PRAMANA
— journal of
physics

© Indian Academy of Sciences

Vol. 74, No. 2
February 2010
pp. 169–175

Harmonic oscillator in Snyder space: The classical case and the quantum case

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MS received 18 August 2009; revised 20 October 2009; accepted 27 October 2009

Abstract. The harmonic oscillator in Snyder space is investigated in its classical and quantum versions. The classical trajectory is obtained and the semiclassical quantization from the phase space trajectories is discussed. An effective cut-off to high frequencies is found. The quantum version is developed and an equivalent usual harmonic oscillator is obtained through an effective mass and an effective frequency introduced in the model. This modified parameters give us a modified energy spectrum also.

Keywords. Harmonic oscillator; Snyder space; non-commutativity.

PACS Nos 02.40.Gh; 03.65.-w; 11.10.Gh

1. Introduction

Today, the idea that the space could be non-commutative has many takers. The non-commutativity is usually set through a constant parameter [1–3]. But there exists another more general formulation called the Snyder space [4]. Snyder investigated these ideas long ago and built a non-commutative Lorentz invariant space-time where the non-commutativity of space operators is proportional to non-linear combinations of phase space operators through a free parameter l, that is usually identified with the Planck longitude $l_{\mathrm{p}} = \sqrt{G/c\hbar}$. Kontsevich [5], worked on these kinds of space and since then, Snyder-like spaces in the sense of non-commutativity are of ever-increasing interest. Snyder space is also interesting because it can be mapped to the k-Minkowski space-time [6]. This space can be canonically and elegantly obtained in its classical version through a Lagrangian and Hamiltonian approach [7], and a dimensional reduction from a $(D+1,2)$ space with two time dimensions, to a $(D,1)$ space with just one time dimension [8].

Nowadays Snyder space is increasingly interesting because it could be seen as an environment where it could be possible to quantize gravity. In fact, it is possible to find a plausible explanation to the Bekenstein conjecture for the area spectrum of a black hole horizon through the area quantum in this kind of space [9].

In this paper the harmonic oscillator is analysed in its classical and quantum versions. The quantum version of this simple but fundamental system was studied in [10], but the treatment here is simpler and we do not need to bet about the right operators and despite that, we can probably shed some light in applications to problems like infinities in quantum field theory (QFT). The importance of building a well-defined harmonic oscillator in this space is that it could be possible to develop a QFT with very desirable properties like a cut-off for high frequencies in order to avoid non-renormalizable infinities. Furthermore, in the paper it is shown that we can build a harmonic oscillator with an effective mass related to the l parameter.

The paper is organized as follows. In §2, the classical version is investigated and some possible quantum consequences postulated, in §3 the quantum version is developed and the energy spectra are obtained. Finally the results are discussed in §4.

2. The classical case

Classical n dimensional Snyder space is characterized by its non-linear commutation relation between the variables of the phase space:

$$\{q_i, q_j\} = -l^2 L_{ij}, \tag{1}$$

$$\{q_i, p_j\} = \delta_{ij} - l^2 p_i p_j, \tag{2}$$

$$\{p_i, p_j\} = 0, \tag{3}$$

where l is a tiny constant parameter (usually identified with Planck longitude), that measures the deformation introduced in the canonical Poisson brackets, and L_{ij}, the angular momentum.

Let us consider the usual Hamiltonian of a harmonic oscillator:

$$H = \frac{1}{2}p^2 + \frac{\omega^2}{2}q^2, \tag{4}$$

where $m = 1$, so $\omega^2 = k$. The canonical equations are

$$\dot{q} = \{q, H\} = p - l^2 p^3, \tag{5}$$

and for p:

$$\dot{p} = \{p, H\} = -\omega^2 q + \omega^2 l^2 q p^2. \tag{6}$$

If we solve $p(q)$, we find the usual relation $p^2 + \omega^2 q^2 = \omega^2$. So the orbits in the phase space are untouched after the deformation of the Poisson brackets.

Solving the simultaneous eqs (5) and (6) we obtain for q:

$$q = \pm \frac{\tan\{(\omega t + d)\sqrt{1 - l^2\omega^2}\}}{\sqrt{\frac{1}{1-l^2\omega^2} + \tan^2\{(\omega t + d)\sqrt{1 - l^2\omega^2}\}}}, \tag{7}$$

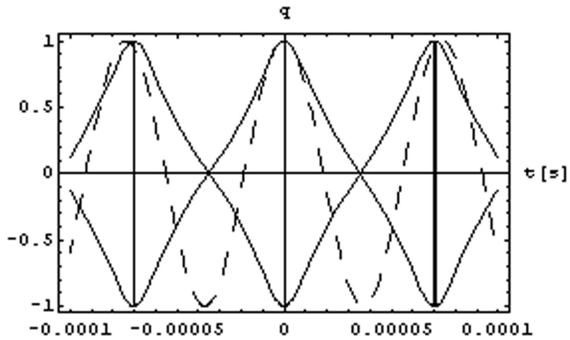

Figure 1. Plot for the two branches of Snyder $q(t)$ with $l = 10^{-5}$ and $\omega = 8.5 \cdot 10^{-4}$ (continuous line), and for normal $q(t)$ (dashed line).

