PRAMANA
— journal of
physics

© Indian Academy of Sciences

Vol. 74, No. 2
February 2010
pp. 207–218

The (G'/G)-expansion method for a discrete nonlinear Schrödinger equation

SHENG ZHANG*, LING DONG, JIN-MEI BA and YING-NA SUN
Department of Mathematics, Bohai University, Jinzhou 121013,
People's Republic of China
*Corresponding author. E-mail: zhshaeng@yahoo.com.cn

MS received 16 August 2009; revised 20 September 2009; accepted 30 September 2009

Abstract. An improved algorithm is devised for using the (G'/G)-expansion method to solve nonlinear differential-difference equations. With the aid of symbolic computation, we choose a discrete nonlinear Schrödinger equation to illustrate the validity and advantages of the improved algorithm. As a result, hyperbolic function solutions, trigonometric function solutions and rational solutions with parameters are obtained, from which some special solutions including the known solitary wave solution are derived by setting the parameters as appropriate values. It is shown that the improved algorithm is effective and can be used for many other nonlinear differential-difference equations in mathematical physics.

Keywords. Nonlinear differential-difference equations; the (G'/G)-expansion method; hyperbolic function solutions; trigonometric function solutions; rational solutions.

PACS Nos 02.30.Jr; 05.45.Yv; 02.30.Ik

1. Introduction

Since the work of Fermi *et al* in the 1950s [1], the investigation of exact solutions of nonlinear differential-difference equations (NLDDEs) has played a crucial role in modelling many phenomena in different fields which include condensed matter physics, biophysics and mechanical engineering. We can also encounter NLDDEs in numerical simulation of soliton dynamics in high energy physics where they arise as approximations of continuum models. Unlike difference equations which are fully discretized, differential-difference equations (DDEs) are semi-discretized with some (or all) of their spacial variables discretized while time is usually kept continuous. In the past decades, many effective methods for obtaining exact solutions of nonlinear evolution equations (NLEEs) have been presented, such as the inverse scattering method [2], Hirota bilinear method [3], Bäcklund transformation [4], Painlevé expansion [5], sine–cosine method [6], homogeneous balance method [7], tanh-function method [8], algebraic method [9], Jacobi elliptic function expansion method [10], F-expansion method [11], auxiliary equation method [12] and exp-function method [13].

Recently, Wang *et al* [14] proposed a new method called the (G'/G)-expansion method to look for travelling wave solutions of NLEEs. By using the (G'/G)-expansion method, Wang *et al* [14,15] have successfully obtained hyperbolic function solutions, trigonometric function solutions and rational solutions of some important NLEEs. Later, Zhang *et al* [16] proposed a generalized (G'/G)-expansion method to improve and extend Wang *et al*'s work [14] for solving variable-coefficient equations and high dimensional equations. Zhang [17] explored new application of this method to some special NLEEs. Generally speaking, it is hard to generalize one method for NLEEs to solve NLDDEs because of the difficulty to search for iterative relations from indices n to $n \pm 1$. More recently, by careful analysis, Zhang *et al* [18] found the iterative relations between the lattice indices and devised an algorithm for using the (G'/G)-expansion method to construct hyperbolic function solutions and trigonometric function solutions of NLDDEs.

The present paper is motivated by the desire to develop and improve the work made in [18]. With this purpose, we will propose an improved algorithm for using the (G'/G)-expansion method to obtain not only hyperbolic function solutions and trigonometric function solutions but also rational solutions of NLDDEs. In order to illustrate the validity and advantages of the improved algorithm, we will apply it to a discrete nonlinear Schrödinger equation [19]:

$$\mathrm{i}\frac{\mathrm{d}u_n}{\mathrm{d}t} = (u_{n+1} + u_{n-1} - 2u_n) - |u_n|^2(u_{n+1} + u_{n-1}), \tag{1}$$

where $\mathrm{i} = \sqrt{-1}$.

The rest of this paper is organized as follows. In §2, we describe the improved algorithm for using the (G'/G)-expansion method to solve NLDDEs. In §3, we use this algorithm to solve the discrete nonlinear Schrödinger equation (1). In §4, some conclusions are given.

