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Solitary wave solutions of selective nonlinear diffusion-reaction equations using homogeneous balance method

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Abstract. An auto-Bäcklund transformation derived in the homogeneous balance method is employed to obtain several new exact solutions of certain kinds of nonlinear diffusion-reaction (D-R) equations. These equations arise in a variety of problems in physical, chemical, biological, social and ecological sciences.

Keywords. Nonlinear diffusion-reaction equation; homogeneous balance method; Bäcklund transformation; solitary wave solutions.

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1. Introduction

There are some privileged classes of partial differential equations in mathematics, which have very wide applications not only in physical but also in biological, chemical, ecological and social sciences. Diffusion-reaction (D-R) equation is one such equation. In fact, a variety of applications of both linear and nonlinear versions of the D-R equation appear in a variety of contexts mainly on the basis of analogy [1–4].

Generally, approximation methods are employed to solve these equation. In recent years, various direct methods were proposed to find exact solutions of nonlinear partial differential equations (NLPDEs) in general. These methods include, Bäcklund transformation (BT) [5], (G'/G)-expansion method [6,7], auxiliary equation method [8–12], exponential function method [13,14], homogeneous balance (HB) method [15–17], variational iteration method [18–20], factorization method [21], algebraic method [22] and Weiss approach [23]. Whereas some of these methods are of general nature in the sense that they can be employed to any NLPDE, others are equation-specific. In this paper, using the homogeneous balance method [24,25], some new solitary wave solutions along with the associated auto-Bäcklund transformations are obtained. According to the HB method, one assumes the solution of a given NLPDE in the form [17]

$$u(x,t) = \frac{\partial^s f(w)}{\partial x^s} + u_0(x,t), \tag{1}$$

where $f = f(w)$, $w = w(x,t)$ and $u_0(x,t)$ are undetermined functions and s is a positive integer which can be determined by balancing the highest-order nonlinear term with the highest-order derivative term in the given equation. In eq. (1) $u_0(x,t)$ is another unknown function which, at times, can be chosen as a constant. In this work, we plan to investigate the exact solutions of the following NL D-R equations:

$$u_t + cu_x = Du_{xx} + \alpha u - \beta u^2 - \gamma u^3, \tag{2}$$
$$u_t + cu_x = Du_{xx} + \alpha u^{n+1} - \beta u^{2n+1}, \tag{3}$$
$$u_t + kuu_x = Du_{xx} + \alpha u - \beta u^2, \tag{4}$$
$$u_t + ku^2u_x = Du_{xx} + \alpha u - \beta u^4, \tag{5}$$
$$u_t = u_{xx} - \frac{m}{1-u}(u_x)^2 - u(1-u), \tag{6}$$
$$u_t + cu_x = ur\left(1 - \frac{u}{k}\right) - \frac{Bu^2}{A+u^2} + Du_{xx}, \tag{7}$$
$$u_t + \alpha uu_x - \nu u_{xx} = \beta u(1-u)(u-\gamma), \tag{8}$$

where D is the diffusion coefficient, u is the concentration or density, c represents the convective velocity, and α, β, γ, m and k are the real constants in different contexts. Equations (2) and (3) are the generalizations of Fisher equation with reference to nonlinearity and eqs (4) and (5) arise in situations where diffusion coefficient D itself becomes density-dependent [3]. Equation (6), expressed in dimensionless form, is also known as generalized Fisher equation [26–28]. These variants of Fisher equations are found to have some close connection with some important phenomena such as neutron action, wave motion in liquid crystal, pattern selection mechanism in nonequilibrium physics [28,29] etc. Equation (7) appears in the context of budworm population dynamics [30] and eq. (8) is known as Burgers–Huxley equation [31] in which ν plays the role of diffusion-like coefficient. Note that eq. (8) reduces to Hodgkin–Huxley equation for $\alpha = 0$ and to Burgers equation for $\beta = 0$ (arises in many physical problems including one-dimensional turbulence, sound and shock waves in viscous medium etc.) [31–33]. Apart from these uncoupled equations, we shall also study the coupled D-R equations, namely,

$$\begin{aligned} u_t - c_1u_x &= D_1u_{xx} + \alpha u - \beta u^2 - \gamma uv, \\ v_t - c_2v_x &= D_2v_{xx} - \mu v + \chi uv, \end{aligned} \tag{9}$$

which basically are the generalization of the Lotka–Volterra equations of the prey–predator system. Here u and v respectively, represent the population densities of the prey and the predator, D_1 and D_2 are the corresponding diffusion coefficients. As before, c_1 and c_2 now are the convective velocities of the two species, and α, β, γ and χ are the real constants.

