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# Temperature evolution during dissipative collapse

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**Abstract.** We investigate the gravitational collapse of a radiating sphere evolving into a final static configuration described by the interior Schwarzschild solution. The temperature profiles of this particular model are obtained within the framework of causal thermodynamics. The overall temperature evolution is enhanced by contributions from the temperature gradient induced by perturbations as well as relaxational effects within the stellar core.

**Keywords.** Gravitational collapse; relativistic stars; thermodynamics.

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## 1. Introduction

Dissipative processes such as heat generation, shear, particle creation and bulk viscosity during stellar collapse have been extensively studied in the past [1–6]. It has been shown in numerous models that noncausal transport equations predict thermodynamical behaviours that can be far different from their noncausal counterparts. In particular, radiating stellar collapse has been shown to yield causal temperature profiles which are always higher within the stellar core. It was well-known that during the later stages of collapse, the core temperature of the stars are of the order of $10^9$ K. Using a perturbative scheme in which the metric functions and thermodynamical variables are perturbed, Herrera and Santos [7] have shown that for a temperature range of $10^6$–$10^9$ K, the relaxation time may vary from as much as $\tau \approx 10^2$ s to as little as $\tau \approx 10^{-4}$ s. Herrera and Santos carried out both Newtonian and post-Newtonian approximations on the causal heat transport equation. They further showed that the causal temperature gradient can differ as much as five orders of magnitude from the noncausal temperature gradient. These results were later confirmed by Govender *et al* [8,9] in both the acceleration-free models and collapse with nonzero acceleration.

In this paper we aim to calculate and study the complete temperature profile of a compact star undergoing nonadiabatic gravitational collapse to a final static equilibrium state. The general framework for such a collapse scenario was first addressed by Govender *et al* [10] in which a collapsing fluid sphere evolved into the Vaidya–Tikekar superdense star [11].

This particular model was shown to fit observational results from the X-ray pulsar Her X-1. However, the Govender *et al* [10] model lacked a thermodynamical treatment of the temperature profile. We are in a position to investigate the effect of the perturbation on the temperature profile as the star in this investigation collapses to a final static configuration. Our treatment also highlights the effect of the relaxation time on the collapsing stellar fluid, again confirming that relaxational effects due to dissipation cannot simply be ignored, especially during the later stages of collapse.

## 2. Interior spacetime

The interior of the star is described by a spherically symmetric line element, with vanishing shear, in comoving and isotropic coordinates

$$\mathrm{d}s^2 = -A^2\,\mathrm{d}t^2 + B^2\left[\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right], \tag{1}$$

where $A = A(t, r)$ and $B = B(t, r)$ are metric functions. The matter distribution for the stellar interior is represented by the energy–momentum tensor of an imperfect fluid

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a, \tag{2}$$

where $\rho$ is the energy density, $p$ is the pressure and $q = (q^a q_a)^{1/2}$ is the magnitude of the heat flux. The fluid four-velocity $u$ is comoving and is given by

$$u^a = \frac{1}{A}\delta^a_0. \tag{3}$$

The heat flow vector takes the form

$$q^a = (0, q, 0, 0), \tag{4}$$

since $q^a u_a = 0$ and the heat is assumed to flow in the radial direction. The fluid collapse rate $\Theta = u^a{}_{;a}$ of the stellar model is given by

$$\Theta = 3\frac{\dot{B}}{AB}. \tag{5}$$

The Einstein field equations reduce to

$$\rho = 3\frac{1}{A^2}\frac{\dot{B}^2}{B^2} - \frac{1}{B^2}\left(2\frac{B''}{B} - \frac{B'^2}{B^2} + \frac{4}{r}\frac{B'}{B}\right), \tag{6}$$

$$p = \frac{1}{A^2}\left(-2\frac{\ddot{B}}{B} - \frac{\dot{B}^2}{B^2} + 2\frac{\dot{A}}{A}\frac{\dot{B}}{B}\right) + \frac{1}{B^2}\left(\frac{B'^2}{B^2} + 2\frac{A'}{A}\frac{B'}{B} + \frac{2}{r}\frac{A'}{A} + \frac{2}{r}\frac{B'}{B}\right), \tag{7}$$