2. Description of the (G'/G)-expansion method for NLDDEs

In this section, we would like to recall the algorithm proposed in [18] and outline the improved algorithm for using the (G'/G)-expansion method to solve NLDDEs step by step. For a given system of M polynomial NLDDEs:

$$\begin{aligned}\Delta\big(&u_{n+p_1}(x), \ldots, u_{n+p_k}(x), \ldots, u'_{n+p_1}(x), \ldots, u'_{n+p_k}(x), \ldots,\\ &u^{(r)}_{n+p_1}(x), \ldots, u^{(r)}_{n+p_k}(x)\big) = 0,\end{aligned} \tag{2}$$

where the dependent variable u has M components $u_{i,n}$, the continuous variable x has N components x_j, the discrete variable n has Q components n_i, the k shift vectors $p_s \in Z^Q$, and $u^{(r)}(x)$ denotes the collection of mixed derivative terms of order r. The main steps of the (G'/G)-expansion method are outlined as follows:

Step 1. When we seek travelling wave solutions of eq. (2), the first step is to introduce the wave transformation:

$$u_{n+p_s}(x) = U_{n+p_s}(\xi_n), \quad \xi_n = \sum_{i=1}^{Q} d_i n_i + \sum_{j=1}^{N} c_j x_j + \zeta, \tag{3}$$

where the coefficients d_i, c_j, and the phase ζ are all constants and $s = 1, 2, \ldots, k$. In this way, eq. (2) becomes

$$\begin{aligned}\Delta\big(&U_{n+p_1}(\xi_n), \ldots, U_{n+p_k}(\xi_n), \ldots, U'_{n+p_1}(\xi_n), \ldots, U'_{n+p_k}(\xi_n), \ldots,\\ &U^{(r)}_{n+p_1}(\xi_n), \ldots, U^{(r)}_{n+p_k}(\xi_n)\big) = 0.\end{aligned} \tag{4}$$

Step 2. We propose the following series expansion as a solution of eq. (4):

$$U_n(\xi_n) = \sum_{l=-m}^{m} \alpha_l \left(\frac{G'(\xi_n)}{G(\xi_n)}\right)^l, \quad \alpha_m^2 + \alpha_{-m}^2 \neq 0, \tag{5}$$

where α_l $(l = 0, 1, 2, \ldots, m)$ are constants to be determined later, $G(\xi_n)$ satisfies a second-order linear ordinary differential equation (LODE):

$$\frac{\mathrm{d}^2 G(\xi_n)}{\mathrm{d}\xi_n^2} + \lambda \frac{\mathrm{d}G(\xi_n)}{\mathrm{d}\xi_n} + \mu G(\xi_n) = 0, \tag{6}$$

where λ and μ are arbitrary constants. Using the general solutions of eq. (6), we have

$$\frac{G'(\xi_n)}{G(\xi_n)} = \begin{cases} \dfrac{\sqrt{\lambda^2-4\mu}}{2}\left(\dfrac{C_1 \sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n) + C_2 \cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)}{C_1 \cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n) + C_2 \sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)}\right) - \dfrac{\lambda}{2}, \\ \quad \text{when} \;\; \lambda^2 - 4\mu > 0, \\ \dfrac{\sqrt{4\mu-\lambda^2}}{2}\left(\dfrac{-C_1 \sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n) + C_2 \cosh(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)}{C_1 \cosh(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n) + C_2 \sinh(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)}\right) - \dfrac{\lambda}{2}, \\ \quad \text{when} \;\; \lambda^2 - 4\mu < 0, \\ \dfrac{C_2}{C_1 + C_2\xi_n} - \dfrac{\lambda}{2}, \quad \text{when} \;\; \lambda^2 - 4\mu = 0. \end{cases} \tag{7}$$