2. Solutions using HB method

Solution of eq. (2)

Using the balancing procedure we obtain $s = 1$ for eq. (2) and for this eq. (1) can be recast in the form

$$u(x,t) = f' w_x + u_0(x,t), \tag{10}$$

where f' represents derivative with respect to w. Substitution of eq. (10) in (2) yields

$$\begin{aligned} &D\left(f''' w_x^3 + 3f'' w_x w_{xx} + f' w_{xxx} + u_{0xx}\right) + \alpha\left(f' w_x + u_0\right) \\ &\quad - \beta\left(f'^2 w_x^2 + 2f' w_x u_0 + u_0^2\right) - c\left(f'' w_x^2 + f' w_{xx} + u_{0x}\right) - u_{0t} \\ &\quad - f' w_{xt} - f'' w_t w_x - \gamma\left(f'^3 w_x^3 + u_0^3 + 3f'^2 w_x^2 u_0 + 3f' w_x u_0^2\right) = 0. \end{aligned} \tag{11}$$

In the spirit of HB method, one requires that the coefficients of w_x^3 in eq. (11) must vanish, viz.,

$$Df''' - \gamma f'^3 = 0. \tag{12}$$

Solution of eq. (12) is given by

$$f = \ln(w), \tag{13}$$

provided $\gamma = 2D$. For this form of f, note the relation $f'^2 = -f''$ and accordingly set the coefficients of f'' and f' as zero in eq. (11). This yields the following set of coupled equations:

$$\begin{aligned} &3D w_x w_{xx} + \beta w_x^2 + 3\gamma u_0 w_x^2 - w_t w_x - c w_x^2 = 0, \\ &D w_{xxx} + \alpha w_x - 2\beta w_x u_0 - 3\gamma w_x u_0^2 - w_{xt} - c w_{xx} = 0. \end{aligned} \tag{14}$$

In addition to eqs (14), an equation of the type

$$D u_{0xx} + \alpha u_0 - \beta u_0^2 - \gamma u_0^3 = u_{0t} + c u_{0x}, \tag{15}$$

also arises when one sets the terms independent of the derivatives of f, as zero. Equation (15) for u_0 is structurally the same as eq. (2) for u implies that the BT (1) is auto-type. Thus, for eq. (2), the defined auto-BT takes the form

$$u = \frac{\partial \ln(w)}{\partial x} + u_0(x,t), \tag{16}$$

where the derivatives of w satisfy eqs (14). Now, as per the prescription of the application of the method of BT [5] one sets $u_0 = 0$ in (14) and obtains

$$\begin{aligned} &3D w_x w_{xx} + \beta w_x^2 - w_t w_x - c w_x^2 = 0, \\ &D w_{xxx} + \alpha w_x - w_{xt} - c w_{xx} = 0. \end{aligned} \tag{17}$$

Next, for the solution of eqs (17), we make an ansatz

$$w(x,t) = 1 + \mathrm{e}^{(px+qt)}, \tag{18}$$

in which the constants p and q are determined by substituting (18) in (17) as $p = (-\beta \pm \sqrt{\beta^2 + 8\alpha D})/4D$ and $q = (1/8D)[12\alpha D + 2c\beta + \beta^2 \mp \sqrt{8\alpha D + \beta^2}(2c+\beta)]$. Thus, eq. (10) in conjunction with (13) will admit the solitary wave solution of eq. (2) as

$$u = \frac{p}{2}\left(1 + \tanh\left(\frac{1}{2}(px+qt)\right)\right). \tag{19}$$

Solution of eq. (3)

In this case, we first make a transformation $u = u^{1/n}$, which converts eq. (3) to the form

$$nuu_t + cnuu_x = Dnuu_{xx} + D(1-n)u_x^2 + \alpha n^2 u^3 - \beta n^2 u^4. \tag{20}$$