$$p = -2\frac{1}{A^2}\frac{\ddot{B}}{B} + 2\frac{\dot{A}}{A^3}\frac{\dot{B}}{B} - \frac{1}{A^2}\frac{\dot{B}^2}{B^2} + \frac{1}{r}\frac{A'}{A}\frac{1}{B^2} + \frac{1}{r}\frac{B'}{B^3} + \frac{A''}{A}\frac{1}{B^2} - \frac{B'^2}{B^4} + \frac{B''}{B^3}, \tag{8}$$

$$q = -\frac{2}{AB^2}\left(-\frac{\dot{B}'}{B} + \frac{B'\dot{B}}{B^2} + \frac{A'}{A}\frac{\dot{B}}{B}\right), \tag{9}$$

for the line element (1). In the above system we have used the convention that overhead dots and primes denote derivatives with respect to the comoving and isotropic coordinates $t$ and $r$ respectively.

## 3. Exterior spacetime

Since the star is radiating energy to the exterior, it is natural that the exterior spacetime be described by Vaidya's outgoing solution [12] given by

$$\mathrm{d}s^2 = -\left(1 - \frac{2m(v)}{\mathsf{r}}\right)\mathrm{d}v^2 - 2\mathrm{d}v\,\mathrm{d}\mathsf{r} + \mathsf{r}^2\big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\big), \tag{10}$$

where $\mathrm{d}m/\mathrm{d}v < 0$. In order to get a complete description of the radiating star, the interior spacetime is matched to the Vaidya exterior across a time-like hypersurface. These junction conditions were first presented by Santos [13] and are widely utilized to model radiating stars in relativistic astrophysics. Here we present the main results of the matching for easy reference. The continuity of the metric functions and the extrinsic curvature across the boundary $\Sigma$ yield

$$A(r_\Sigma, t)\mathrm{d}t = \mathrm{d}\tau, \tag{11}$$

$$r_\Sigma B(r_\Sigma, t) = \mathcal{Y}(\tau), \tag{12}$$

$$m(v) = \left(\frac{r^3 B}{2A^2} B_t^2 - r^2 B_r - \frac{r^3}{2B} B_r^2\right)_\Sigma, \tag{13}$$

$$p_\Sigma = (qB)_\Sigma, \tag{14}$$

where $m(v)$ represents the total mass contained within a sphere of radius $r$, $\tau$ and $\mathcal{Y}$ are coordinates on the stellar surface and (14) ensures the conservation of momentum across the boundary.

## 4. A radiating model

Following Govender *et al* [10] we consider a model in which the star undergoes dissipative collapse and evolves to a stable equilibrium state. To obtain an analytical model we impose the following conditions on the metric functions and thermodynamical variables:

$$A(r,t) = A_0(r) + \epsilon a(r)T(t), \tag{15}$$

$$B(r,t) = B_0(r) + \epsilon b(r)T(t), \tag{16}$$

$$\rho(r,t) = \rho_0(r) + \epsilon\bar{\rho}(r,t), \tag{17}$$

$$p(r,t) = p_0(r) + \epsilon\bar{p}(r,t), \tag{18}$$

where the heat flux is of the order of $\epsilon$ $(0 < \epsilon \ll 1)$ and $(A_0, B_0)$ represent the final static configuration, for which we have

$$\rho_0 = -\frac{1}{B_0^2}\left[2\frac{B_0''}{B_0} - \left(\frac{B_0'}{B_0}\right)^2 + \frac{4}{r}\frac{B_0'}{B_0}\right], \tag{19}$$

$$p_0 = \frac{1}{B_0^2}\left[\left(\frac{B_0'}{B_0}\right)^2 + \frac{2}{r}\frac{B_0'}{B_0} + \frac{2}{r}\frac{A_0'}{A_0} + 2\frac{A_0'}{A_0}\frac{B_0'}{B_0}\right]. \tag{20}$$