With the help of eq. (7) and the properties of hyperbolic functions and trigonometric functions, we can easily construct a uniform formula:

$$\begin{aligned}\frac{G'(\xi_{n\pm p_s})}{G(\xi_{n\pm p_s})} &= \frac{\sqrt{\delta + \epsilon(\lambda^2-4\mu)}}{2}\\ &\times \left(\frac{\frac{2}{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}\left(\frac{G'(\xi_n)}{G(\xi_n)} + \frac{\lambda}{2}\right) \pm \epsilon f(\frac{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}{2}\varphi_s)}{1 \pm \left(\frac{2}{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}\left(\frac{G'(\xi_n)}{G(\xi_n)} + \frac{\lambda}{2}\right)\right) f(\frac{\sqrt{4\delta+\epsilon(\lambda^2-4\mu)}}{2}\varphi_s)}\right) - \frac{\lambda}{2},\end{aligned} \tag{8}$$

where $\epsilon = \{1, -1, 0\}$, $\delta = \{4, 0\}$, $\varphi_s = p_{s1}d_1 + p_{s2}d_2 + \cdots + p_{sQ}d_Q$, p_{si} is the ith component of shift vector p_s, and

$$f\left(\frac{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}{2}\varphi_s\right)$$
$$= \begin{cases} \tanh\left(\dfrac{\sqrt{\lambda^2-4\mu}}{2}\varphi_s\right), & \text{when } \epsilon = 1, \delta = 0, \lambda^2 - 4\mu > 0, \\ \tan\left(\dfrac{\sqrt{4\mu-\lambda^2}}{2}\varphi_s\right), & \text{when } \epsilon = -1, \delta = 0, \lambda^2 - 4\mu < 0, \\ \varphi_s, \quad \text{when } \epsilon = 0, \delta = 4, \lambda^2 - 4\mu = 0. & \end{cases} \tag{9}$$

Thus we have

$$U_n(\xi_{n\pm p_s}) = \sum_{l=-m}^{m} \alpha_l \left[\frac{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}{2} \times \left(\frac{\frac{2}{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}\left(\frac{G'(\xi_n)}{G(\xi_n)} + \frac{\lambda}{2}\right) \pm \epsilon f(\frac{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}{2}\varphi_s)}{1 \pm \left(\frac{2}{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}\left(\frac{G'(\xi_n)}{G(\xi_n)} + \frac{\lambda}{2}\right)\right) f(\frac{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}{2}\varphi_s)}\right) - \frac{\lambda}{2}\right]^l . \tag{10}$$

Step 3. Determine the degree m of eqs (5) and (10) by balancing the highest-order nonlinear term(s) and the highest-order partial derivative of $U_n(\xi_n)$ in eq. (4). It should be noted that the leading terms of $U_n(\xi_{n+p_s})$ ($p_s \neq 0$) will not affect the balance because we are interested in balancing the terms of $(G'(\xi_n)/G(\xi_n))$.

Step 4. Substituting eqs (5) and (10) given the value of m determined in Step 3 along with (6) into (4) and clearing the denominator and collecting all terms with the same order of $(G'(\xi_n)/G(\xi_n))$ together, the left-hand side of (4) is converted into a polynomial in $(G'(\xi_n)/G(\xi_n))$. Then setting each coefficient of this polynomial to zero, we will derive a set of algebraic equations, from which the constants α_l, d_i and c_j can be explicitly determined using *Mathematica*.

Step 5. Using the results obtained in the above steps, we can finally obtain exact solutions of eq. (2).

Remark 1. It can be easily found that when $\alpha_{-1} = \alpha_{-2} = \cdots = \alpha_{-m} = 0$ and $\delta = 0$, the formula (8) exactly becomes the formula (9) constructed in [18]. More importantly, formula (8) contains the rational solution $\frac{C_2}{C_1+C_2\xi_n} - \frac{\lambda}{2}$ as a special case so that it can be used to obtain not only hyperbolic function solutions and trigonometric function solutions but also rational solutions of NLDDEs if such formal solutions exist.

3. Exact travelling wave solutions of eq. (1)

In this section, we apply the improved algorithm developed in §2 to eq. (1). To extend this improved algorithm to eq. (1), we use the transformations [19]

$$u_n = \mathrm{e}^{\mathrm{i}\theta_n} v_n(\xi_n), \quad \theta_n = pn + qt + \varsigma, \quad \xi_n = d_1 n + c_1 t + \zeta, \tag{11}$$

and

$$u_{n+1} = \mathrm{e}^{\mathrm{i}\theta_n}\mathrm{e}^{\mathrm{i}p} v_{n+1}(\xi_n), \quad u_{n-1} = \mathrm{e}^{\mathrm{i}\theta_n}\mathrm{e}^{-\mathrm{i}p} v_{n-1}(\xi_n), \tag{12}$$