As the balancing procedure leads to $s = 1$ for this case, eq. (10) for $u(x,t)$ can be restored even for this case. The use of eq. (10) in (20) gives rise to

$$\begin{aligned} &D(1-n)(f''^2 w_x^4 + f'^2 w_{xx}^2 + u_{0x}^2 + 2f'f''w_x^2 w_{xx} + 2f''w_x^2 u_{0x} + 2f'w_{xx}u_{0x})\\ &\quad + Dn(f'f'''w_x^4 + 3f'f''w_x^2 w_{xx} + f'^2 w_x w_{xxx} + f'w_x u_{0xx})\\ &\quad + Dnu_0(f'''w_x^3 + 3f''w_x w_{xx} + f'w_{xxx} + u_{0xx})\\ &\quad - n(f'f''w_t w_x^2 + f'^2 w_x w_{xt} + f'w_x u_{0t}) - nu_0(f''w_t w_x + f'w_{xt} + u_{0t})\\ &\quad - cn(f'f''w_x^3 + f'^2 w_x w_{xx} + f'w_x u_{0x}) - ncu_0(f''w_x^2 + f'w_{xx} + u_{0x})\\ &\quad + \alpha n^2(f'^3 w_x^3 + u_0^3 + 3f'^2 u_0 w_x^2 + 3f'w_x u_0^2)\\ &\quad - \beta n^2(f'^4 w_x^4 + 4f'^3 w_x^3 u_0 + 4f'w_x u_0^3 + 6f'^2 w_x^2 u_0^2 + u_0^4) = 0. \end{aligned} \tag{21}$$

Again, by equating the coefficients of w_x^4 in (21) to zero one obtains

$$D(1-n)f''^2 + Dnf'f''' - \beta n^2 f'^4 = 0, \tag{22}$$

which has the solution $f = \ln(w)$ provided $\beta = D(1+n)/n^2$. Using the relations $f'f'' = -\frac{1}{2}f'''$, $f'^3 = \frac{1}{2}f'''$ and $f'^2 = -f''$ and setting the coefficients of f''', f'' and f' to zero as before we get the following set of equations:

$$\begin{aligned} &-Dw_x^2 w_{xx} - \frac{1}{2}Dnw_x^2 w_{xx} - Dnu_0 w_x^3 - 2Du_0 w_x^3 + \frac{n}{2}w_t w_x^2\\ &\quad + \frac{nc}{2}w_x^3 + \frac{\alpha n^2}{2}w_x^3 = 0, \end{aligned} \tag{23}$$

$$\begin{aligned} &-Dw_{xx}^2 + Dnw_{xx}^2 + 2Dw_x^2 u_{0x} - 2Dnw_x^2 u_{0x} - Dnw_x w_{xxx}\\ &\quad + 3Dnu_0 w_x w_{xx} + nw_x w_{xt} - nu_0 w_t w_x + cnw_x w_{xx} - cnu_0 w_x^2\\ &\quad - 3\alpha n^2 u_0 w_x^2 + 6D(1+n)w_x^2 u_0^2 = 0, \end{aligned} \tag{24}$$

$$\begin{aligned}&2D(1-n)w_{xx}u_{0x} + Dnw_x u_{0xx} + Dnu_0 w_{xxx} - nw_x u_{0t}\\&\quad - nu_0 w_{xt} - ncw_x u_{0x} - ncw_x u_{0x} - ncu_0 w_{xx}\\&\quad + 3\alpha n^2 w_x u_0^2 - 4D(1+n)w_x u_0^3 = 0,\end{aligned} \tag{25}$$

and an equation for u_0, analogous to (20), viz.,

$$Dnu_0 u_{0xx} + D(1-n)u_{0x}^2 + \alpha n^2 u_0^3 - \beta n^2 u_0^4 = nu_0 u_{0t} + cnu_0 u_{0x}. \tag{26}$$

As before, the ansatz for w, viz., $w = 1 + \mathrm{e}^{(px+qt)}$ finally leads to the solution of (3) in the form

$$u = \left[\frac{p}{2}\left\{1 + \tanh\left(\frac{1}{2}(px+qt)\right)\right\}\right]^{1/n}, \tag{27}$$

where p and q are found to be

$$\begin{aligned}p &= \frac{n^2\alpha}{D(1+n)}\\ q &= \frac{-cn^2\alpha - cn^3\alpha + n^3\alpha^2}{D(1+n)^2}.\end{aligned} \tag{28}$$

Solution of eq. (4)