The pressure isotropy equation for the static configuration is

$$\left(\frac{A_0'}{A_0} + \frac{B_0'}{B_0}\right)' - \left(\frac{A_0'}{A_0} + \frac{B_0'}{B_0}\right)^2 - \frac{1}{r}\left(\frac{A_0'}{A_0} + \frac{B_0'}{B_0}\right) + 2\left(\frac{A_0'}{A_0}\right)^2 = 0, \tag{21}$$

and the perturbed quantities up to first order in $\epsilon$ are

$$\bar{\rho} = -3\rho_0\frac{b}{B_0}T + \frac{1}{B_0^3}\left[-\left(\frac{B_0'}{B_0}\right)^2 b + 2\left(\frac{B_0'}{B_0} - \frac{2}{r}\right)b' - 2b''\right]T, \tag{22}$$

$$\bar{p} = -2p_0\frac{b}{B_0}T - 2\frac{b}{A_0^2B_0}\ddot{T}$$
$$+ \frac{2}{B_0^2}\left[\left(\frac{B_0'}{B_0} + \frac{1}{r} + \frac{A_0'}{A_0}\right)\left(\frac{b}{B_0}\right)' + \left(\frac{B_0'}{B_0} + \frac{1}{r}\right)\left(\frac{a}{A_0}\right)'\right]T, \tag{23}$$

$$\bar{q} = \frac{2\epsilon}{B_0^2}\left(\frac{b}{A_0B_0}\right)'\dot{T}. \tag{24}$$

The condition of pressure isotropy for the perturbed matter distribution yields

$$\left[\left(\frac{a}{A_0}\right)' + \left(\frac{b}{B_0}\right)'\right]' - 2\left[\left(\frac{a}{A_0}\right)' + \left(\frac{b}{B_0}\right)'\right]\left(\frac{A_0'}{A_0} + \frac{B_0'}{B_0}\right)$$
$$- \frac{1}{r}\left[\left(\frac{a}{A_0}\right)' + \left(\frac{b}{B_0}\right)'\right] + 4\frac{A_0'}{A_0}\left(\frac{a}{A_0}\right)' = 0. \tag{25}$$

Introducing the following parameters

$$\alpha = \frac{A_0^2}{bB_0}\left[\left(\frac{B_0'}{B_0} + \frac{1}{r} + \frac{A_0'}{A_0}\right)\left(\frac{b}{B_0}\right)' + \left(\frac{B_0'}{B_0} + \frac{1}{r}\right)\left(\frac{a}{A_0}\right)'\right], \tag{26}$$

$$\beta = \frac{A_0^2}{2b}\left(\frac{b}{A_0B_0}\right)', \tag{27}$$

allows us to write (23) and (24) as

$$\bar{p} = -2p_0\frac{b}{B_0}T + 2\frac{\alpha b}{A_0^2B_0}T - 2\frac{b}{A_0^2B_0}\ddot{T}, \tag{28}$$

$$\bar{q} = \frac{4\epsilon b}{A_0^2B_0^2}\beta\dot{T}. \tag{29}$$

By substituting (28) and (29) into the boundary condition (14), we obtain the temporal evolution equation

$$\ddot{T} + 2\beta\dot{T} - \alpha T = 0, \tag{30}$$

where we have taken $(p_0)_\Sigma = 0$. This is a linear second-order equation with solutions in terms of elementary functions. Bearing in mind that we are investigating a collapse scenario, more specifically the evolution leading to a final equilibrium configuration, we take

$$T(t) = T_0 \mathrm{e}^{-(\beta_\Sigma + \sqrt{\alpha_\Sigma + \beta_\Sigma^2})t} \tag{31}$$

which obeys the following boundary conditions:

$$T(t)|_{t=\infty} = 0, \quad T(t)|_{t=0} = T_0,$$

where $T_0$ is a constant. We want $T(t)$ to decrease with time for a final static state. So we must have $\alpha_\Sigma > 0$.

## 5. Thermodynamics

Our primary interest is to investigate the physical viability of a collapsing star evolving into a final static configuration. To this end, we seek to obtain the temperature profile of our model within the context of extended irreversible thermodynamics. The role of relaxational effects during dissipative gravitational collapse have been highlighted in many studies. Previous works have shown that the inclusion of relaxation effects, especially during the late stages of collapse (when the stellar fluid is far from hydrostatic equilibrium) lead to higher temperatures within the stellar core. The causal transport equation in the absence of rotation and viscous stress is

$$\tau h_a{}^b \dot{q}_b + q_a = -\kappa \left( h_a{}^b \nabla_b T + T \dot{u}_a \right), \tag{32}$$

where $h_{ab} = g_{ab} + u_a u_b$ projects into the comoving rest space, $T$ is the local equilibrium temperature, $\kappa$ ($\geq 0$) is the thermal conductivity and $\tau$ ($\geq 0$) is the relaxational time-scale which gives rise to the causal and stable behaviour of the theory. To obtain the noncausal Fourier heat transport equation we must set $\tau = 0$ in (32). For the metric (1), eq. (32) becomes