where p, q and ς are all constants. With the expression $\mathrm{e}^{\pm\mathrm{i}p} = \cos(p) \pm \mathrm{i}\sin(p)$, eq. (1) is reduced to

$$\begin{aligned} &-qv_n - \cos(p)(1 - v_n^2)(v_{n+1} + v_{n-1}) + 2v_n \\ &\qquad + \mathrm{i}[c_1 v_n' - \sin(p)(1 - v_n^2)(v_{n+1} - v_{n-1})] = 0, \end{aligned} \tag{13}$$

further separating the real and imaginary parts, we get

$$qv_n + \cos(p)(1 - v_n^2)(v_{n+1} + v_{n-1}) - 2v_n = 0, \tag{14}$$

$$c_1 v_n' - \sin(p)(1 - v_n^2)(v_{n+1} - v_{n-1}) = 0. \tag{15}$$

Suppose the solutions of eqs (14) and (15) are in the form of eqs (5) and (10), and according to the homogeneous balance procedure, let eqs (14) and (15) have the following formal solution:

$$v_n = \sum_{l=-1}^{1} \alpha_l \left(\frac{G'(\xi_n)}{G(\xi_n)}\right)^l, \quad \alpha_1^2 + \alpha_{-1}^2 \neq 0, \tag{16}$$

$$\begin{aligned} v_{n+1} = \sum_{l=-1}^{1} \alpha_l \Bigg[&\frac{\sqrt{\delta + \epsilon(\lambda^2 - 4\mu)}}{2} \\ &\times \left(\frac{\frac{2}{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}\left(\frac{G'(\xi_n)}{G(\xi_n)} + \frac{\lambda}{2}\right) + \epsilon f(\frac{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}{2} d_1)}{1 + \left(\frac{2}{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}\left(\frac{G'(\xi_n)}{G(\xi_n)} + \frac{\lambda}{2}\right)\right) f(\frac{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}{2} d_1)} \right) - \frac{\lambda}{2} \Bigg]^l, \end{aligned} \tag{17}$$

$$\begin{aligned} v_{n-1} = \sum_{l=-1}^{1} \alpha_l \Bigg[&\frac{\sqrt{\delta + \epsilon(\lambda^2 - 4\mu)}}{2} \\ &\times \left(\frac{\frac{2}{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}\left(\frac{G'(\xi_n)}{G(\xi_n)} + \frac{\lambda}{2}\right) - \epsilon f(\frac{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}{2} d_1)}{1 - \left(\frac{2}{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}\left(\frac{G'(\xi_n)}{G(\xi_n)} + \frac{\lambda}{2}\right)\right) f(\frac{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}{2} d_1)} \right) - \frac{\lambda}{2} \Bigg]^l. \end{aligned} \tag{18}$$

Substituting (16)–(18) along with (6) into eqs (14) and (15), and clearing the denominator and collecting all terms with the same order of $(G'(\xi_n)/G(\xi_n))$ together, the left-hand sides of eqs (14) and (15) are converted into two polynomials in $(G'(\xi_n)/G(\xi_n))$. Setting each coefficient of these polynomials to zero, we derive a set of algebraic equations for α_0, α_1, d_1 and c_1. Solving the set of algebraic equations using *Mathematica*, we have

Case 1:

$$\alpha_0 = \pm\frac{\lambda\tanh(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1)}{\sqrt{\lambda^2-4\mu}}, \quad c_1 = \frac{4\sin(p)\tanh(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1)}{\sqrt{\lambda^2-4\mu}}, \tag{19}$$

$$\alpha_1 = \pm\frac{2\tanh(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1)}{\sqrt{\lambda^2-4\mu}}, \quad q = 2 - 2\cos(p)\,\mathrm{sech}^2\left(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1\right), \tag{20}$$

$$\alpha_{-1} = 0, \quad p = p, \quad d_1 = d_1, \quad \epsilon = 1, \quad \delta = 0, \quad \lambda^2 - 4\mu > 0. \tag{21}$$