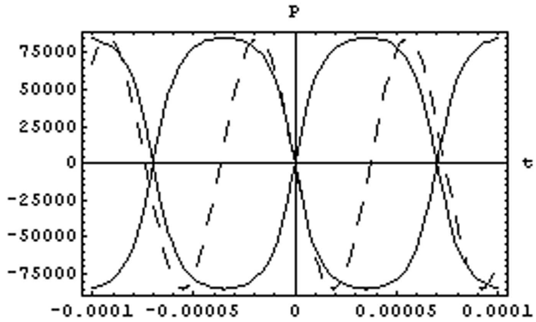

Figure 2. Plot for the two branches of Snyder $p(t)$ with $l = 10^{-5}$ and $\omega = 8.5 \cdot 10^{-4}$ (continuous line), and for normal $p(t)$ (dashed line).

where d is a suitable constant in order to achieve the initial condition $q(t = 0) = 1$, and p can be expressed in terms of q:

$$p = \pm\omega\sqrt{1 - q^2}. \tag{8}$$

Figures 1 and 2 show the behaviour of the positions q and momentum p. We can see from q graph, that Snyder oscillator is periodic but contains harmonics that deform the trajectory. Furthermore, it is possible to see that the Snyder oscillator has a different equivalent period from the normal one.

Another conspicuous feature is that $\omega = 1/l$ is effectively a cut-off to high frequencies. Indeed this is a good news in the search of possible QFT theory in Snyder space, because there is some hope of avoiding infinities.

Because the orbits are not affected by the non-linear version of Poisson brackets, we expect that the energy spectra from the Sommerfeld–Wilson quantization method $\int p\,\mathrm{d}q = n\hbar$, should formally be the same as that of the linear oscillator: $E_n = n\hbar w$, but as we have seen, the real equivalent period is modified.

Of course, Snyder oscillator is no longer an expression in single sin or cos functions, i.e. it is not a pure oscillator, but we still can express it as a Fourier transform and formulate it as a linear infinite combination of harmonics of frequency ω. That is, the single Snyder oscillator looks like a set of normal coupled oscillators.

In fact, using the fact that l^2 must be considered as a tiny parameter in relation with all the other quantities, we can use a perturbation method, among others, to solve these equations.

Let us start with the usual solution to p:

$$p_0 = -\omega \sin(\omega t), \tag{9}$$

where we have normalized the initial q perturbation, that is, $q(t = 0) = 1$. With p_0, we can integrate $\dot{q}$ in order to obtain the first order q_1:

$$q_1(t) = \left(1 - \frac{3}{4} l^2 \omega^2\right) \cos(\omega t) + \frac{1}{12} l^2 \omega^2 \cos(3\omega t) + q_1^0, \tag{10}$$

where q_1^0 is the constant evaluated in order of having the initial value of q. Now, we can introduce q_1 in (6) and integrate to obtain p_1

$$\begin{aligned} p_1(t) = &\left(-1 + l^2\omega^2 - \frac{5}{24} l^4 \omega^5\right) \sin(\omega t) \\ &+ \left(-\frac{1}{9} l^2 \omega^3 + \frac{11}{144} l^4 \omega^5\right) \sin(3\omega t) \\ &- \frac{1}{240} l^4 \omega^5 \sin(5\omega t), \end{aligned} \tag{11}$$

and so on. The method gives us, as we expected, the expansion in terms of harmonic functions in harmonic frequencies of ω.

3. The quantum case

After the Dirac quantization recipe, we can postulate the commutation relations of the Snyder space:

$$[\hat{Q}_i, \hat{Q}_j] = -i l^2 \hat{L}_{ij}, \tag{12}$$

$$[\hat{Q}_i, \hat{P}_j] = i\delta_{ij} - i l^2 \hat{P}_i \hat{P}_j, \tag{13}$$

$$[\hat{P}_i, \hat{P}_j] = 0. \tag{14}$$

Equation (12) is a non-linear version of the usual non-commutativity one, where the commutator of the position operators is proportional to a constant [2,3]. Here it is proportional to the angular momentum operator, $\hat{L}_{ij}$. Equation (13) is related to the different models with generalized commutation relations [11–13].

