_{11} + \alpha_{11}\delta_{12})$$
$$-8(\alpha_{21}\delta_{22} + \alpha_{22}\delta_{21}) - 4(\beta^2_{12}\gamma^1_{21} + \beta^2_{11}\gamma^1_{22})$$
$$+10(\beta^1_{21}\gamma^2_{12} + \beta^1_{22}\gamma^2_{11}) + 12\beta^2_{22,x} - 3(\gamma^2_{12,f_3} - \gamma^2_{11,f_4})$$
$$+6(\beta^2_{22}\gamma^2_{21} + \beta^2_{21}\gamma^2_{22} - \beta^2_{22}\gamma^1_{11} - \beta^2_{21}\gamma^1_{12}) = 0,$$

$$-6\alpha_{21,x} + 2(\beta^2_{31,f_1} + \beta^2_{32,f_2}) + (\alpha_{21}\gamma^1_{11} - \alpha_{22}\gamma^1_{12}) + 5(\alpha_{21}\gamma^2_{21} - \alpha_{22}\gamma^2_{22})$$
$$+2(\alpha_{11}\gamma^1_{21} - \alpha_{12}\gamma^1_{22} - \beta^2_{21,f_3} - \beta^2_{22,f_4} - \alpha_{31}\gamma^2_{11} + \alpha_{32}\gamma^2_{12})$$
$$+6(\beta^2_{11}\beta^1_{31} - \beta^2_{12}\beta^1_{32} - \beta^2_{21}\beta^1_{21} + \beta^2_{22}\beta^1_{22})$$
$$-(\beta^1_{21,f_1} + \beta^1_{22,f_2} - \beta^1_{11,f_3} - \beta^1_{12,f_4}) = 0,$$

$$-6\alpha_{22,x} + 2(\beta^2_{32,f_1} - \beta^2_{31,f_2}) + (\alpha_{21}\gamma^1_{12} + \alpha_{22}\gamma^1_{11}) + 5(\alpha_{21}\gamma^2_{22} + \alpha_{22}\gamma^2_{21})$$
$$+2(\alpha_{11}\gamma^1_{22} + \alpha_{12}\gamma^1_{21}) - 2(\beta^2_{22,f_3} - \beta^2_{21,f_4} + \alpha_{31}\gamma^2_{12} + \alpha_{32}\gamma^2_{11})$$
$$+6(\beta^2_{12}\beta^1_{31} + \beta^2_{11}\beta^1_{32} - \beta^2_{21}\beta^1_{22} - \beta^2_{22}\beta^1_{21}) - \beta^1_{22,f_1}$$
$$+\beta^1_{21,f_2} + \beta^1_{12,f_3} - \beta^1_{11,f_4} = 0,$$

$$4\beta^1_{21,x} + \gamma^1_{21,f_1} + \gamma^1_{22,f_2} - 8(\alpha_{21}\delta_{11} - \alpha_{22}\delta_{12})$$
$$+2(\beta^2_{21}\gamma^1_{21} - \beta^2_{22}\gamma^1_{22} + \beta^1_{11}\gamma^1_{21} - \beta^1_{12}\gamma^1_{22} + \beta^1_{21}\gamma^2_{21} - \beta^1_{22}\gamma^2_{22} - \beta^1_{12}\gamma^1_{11} + \beta^1_{22}\gamma^1_{12})$$
$$-4(\beta^1_{31}\gamma^2_{11} - \beta^1_{32}\gamma^2_{12}) - 2(\gamma^1_{11,f_3} + \gamma^1_{12,f_4}) - 4(\alpha_{31}\delta_{21} - \alpha_{32}\delta_{22}) = 0,$$