Case 2:

$$\alpha_0 = \pm\frac{\lambda\tan(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1)}{\sqrt{4\mu-\lambda^2}}, \quad c_1 = \frac{4\sin(p)\tan(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1)}{\sqrt{4\mu-\lambda^2}}, \tag{22}$$

$$\alpha_1 = \pm\frac{2\tan(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1)}{\sqrt{4\mu-\lambda^2}}, \quad q = 2 - 2\cos(p)\sec^2\left(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1\right), \tag{23}$$

$$\alpha_{-1} = 0, \quad p = p, \quad d_1 = d_1, \quad \epsilon = -1, \quad \delta = 0, \quad \lambda^2 - 4\mu < 0. \tag{24}$$

Case 3:

$$\alpha_0 = \pm\frac{d_1\lambda}{2}, \quad c_1 = 2d_1\sin(p), \quad \alpha_1 = \pm d_1, \quad d_1 = d_1, \quad p = p, \tag{25}$$

$$\alpha_{-1} = 0, \quad q = 2 - 2\cos(p), \quad \epsilon = 0, \quad \delta = 4, \quad \lambda^2 - 4\mu = 0. \tag{26}$$

Case 4:

$$\alpha_0 = \pm\frac{d_1\lambda}{2}, \quad c_1 = 2d_1\sin(p), \quad \alpha_1 = 0, \quad d_1 = d_1, \quad p = p, \tag{27}$$

$$\alpha_{-1} = \pm\frac{d_1\lambda^2}{4}, \quad q = 2 - 2\cos(p), \quad \epsilon = 0, \quad \delta = 4, \quad \lambda^2 - 4\mu = 0. \tag{28}$$

When $\lambda^2 - 4\mu > 0$, from Case 1 we obtain a hyperbolic function solution of eq. (1):

$$u_n = \pm e^{i\theta_n} \tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2} d_1\right) \times \left(\frac{C_1 \sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n) + C_2 \cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)}{C_1 \cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n) + C_2 \sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)}\right), \tag{29}$$

where

$$\theta_n = pn + \left[2 - 2\cos(p)\,\mathrm{sech}^2\left(\frac{\sqrt{\lambda^2-4\mu}}{2} d_1\right)\right] t + \varsigma,$$

$$\xi_n = d_1 n + \frac{4\sin(p)\tanh(\frac{\sqrt{\lambda^2-4\mu}}{2} d_1)}{\sqrt{\lambda^2-4\mu}} t + \zeta.$$

If we set $\mu = \frac{\lambda^2-4}{4}$ and $C_2 = 0$, solution (29) becomes

$$u_n = \pm e^{i\{pn+[2-2\cos(p)\,\mathrm{sech}^2(d_1)]t+\varsigma\}} \tanh(d_1) \times \tanh[d_1 n + 2\sin(p)\tanh(d_1)t + \zeta], \tag{30}$$

which is the dark solitary wave solution found by Dai and Zhang, i.e. the solution (20) in [19].

Setting again $\mu = (\lambda^2-4)/4$ and $C_1 = 0$, solution (29) becomes

$$u_n = \pm e^{i\{pn+[2-2\cos(p)\,\mathrm{sech}^2(d_1)]t+\varsigma\}} \tanh(d_1) \times \coth[d_1 n + 2\sin(p)\tanh(d_1)t + \zeta], \tag{31}$$

which is a singular travelling wave solution.

When $\lambda^2 - 4\mu < 0$, from Case 2 we get a trigonometric function solution of eq. (1):

$$u_n = \pm e^{i\theta_n} \tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2} d_1\right) \times \left(\frac{-C_1 \sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n) + C_2 \cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)}{C_1 \cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n) + C_2 \sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)}\right), \tag{32}$$

where

$$\theta_n = pn + \left[2 - 2\cos(p)\sec^2\left(\frac{\sqrt{4\mu-\lambda^2}}{2} d_1\right)\right] t + \varsigma,$$

$$\xi_n = d_1 n + \frac{4\sin(p)\tan(\frac{\sqrt{4\mu-\lambda^2}}{2} d_1)}{\sqrt{4\mu-\lambda^2}} t + \zeta.$$