The balancing procedure for this case will suggest $s = 1$ and accordingly one arrives at eq. (10) using (1). After substituting eq. (10) into (4) one obtains

$$\begin{aligned}&D\left(f'''w_x^3 + 3f''w_x w_{xx} + f'w_{xxx} + u_{0xx}\right) + \alpha\left(f'w_x + u_0\right)\\&\quad - \beta\left(f'^2 w_x^2 + u_0^2 + 2f'w_x u_0\right) - f''w_t w_x - f'w_{xt} - u_{0t}\\&\quad - k\left(f'f''w_x^3 + f'^2 w_x w_{xx} + f'w_x u_{0x} + f''w_x^2 u_0\right.\\&\quad \left.+ f'w_{xx}u_0 + u_0 u_{0x}\right) = 0.\end{aligned} \tag{29}$$

Now setting the coefficients of w_x^3 to zero in the above equations one obtains the following equation:

$$Df''' - kf'f'' = 0, \tag{30}$$

which admits the solution $f = \ln(w)$ provided $k = -2D$. As before, using the relation $f'^2 = -f''$ and again setting the coefficients of f'', f' and the term independent of the derivative of f to zero and assuming the solution of the resulting equation as $w = 1 + \mathrm{e}^{(px+qt)}$ we get the following solution of (4):

$$u = \frac{p}{2}\left(1 + \tanh\left(\frac{1}{2}(px+qt)\right)\right), \tag{31}$$

where

$$p = \frac{\alpha}{\beta}, \quad q = \frac{\alpha^2 D}{\beta^2} + \alpha.$$

Equation (31) is a solitary wave solution of eq. (4). Note that for $D = 1$, and for certain choices of the parameters in eq. (4), the factorization method is employed [21] and exponential solutions are obtained.

Solution of eq. (5)

For this equation the balancing procedure suggests $s = 1$ and a solution u as in eq. (10). Substituting solution (10) in (5) leads to the equation

$$\begin{aligned} & D\left(f'''w_x^3 + 3f''w_xw_{xx} + f'w_{xxx} + u_{0xx}\right) \\ & \quad + \alpha\left(f'w_x + u_0\right) - f''w_tw_x - f'w_{xt} - u_{0t} \\ & \quad - k(f'^2f''w_x^4 + f'^3w_x^2w_{xx} + f'^2w_x^2u_{0x} + f''w_x^2u_0^2 + f'w_{xx}u_0^2 \\ & \quad + u_0^2u_{0x} + 2f'f''u_0w_x^3 + 2f'^2w_xw_{xx}u_0 + 2f'w_xu_0u_{0x}) \\ & \quad - \beta\left(f'^4w_x^4 + u_0^4 + 4f'^3w_x^3u_0 + 4f'w_xu_0^3 + 6f'^2w_x^2u_0^2\right) = 0. \end{aligned} \tag{32}$$

Now equating the coefficients of w_x^4 to zero in (32) leads to

$$kf'' + \beta f'^2 = 0, \tag{33}$$

which has the solution $f = \ln(w)$ provided $\beta = k$. Again, following the same procedure as before, the solution of eq. (5) turns out to be

$$u = \frac{p}{2}\left(1 + \tanh\left(\frac{1}{2}\left(px + qt\right)\right)\right), \tag{34}$$

where $p = 2D/k$, $q = 12D^3/k^2$ and a constraining relation $\alpha = 8D^3/k^2$.

Solution of eq. (6)

When one applies the balancing procedure in this case, one obtains $s = 2$ for which the final solution does not exist. Therefore, we resort to use $s = 1$ – a case investigated by Bindu *et al* [28] in the context of Painlevé method. This value of s again allows us to choose the same form of u as in (10). After substituting (10) in (6) and setting the coefficients of w_x^4 to zero we get the following equation:

$$f'f''' - mf''^2 = 0, \tag{35}$$

which admits the solution as $f = \ln w$ provided $m = 2$. This value of m also conforms the one obtained by Painlevé method [28].