$$4\beta^1_{22,x} + \gamma^1_{22,f_1} - \gamma^1_{21,f_2} - 8(\alpha_{22}\delta_{11} + \alpha_{21}\delta_{12})$$
$$+2(\beta^2_{22}\gamma^1_{21} + \beta^2_{21}\gamma^1_{22} + \beta^1_{12}\gamma^1_{21} + \beta^1_{11}\gamma^1_{22} + \beta^1_{21}\gamma^2_{22}$$
$$+\beta^1_{22}\gamma^2_{21} - \beta^1_{21}\gamma^1_{12} - \beta^1_{22}\gamma^1_{11})$$
$$-4(\beta^1_{31}\gamma^2_{12} + \beta^1_{32}\gamma^2_{11}) - 2(\gamma^1_{12,f_3} - \gamma^1_{11,f_4})$$
$$-4(\alpha_{32}\delta_{21} + \alpha_{31}\delta_{22}) = 0,$$

$$-4\beta^1_{21,x} + 4\beta^2_{31,x} + 2(\gamma^1_{21,f_1} + \gamma^1_{22,f_2}) - 4(\gamma^1_{21}\beta^2_{21} - \gamma^1_{22}\beta^2_{22}$$
$$-\beta^1_{11}\gamma^1_{21} + \beta^1_{12}\gamma^1_{22} - \beta^1_{21}\gamma^2_{21} + \beta^1_{22}\gamma^2_{22} + \beta^1_{21}\gamma^1_{11} - \beta^1_{22}\gamma^1_{12})$$
$$-\gamma^2_{21,f_3} - \gamma^2_{22,f_4} - \gamma^1_{11,f_3} - \gamma^1_{12,f_4} - 8(\alpha_{31}\delta_{21} - \alpha_{32}\delta_{22}) = 0,$$

$$-4\beta^1_{22,x} + 4\beta^2_{32,x} + 2(\gamma^1_{22,f_1} - \gamma^1_{21,f_2})$$
$$-4(\gamma^1_{22}\beta^2_{21} + \gamma^1_{21}\beta^2_{22} - \beta^1_{11}\gamma^1_{22} - \beta^1_{12}\gamma^1_{21} - \beta^1_{21}\gamma^2_{22}$$
$$-\beta^1_{22}\gamma^2_{21} + \beta^1_{21}\gamma^1_{12} + \beta^1_{22}\gamma^1_{11})$$
$$-\gamma^2_{22,f_3} + \gamma^2_{21,f_4} - \gamma^1_{12,f_3} + \gamma^1_{11,f_4} - 8(\alpha_{32}\delta_{21} - \alpha_{31}\delta_{22}) = 0,$$

$$2(\delta_{11,f_1} + \delta_{12,f_2}) + 4(\beta^1_{11}\delta_{11} - \beta^1_{12}\delta_{12} + \beta^1_{21}\delta_{21} - \beta^1_{22}\delta_{22} - \beta^2_{31}\delta_{21} + \beta^2_{32}\delta_{22})$$
$$+2(\gamma^2_{21,x} - \gamma^1_{11,x} - \delta_{21,f_3} - \delta_{22,f_4})$$
$$+\gamma^2_{21}\gamma^2_{21} - \gamma^2_{22}\gamma^2_{22} - \gamma^1_{11}\gamma^1_{11} + \gamma^1_{12}\gamma^1_{12} - 4(\beta^2_{21}\delta_{11} - \beta^2_{22}\delta_{12}) = 0,$$

$$2(\delta_{12,f_1} - \delta_{11,f_2}) + 4(\beta^1_{11}\delta_{12} + \beta^1_{12}\delta_{11} + \beta^1_{21}\delta_{22} + \beta^1_{22}\delta_{21} - \beta^2_{31}\delta_{22} - \beta^2_{32}\delta_{21})$$
$$+2(\gamma^2_{22,x} - \gamma^1_{12,x} - \delta_{22,f_3} + \delta_{21,f_4} + \gamma^2_{21}\gamma^2_{22} - \gamma^1_{11}\gamma^1_{12})$$
$$-4(\beta^2_{21}\delta_{12} + \beta^2_{22}\delta_{11}) = 0,$$

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