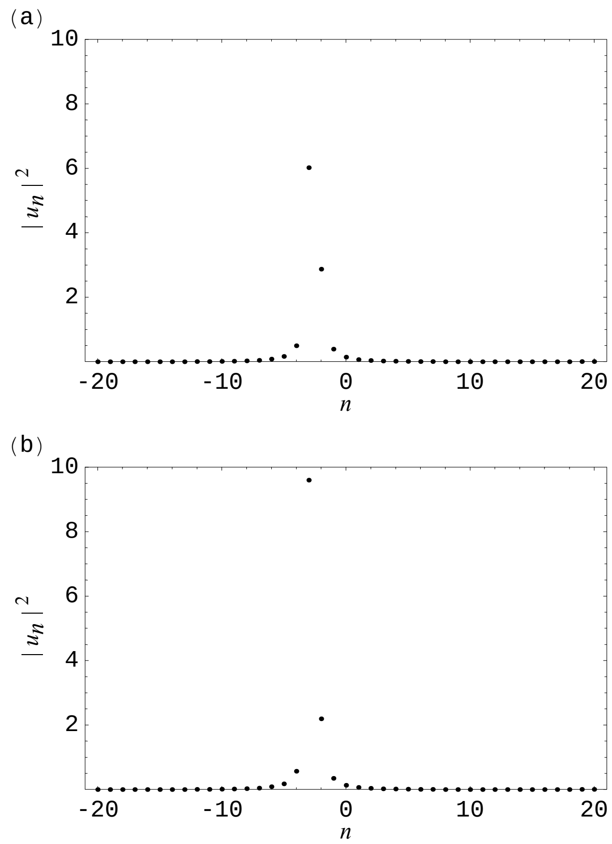

Figure 1. Plots of solution (32) with (+) branch for parameters $d_1 = 1$, $p = 3$, $\zeta = 0$, $\varsigma = 0$, $C_1 = 1$, $C_2 = 4$, $\lambda = 0.1$, $\mu = 0.0125$. (**a**) $t = 0.5$, (**b**) $t = 0.8$.

Figure 1 shows the asymptotic properties of solution (32) for two times. When $t = 1.94295$, the sharp spike is $|u_n|^2 = 5.17475 \times 10^{13}$ at $n = -3$. Different from the continuous case, each exotic wave in figure 1 does not exhibit a singular property, although solution (32) possesses singular points. Such exotic waves with similar property were found in [20]. The parameters d_1 and p determine the velocity and frequency of the travelling wave

$$v = \frac{4\sin(p)\tan(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1)}{\sqrt{4\mu-\lambda^2}}, \tag{33}$$

$$\omega = 2 - 2\cos(p)\sec^2\left(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1\right), \tag{34}$$

which have different properties from those of [20] (see figure 2).

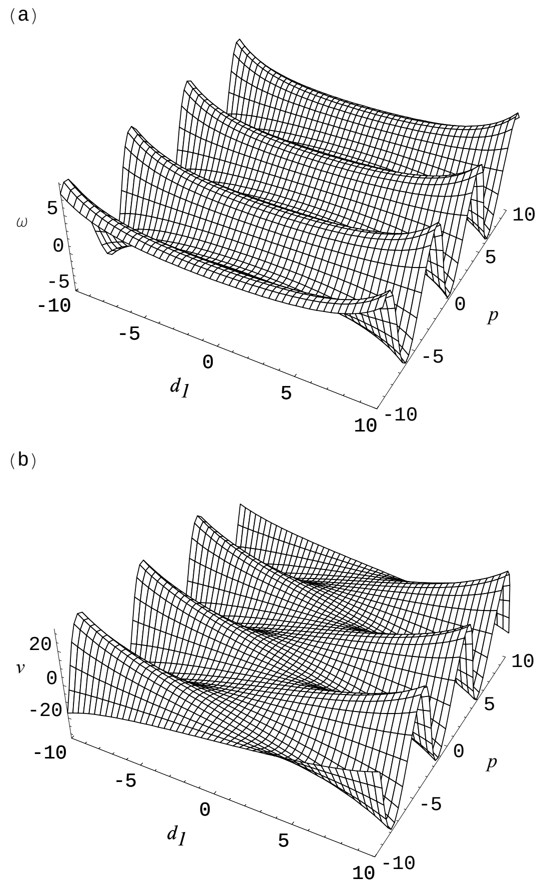

Figure 2. Plots of (**a**) frequency ω and (**b**) wave velocity v of solution (32) for parameters $\lambda = 0.1$, $\mu = 0.0125$.