Solution of eq. (7)

First we recast eq. (7) in the form

$$\begin{aligned} & ADu_{xx} + Du^2u_{xx} + Aru - \frac{Ar}{k}u^2 + ru^3 - \frac{r}{k}u^4 - Bu^2 \\ & \quad - \left(Au_t + Acu_x + u^2u_t + cu^2u_x\right) = 0, \end{aligned} \tag{36}$$

and then use the balancing procedure which yields $s = 2$. This in turn, allows us to use the form of u from (1) as

$$u(x,t) = f''w_x^2 + f'w_{xx} + u_0(x,t). \tag{37}$$

As such, the solution of (36) in the present scheme of methodology turns out to be a bit lengthy when compared with the other cases discussed above. This is mainly because in this case when one substitutes (37) in (36), then one has to rationalize the resultant expression until the seventh derivative of f, contrary to the earlier cases where the maximum derivative of f turned out to be four. Therefore, in this case one has to handle more number of equations before arriving at the final solution in the form

$$u(x,t) = \frac{3k}{2}\mathrm{sech}^2\left(\pm\sqrt{\frac{3k}{2}}\,(x-ct)\right), \tag{38}$$

with the constraints $r = -6kD$ and $B = 0$. It is interesting to note that eq. (7) admits the solution (38) if and only if $B = 0$. In fact, this reduction here appears more in a natural way in the present procedure.

Solution of eq. (8)

Balancing procedure for this case yields $s = 1$, which in turn leads to the choice (10) for u. Again, substitution of eq. (10) in (8) leads to the equation

$$\begin{aligned}
&\nu(f'''w_x^3 + 3f''w_xw_{xx} + f'w_{xxx} + u_{0xx}) - \gamma\beta(f'w_x + u_0)\\
&\quad + \beta(1+\gamma)(f'^2w_x^2 + 2f'w_xu_0 + u_0^2) - f''w_tw_x - f'w_{xt} - u_{0t}\\
&\quad - \beta(f'^3w_x^3 + u_0^3 + 3f'^2w_x^2u_0 + 3f'w_xu_0^2) - \alpha u_0(f''w_x^2 + f'w_{xx} + u_{0x})\\
&\quad - \alpha(f'f''w_x^3 + f'^2w_xw_{xx} + f'w_xu_{0x}) = 0.
\end{aligned} \tag{39}$$

Setting the coefficients of w_x^4 to zero in (39) yields

$$\nu f''' - \beta f'^3 - \alpha f'f'' = 0, \tag{40}$$

which admits the solution $f = \ln w$ provided $\beta = 2\nu + \alpha$. Thus, using the same procedure as before, one obtains the solution of eq. (8) as

$$u = \frac{p}{2}\left(1 + \tanh\left(\frac{1}{2}\,(px + qt)\right)\right), \tag{41}$$

with $p = 1$ and γ and accordingly $q = \nu - \gamma\beta$ and $\nu\gamma^2 - \gamma\beta$.

Solution of eq. (9)

When one applies the balancing procedure to equations in (9), one gets $s = 2$ for both the equations. This results in the following choices for u and v:

$$\begin{aligned}
u(x,t) &= f''w_x^2 + f'w_{xx} + u_0(x,t),\\
v(x,t) &= f''w_x^2 + f'w_{xx} + v_0(x,t).
\end{aligned} \tag{42}$$

Using eqs (42) in (9) yields the following equations:

$$
\begin{aligned}
&D_1(f''''w_x^4 + 6f'''w_x^2w_{xx} + 3f''w_{xx}^2 + 4f''w_xw_{xxx} + f'w_{xxxx} + u_{0xx})\\
&\quad + \alpha(f''w_x^2 + f'w_{xx} + u_0) + c_1(f'''w_x^3 + 3f''w_xw_{xx} + f'w_{xxx} + u_{0x})\\
&\quad - \beta(f''^2w_x^4 + f'^2w_{xx}^2 + u_0^2 + 2f'f''w_x^2w_{xx} + 2u_0f''w_x^2 + 2u_0f'w_{xx})\\
&\quad - \gamma(f''^2w_x^4 + f'^2w_{xx}^2 + 2f'f''w_x^2w_{xx} + f''w_x^2v_0\\
&\quad + f'w_{xx}v_0 + f''w_x^2u_0 + f'w_{xx}u_0 + u_0v_0)\\
&\quad \times(f'''w_tw_x^2 + 2f''w_xw_{xt} + f''w_tw_{xx} + f'w_{xxt} + u_{0t}) = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&D_2(f''''w_x^4 + 6f'''w_x^2w_{xx} + 3f''w_{xx}^2 + 4f''w_xw_{xxx} + f'w_{xxxx} + v_{0xx})\\
&\quad - \mu(f''w_x^2 + f'w_{xx} + v_0) + c_2(f'''w_x^3 + 3f''w_xw_{xx} + f'w_{xxx} + v_{0x})\\
&\quad + \chi(f''^2w_x^4 + f'^2w_{xx}^2 + 2f'f''w_x^2w_{xx} + f''w_x^2v_0\\
&\quad + f'w_{xx}v_0 + f''w_x^2u_0 + f'w_{xx}u_0 + u_0v_0)\\
&\quad - (f'''w_tw_x^2 + 2f''w_xw_{xt} + f''w_tw_{xx} + f'w_{xxt} + v_{0t}) = 0.
\end{aligned}
$$