To study the one-dimensional harmonic oscillator we start considering a standard Hamiltonian:

$$\hat{H} = \frac{1}{2m} \hat{P}^2 + \frac{1}{2} m\omega^2 \hat{X}^2. \tag{15}$$

We define the usual creation and annihilation operators, where $\hbar = 1$:

$$a = \sqrt{\frac{m\omega}{2}}\hat{Q} + i\sqrt{\frac{1}{2m\omega}}\hat{P}, \tag{16}$$

$$a^\dagger = \sqrt{\frac{m\omega}{2}}\hat{Q} - i\sqrt{\frac{1}{2m\omega}}\hat{P}. \tag{17}$$

Using the commutation rules between operators of position and momentum, the commutation rules of the operators a and $a^\dagger$ are $[a, a] = [a^\dagger, a^\dagger] = 0$, and

$$[a, a^\dagger] = 1 - l^2\hat{P}^2 = 1 + \frac{l^2}{2}(a^\dagger - a)^2. \tag{18}$$

Writing the Hamiltonian in terms of the creation and annihilation operators and using the commutation relation between a and $a^\dagger$, we obtain

$$H = \omega\left(a^\dagger a + \frac{1}{2}\right) + \frac{\omega l^2}{2}(a^\dagger a^\dagger - a^\dagger a - aa^\dagger + a^2). \tag{19}$$

Due the structure of the Hamiltonian, $|n\rangle$ is no longer an eigenvalue of the Hamiltonian. In fact

$$\begin{aligned} H|n\rangle =& \,\omega\left\{n\left[1 - \frac{l^2}{(1+l^2)}\right] + \frac{1}{2}\left[1 + \frac{l^2}{(1+l^2)}\right]\right\}|n\rangle \\ &+\omega\left\{\frac{l^2}{2(1+l^2)}\sqrt{n+1}\sqrt{n+2}\right\}|n+2\rangle \\ &+\omega\left\{\frac{l^2}{2(1+l^2)}\sqrt{n}\sqrt{n-1}\right\}|n-2\rangle. \end{aligned} \tag{20}$$

So, the Snyder oscillator mixes states as we expected from the classical version.

But, encouraged by the semiclassical quantization that says that we could find a standard spectrum of the energy, we will use a QFT trick: because the extra term in the Hamiltonian induced by the non-linearity of the commutators of a and $a^\dagger$ is proportional to the dynamic term, we can add a counter term to the original Hamiltonian:

$$\widetilde{H} = \frac{1}{2m}\hat{P}^2 + \frac{1}{2}m\omega^2\hat{X}^2 + \frac{l^2}{2}\hat{P}^2. \tag{21}$$

Now, it is possible to define a new mass parameter, $\widetilde{m} = m/(1+ml^2)$ and modify the frequency, $\widetilde{\omega} = \omega\sqrt{(1+ml^2)}$, then $\widetilde{H}$ becomes

$$\widetilde{H} = \frac{1}{2\widetilde{m}}\hat{P}^2 + \frac{1}{2}\widetilde{m}\widetilde{\omega}^2\hat{X}^2. \tag{22}$$

This Hamiltonian has an eigenvector $|n-2\rangle$ and an eigenvalue n, and its spectra is

$$E = \widetilde{\omega}\left(n + \frac{1}{2}\right). \tag{23}$$

This mass renormalization-like procedure allows us to see the Snyder oscillator as the usual one, at least in the energy spectra, but with an effective mass. The energy spectra have been modified due the l parameter. So, the zero energy $\widetilde{E}_0 = \widetilde{\omega}/2$ and $\Delta E = \widetilde{\omega}$ (remember that $\hbar = 1$).

4. Discussion and outlook

In this work we have found the classical trajectory of an oscillator in Snyder space and found that we can see it as a set of coupled oscillators that can be described by an expansion in harmonic functions in the harmonic frequencies of ω. We found also that there is a high-frequency cut-off, because beyond it the oscillator has no response. Due to this, we can hope that infinities in QFT theories could be avoided in Snyder space. Furthermore, we could see that the Sommerfeld–Wilson quantization method indicates that the spectra should be formally like the usual harmonic oscillator, and consequently in the quantum formulation, we saw that the oscillator in this space effectively mixes states, but through a QFT of mass renormalization, we could build a standard harmonic oscillator with an energy spectrum modified due the presence of the non-commutative parameter l. We expect, in the following, to couple infinite oscillators in order to built an effective QFT in Snyder space. On the other hand, it will be worth investigating the movement of integrals of these kinds of systems in higher dimensions and, as an applied case, to study the calculation of the hydrogen atom spectra and the shift in the energy spectrum.

Acknowledgements

The author acknowledges the referee for useful suggestions in order to improve the presentation of the results of this paper and suggestions to continue the research. This paper was supported by Grant UTA No. 4722-09

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