Similarly, if we set μ, C_1 and C_2 as some appropriate values, solution (32) gives the following formal periodic wave solutions:

$$u_n = \mp e^{i\{pn+[2-2\cos(p)\sec^2(d_1)]t+\varsigma\}} \times \tan(d_1)\tan[d_1 n + 2\sin(p)\tan(d_1)t + \zeta], \tag{35}$$

and

$$u_n = \pm e^{i\{pn+[2-2\cos(p)\sec^2(d_1)]t+\varsigma\}}\tan(d_1) \times \cot[d_1 n + 2\sin(p)\tan(d_1)t + \zeta]. \tag{36}$$

When $\lambda^2 - 4\mu = 0$, from Cases 3 and 4 we get two rational solutions of eq. (1):

$$u_n = \pm e^{i\{pn+[2-2\cos(p)]t+\varsigma\}}\frac{d_1 C_2}{C_1 + C_2\xi_n}, \tag{37}$$

where $\xi_n = d_1 n + 2d_1 \sin(p)t + \zeta$.

$$u_n = \pm e^{i\{pn+[2-2\cos(p)]t+\varsigma\}} \frac{d_1 C_2 \lambda}{2C_2 - C_1\lambda - C_2\lambda\xi_n}, \tag{38}$$

where $\xi_n = d_1 n + 2d_1 \sin(p)t + \zeta$. Obviously, solution (38) is equivalent to (37). However, if we use the improved algorithm to solve the first lattice equation [18]:

$$\frac{\partial^2 u_n}{\partial x \partial t} = \left(\frac{\partial u_n}{\partial t} + 1\right)(u_{n-1} - 2u_n + u_{n+1}), \tag{39}$$

besides the hyperbolic function solutions and trigonometric function solutions reported in [18], another two rational solutions of eq. (39) can be obtained as follows:

$$u_n = \alpha_0 - \frac{d_1^2 \lambda}{2c_2} + \frac{d_1^2 C_2}{c_2(C_1 + C_2\xi_n)}, \tag{40}$$

where $\xi_n = d_1 n + \frac{d_1^2}{c_2} x + c_2 t + \zeta$.

$$u_n = \alpha_0 + \frac{d_1^2 C_2(1 + \alpha_{-1}c_2)}{c_2(C_1 + C_2\xi_n)} + \frac{\alpha_{-1}(C_1 + C_2\xi_n)}{C_2}, \tag{41}$$

where $\xi_n = d_1 n + \frac{d_1^2(1+\alpha_{-1}c_2)}{c_2} x + c_2 t + \zeta$. Solution (41) is more general than (40) because of the arbitrary constant α_{-1}.

In addition, if we consider the second lattice equation [18]:

$$\frac{du_n}{dt} = (\alpha + \beta u_n + \gamma u_n^2)(u_{n-1} - u_{n+1}), \tag{42}$$

another one rational solution of eq. (42) can be obtained as follows:

$$u_n = -\frac{\beta}{2\gamma} \pm \frac{d_1 C_2 \sqrt{\beta^2 - 4\alpha\gamma}}{2\gamma(C_1 + C_2\xi_n)}, \tag{43}$$

where $\xi_n = d_1 n + \frac{d_1(\beta^2 - 4\alpha\gamma)}{2\gamma} t + \zeta$.

Dai and Wang [20] devised a general algorithm for using the exp-function method [13] to derive exact travelling wave solutions of NLDDEs. By applying the devised algorithm to eq. (1), Dai and Wang have successfully obtained some hyperbolic function solutions. We would like to point out here that solutions (37) or (38), (40), (41) and (43) cannot be obtained by the methods [18,20]. To the best of our knowledge, these solutions with arbitrary parameters are novel. It illustrates that the improved algorithm proposed in this paper is effective and more powerful for NLDDEs.

Remark 2. All solutions presented in this paper have been checked with *Mathematica* by putting them back into the original eqs (1), (39) and (42), respectively.