After equating the coefficients of w_x^4 to zero separately in the above equations one obtains

$$D_1f'''' - (\beta + \gamma)f''^2 = 0, \tag{43}$$

and

$$D_2f'''' + \chi f''^2 = 0. \tag{44}$$

Interestingly, both the equations in (43) admit the same solution, $f = \ln(w)$, provided $(\beta + \gamma) = -6D_1$ and $\chi = 6D_2$. Following the same procedure as for the earlier cases, one arrives at the constraints $c_1 = c_2$ and $\alpha\chi = \mu(\beta+\gamma)$ in a nontrivial manner. For this case, eqs (9) admit the exact solution

$$u(x,t) = v(x,t) = \frac{p^2}{4}\mathrm{sech}^2\left(\sqrt{\frac{p^2}{4}}(x + c_1t)\right), \tag{45}$$

with $p = \pm\sqrt{6\alpha/(\beta+\gamma)}$.

3. Applicational aspects

Whereas the NL D-R equations studied in this work have very wide applications in various branches of science and engineering, here we highlight the one pertaining to biology, particularly in the studies of cancer growth [34]. These studies may also be useful in understanding the dynamics of tumor growth and drug response in the organism. Martins *et al* [34] beautifully reviewed the possibility of such applications of NL D-R equations.

According to the continuum growth model of cancer cells one considers a small element of cancerous area and considers the rate of change of cells within this element which is given by the following symbolic equation [33]:

Rate of change of the density of cells within the element
= (Generation of cells within the element)
– (Advective outflow through its surface)
– (Diffusive outflow through its surface)
– (Death of cells in it).

In fact, the above equality follows from the mass balance principle applied to each volume element. After translating the above mass balance principle in mathematical terms, one can write

$$\frac{\partial u_j}{\partial t} = -\nabla \cdot (\vec{v} u_j) - \nabla \cdot (D \nabla u_j) + \Gamma(u_j) - \delta u_j, \tag{46}$$

where $\vec{v}$ and D as before are the convective velocity and the diffusion coefficient respectively, $\Gamma(u)$ is the proliferation term per unit volume and δ is the death coefficient of the jth cell type population density or concentration. Note that in eq. (45) the convective velocity $\vec{v}$ can be both space- and time-dependent and also the diffusion coefficient D can be density-dependent. Further, for various nonlinear choices of the proliferation term $\Gamma(u)$ one can understand the underlying mechanism leading to cancer growth cells. Therefore, the results obtained here (see eqs (2)–(9)) are of immediate use in such studies, particularly for a fixed-cell type population density and constant v and D.

4. Concluding remarks

In this paper, an attempt is made to obtain exact solutions of selective NL D-R equations (see eqs (2)–(9)) using the auto-BT as derived within the framework of HB method. The method, employed perhaps for the first time to this category of D-R equations, suggested several interesting features in some of the solutions (see eqs (7) and (9)). Even though the solutions of some of the equations were investigated in the literature in a localized space-time domain using the so-called approximation methods [3], the exact solutions obtained here, of course by using a common ansatz of the type $w = 1 + \mathrm{e}^{(px+qt)}$, for all the nine cases consistently, have their own beauty in mathematical terms. Note that the exact solutions of some of the equations like eqs (6) and (8) obtained here also conform to the results obtained earlier using other methods. Particularly, for eq. (6) the present method confirms its integrability [28] for $m = 2$. Further, eq. (7) is found to admit solution only for $B = 0$. It will be interesting to look for the exact solutions of some of these equations when the parameters in an equation become time- and/or space-dependent. Such studies are in progress.

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