$$\tau (qB)^{\dot{}} + AqB = -\frac{\kappa (AT)'}{B}. \tag{33}$$

To obtain a physically reasonable stellar model, we shall adopt the thermodynamic coefficients for radiative transfer. Hence we are considering the situation where energy is transported away from the stellar interior by massless particles (photons), moving with a long mean free path through matter that is effectively in hydrodynamic equilibrium, and that is dynamically dominant. Govender *et al* [8,9] have shown that the choice

$$\kappa = \gamma T^3 \tau_\mathrm{c}, \quad \tau_\mathrm{c} = \left(\frac{\alpha}{\gamma}\right) T^{-\sigma}, \quad \tau = \left(\frac{\beta\gamma}{\alpha}\right) \tau_\mathrm{c}, \tag{34}$$

is physically reasonable for the thermal conductivity $\kappa$, the mean collision time between massive and massless particles $\tau_\mathrm{c}$ and the relaxation time $\tau$. The quantities $\alpha \geq 0$, $\beta \geq 0$,

$\gamma \geq 0$ and $\sigma \geq 0$ are constants. Note that the mean collision time decreases with increasing temperature as expected except for the special case $\sigma = 0$, when it is constant. With these assumptions the causal heat transport eq. (33) becomes

$$\beta (qB)\dot{} \, T^{-\sigma} + A(qB) = -\alpha \frac{T^{3-\sigma}(AT)'}{B}. \tag{35}$$

In (34) we can think of $\beta$ as the 'causality' index, measuring the strength of relaxational effects, with $\beta = 0$ giving the noncausal case. For our perturbative model, we write

$$\mathcal{T} = \mathcal{T}_0 + \epsilon \bar{\mathcal{T}} T, \tag{36}$$

where $\mathcal{T}_0$ represents the equilibrium temperature. Utilizing (36) in (35) we obtain

$$\begin{aligned} \bar{\mathcal{T}}(r) = {} & \frac{-2\beta}{\eta A_0} \frac{\ddot{T}}{T} \int \left( \frac{b}{A_0 B_0} \right)' \mathcal{T}_0^{-3} \mathrm{d}r \\ & - \frac{2}{\eta A_0} \frac{\dot{T}}{T} \int A_0 \mathcal{T}_0^{\sigma-3} \left( \frac{b}{A_0 B_0} \right)' \mathrm{d}r - \frac{a\mathcal{T}_0}{A_0} + \frac{C_1}{A_0}, \end{aligned} \tag{37}$$

where $C_1$ is an integration constant and

$$(A_0 \mathcal{T}_0)' = 0. \tag{38}$$

Relation (38) leads to

$$\mathcal{T}_0 = \frac{C_0}{A_0}, \tag{39}$$

where $C_0 > 0$ is a constant. As pointed out by Herrera and Santos [7,14], this is a well-known result first obtained by Tolman which ensures the existence of a temperature gradient that prevents heat flux from regions of higher to regions of lower gravitational field intensity during thermal equilibrium. To investigate the evolution of the temperature we assume that our end-state of collapse is described by the static Schwarzschild interior solution in isotropic coordinates

$$A_0 = \zeta_1 - \zeta_2 \frac{(1 - r^2)}{(1 + r^2)}, \tag{40}$$

$$B_0 = \frac{2R}{(1 + r^2)}, \tag{41}$$

where $\zeta_1$, $\zeta_2$ and $R$ are constants. We can easily calculate the energy density and pressure for the static configuration as

$$\rho_0 = \frac{3}{R^2}, \tag{42}$$

$$p_0 = \frac{1}{R^2} \left( -1 + \frac{2\zeta_2 (1 - r^2)}{\zeta_1 (1 + r^2) - \zeta_2 (1 - r^2)} \right). \tag{43}$$