4. Conclusion

The (G'/G)-expansion method is first developed and improved to obtain not only hyperbolic function solutions and trigonometric function solutions but also to obtain rational solutions of NLDDEs owing to the improved algorithm devised in this paper. In order to illustrate the validity and advantages of the improved algorithm, we apply it to the discrete nonlinear Schrödinger equation. As a result, hyperbolic function solutions, trigonometric function solutions and rational solutions with parameters are obtained, from which some special solutions including the known solitary wave solution are derived by setting appropriate values for the parameters. These obtained solutions with free parameters may be important to explain some physical phenomena. Considering the connection between the method proposed in this paper and Hirota bilinear method [21] and similarity reductions [22], we can conclude that these methods have a common point. That is, reducing the given NLDDE(s) which are difficult to solve, into solvable equation(s) by using suitable transformation(s). The paper shows that the improved algorithm is effective and more powerful and it can be used for other NLDDEs in mathematical physics, for instance the discrete mKdV equation [23], the Toda equation [24], the modified Volterra lattice [25], the Ablow–Ladik lattice equations [26], and so on. Employing the improved algorithm to study these equations is our task in the future.

Acknowledgements

The authors would like to express their sincere thanks to editors and referees for their valuable suggestions and comments. This work was supported by the Natural Science Foundation of Educational Committee of Liaoning Province of People's Republic of China.

References

[1] E Fermi, J Pasta and S Ulam, *Collected papers of Enrico Fermi* (Chicago University Press, Chicago, 1965)
[2] M J Ablowitz and P A Clarkson, *Soliton, nonlinear evolution equations and inverse scattering* (Cambridge University Press, Cambridge, 1991)
[3] R Hirota, *Phys. Rev. Lett.* **27**, 1192 (1971)
[4] M R Miurs, *Bäcklund transformation* (Springer, Berlin, 1978)
[5] J Weiss, M Tabor and G Carnevale, *J. Math. Phys.* **24**, 522 (1983)
[6] C T Yan, *Phys. Lett.* **A224**, 77 (1996)
[7] M L Wang, *Phys. Lett.* **A213**, 279 (1996)
[8] S Zhang and T C Xia, *Commun. Theor. Phys. (Beijing, China)* **45**, 985 (2006)
[9] J Q Hu, *Chaos, Solitons Fractals* **23**, 391 (2005)
[10] S K Liu *et al*, *Phys. Lett.* **A289**, 69 (2001)
[11] M L Wang and Y B Zhou, *Phys. Lett.* **A318**, 84 (2003)
[12] S Zhang and T C Xia, *J. Phys. A: Math. Theor.* **40**, 227 (2007)
[13] J H He and X H Wu, *Chaos, Solitons and Fractals* **30**, 700 (2006)

[14] M L Wang, X Z Li and J L Zhang, *Phys. Lett.* **A372**, 417 (2008)
[15] M L Wang, J L Zhang and X Z Li, *Appl. Math. Comput.* **206**, 321 (2008)
[16] S Zhang, J L Tong and W Wang, *Phys. Lett.* **A372**, 2254 (2008)
[17] H Q Zhang, *Commun. Nonlinear Sci. Numer. Simul.* **14**, 3220 (2009)
[18] S Zhang *et al*, *Phys. Lett.* **A37**, 905 (2009)
[19] C Q Dai and J F Zhang, *Chaos, Solitons and Fractals* **27**, 1042 (2006)
[20] C Q Dai and Y Y Wang, *Phys. Scr.* **78**, 015013 (2008)
[21] D Y Chen, *Soliton, Introduction* (Science Press, Beijing, 2006)
[22] S F Shen, *J. Phys. A: Math. Theor.* **40**, 1775 (2007)
[23] M J Ablowitz and J F Ladik, *Stud. Appl. Math.* **57**, 1 (1977)
[24] M Toda, *Theory of nonlinear lattices* (Springer, Berlin, 1981)
[25] V E Adler, S I Svinolupov and R I Yamilov, *Phys. Lett.* **A254**, 24 (1999)
[26] X J Lai and J F Zhang, *Z. Naturforsch.* **A60**, 573 (2005)