The vanishing of the pressure at the boundary and the continuity of the metric functions across $\Sigma$ lead to

$$\frac{\zeta_1}{\zeta_2} = 3 \frac{(1 - r_\Sigma^2)}{(1 + r_\Sigma^2)},$$

and

$$\zeta_2 = \frac{1}{2}.$$

As demonstrated by Bonner *et al* [3], the physical requirements $\rho_0 > 0$, $p_0 > 0$, $p_0 < \rho_0$ and $0 \le r \le r_\Sigma$ are satisfied provided that

$$r_\Sigma^2 < \frac{1}{3}, \tag{44}$$

$$\frac{2m_0}{r_\Sigma} = \frac{4r_\Sigma^2}{\left(1+r_\Sigma^2\right)^2} < \frac{3}{4}, \tag{45}$$

where

$$m_0 = \frac{4Rr_\Sigma^3}{\left(1+r_\Sigma^2\right)^3}$$

represents the total mass within the static sphere up to the boundary $\Sigma$. Furthermore, the pressure isotropy condition for the nonstatic configuration is ensured by choosing

$$\frac{a}{A_0} = \frac{b}{B_0} = \frac{k_1}{2}\int rB_0^2 \mathrm{d}r + k_2, \tag{46}$$

where $k_1$ and $k_2$ are constants of integration as shown by Govender *et al* [10].

It is clear from figure 1 that the causal temperature is greater than the noncausal temperature everywhere within the stellar interior. We must point out that the contributions from $\bar{\mathcal{T}}$ in (36) to the overall temperature profile $\mathcal{T}$ are due to the positive contribution of $\bar{\mathcal{T}}$ and the relaxational effects in $\bar{\mathcal{T}}$. Plots of $\bar{\mathcal{T}}$ in both the causal and noncausal cases indicate that $\bar{\mathcal{T}}$ is a positive decreasing function from the centre of the star to the stellar surface. This perturbative contribution is greatly enhanced by relaxational effects. Figure 1 clearly indicates that the causal temperature gradient is steeper than it's noncausal counterpart closer to the core with the difference dropping off as one gets to the stellar surface.

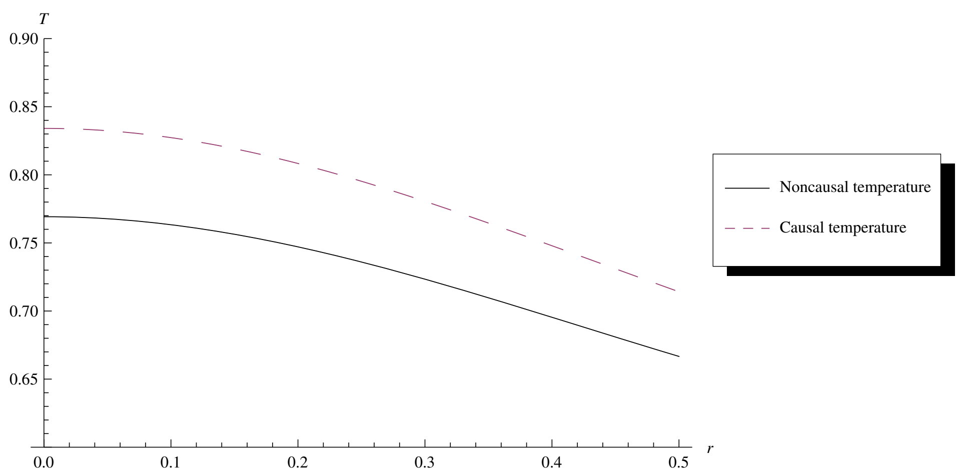


**Figure 1.** Causal (dashed line) and noncausal (solid line) temperature profiles vs. radial coordinate.

## 6. Conclusion

We have provided a complete description of a radiating star undergoing dissipative gravitational collapse in the form of radial heat emission, with the final end-state being the static, interior Schwarzschild model. The temperature profile is obtained for both the casual and noncausal regimes. It is clear that the temperature induced by perturbations leads to higher core temperatures. This effect is enhanced by relaxational effects which is noticeable at later stages of collapse (when the system is far from hydrostatic equilibrium). Our results confirm the earlier findings by Herrera and Santos [7] in which it was shown that the causal temperature gradient can be five orders of magnitude greater than its noncausal counterpart. Our results are also in keeping with the perturbative results of Govender *et al* [9] in which it was shown that for a sphere collapsing from an initial static configuration, relaxational effects become dominant during the later stages of collapse. We point out that causal thermodynamics, in the context of the Israel–Stewart theory, has been utilized by Herrera *et al* [15] and Di Prisco *et al* [16] to study viscous dissipative collapse in the streaming and diffusion approximations respectively. It would be interesting to consider an extension of our model to these physical situations.

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