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From bell-shaped solitary wave to W/M-shaped solitary wave solutions in an integrable nonlinear wave equation

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Abstract. The bifurcation theory of dynamical systems is applied to an integrable nonlinear wave equation. As a result, it is pointed out that the solitary waves of this equation evolve from bell-shaped solitary waves to W/M-shaped solitary waves when wave speed passes certain critical wave speed. Under different parameter conditions, all exact explicit parametric representations of solitary wave solutions are obtained.

Keywords. Bifurcation method; periodic wave solution; solitary wave solution; W/M-shaped solitary wave solutions.

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1. Introduction

The KdV equation

$$u_t = 6uu_x + u_{xxx} \tag{1.1}$$

is a well-known nonlinear partial differential equation originally formulated to model unidirectional propagation of shallow water gravity waves in one dimension [1]. It describes the long time evolution of weakly nonlinear dispersive waves of small but finite amplitude. The original experimental observations of Russell [2] in 1844 and the pioneering studies by Korteweg and de Vries [1] in 1895 showed the balance between the weak nonlinear term $6uu_x$ and the dispersion term u_{xxx} which gave rise to unidirectional solitary wave.

Because of its role as a model equation in describing a variety of physical systems, the KdV equation has been widely investigated in recent decades. Some similar models similar to the KdV equations were proposed in [3,4]. In 1997, Rosenau [4] studied the nonanalytic solitary waves of the following integrable nonlinear wave equation:

$$(u - u_{xx})_t = au_x + \frac{1}{2}[(u^2 - u_x^2)(u - u_{xx})]_x, \tag{1.2}$$

where $a(\neq 0)$ is a constant. The equation is obtained through a reshuffling procedure of the Hamiltonian operators underlying the bi-Hamiltonian structure of mKdV equations

$$u_t = u_{xxx} + \frac{3}{2}u^2 u_x. \tag{1.3}$$

Equation (1.2) supports peakons. Among the nonanalytic entities, the peakon, a soliton with a finite discontinuity in gradient at its crest, is perhaps the weakest nonanalyticity observable by the eye. In [4], the author has studied the peakons of eq. (1.2) and pointed out that the interaction of nonlinear dispersion with nonlinear convection generates exactly compact structures. Unfortunately, as the author has pointed out in [4], 'a lack of proper mathematical tools makes this goal at the present time pretty much beyond our reach'. In this paper, we shall point out that the existence of singular curves in the phase plane of the travelling wave system is the original reason for the appearance of nonsmooth travelling wave solutions in our travelling wave models by using the theory of dynamical systems.

Recently, Qiao [5] proposed the following completely integrable wave equation:

$$(u - u_{xx})_t + (u - u_{xx})_x(u^2 - u_x^2) + 2(u - u_{xx})^2 u_x = 0, \tag{1.4}$$

where u is the fluid velocity and subscripts denote the partial derivatives. This equation can also be derived from the two-dimensional Euler equation using the approximation procedure. The author obtained the so-called 'W/M'-shaped-peak solitons. More recently, Li and Zhang [6] used the method of dynamical systems to eq. (1.4), and explained why the so-called 'W/M'-shaped-peak solitons can be created.

In this paper, we study the solitary wave solutions of eq. (1.2) using the bifurcation theory of dynamical systems, which was developed by Li *et al* in [6–9]. We show that there exist smooth solitary wave solutions of eq. (1.2) when some parameter conditions are satisfied. In addition, we point out that the solitary waves of eq. (1.2) evolve from bell-shaped solitary waves to W/M-shaped solitary waves when wave speed passes certain critical wave speed.

To investigate the travelling wave solutions of eq. (1.2), substituting $u = u(x - ct) = u(\xi)$ into eq. (1.2), we obtain

$$-c(\phi - \phi'')' = a\phi' + \frac{1}{2}[(\phi^2 - \phi'^2)(\phi - \phi'')]', \tag{1.5}$$

where ϕ' is the derivative with respect to ξ. Integrating eq. (1.5) once and neglecting the constant of integration we find

$$(\phi^2 - \phi'^2 + 2c)\phi'' = \phi^3 + 2(a+c)\phi - \phi\phi'^2. \tag{1.6}$$

Clearly, eq. (1.6) is equivalent to the two-dimensional system

$$\begin{cases} \dfrac{\mathrm{d}\phi}{\mathrm{d}\xi} = y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}\xi} = \dfrac{\phi(\phi^2 - y^2 + 2(a+c))}{\phi^2 - y^2 + 2c}, \end{cases} \tag{1.7}$$

which has the first integral

$$H(\phi, y) = (y^2 - \phi^2 - 2c)^2 + 4a\phi^2 = h. \tag{1.8}$$

On the singular curve $\phi^2 - y^2 + 2c = 0$, system (1.7) is discontinuous. Such a system is called a singular travelling wave system by Li and Zhang [6].

2. Bifurcations of phase portraits of system (1.7)

In this section, we discuss bifurcations of phase portraits of system (1.7) and the existence of critical wave speed.

It is known that system (1.7) has the same phase portraits as the system

$$\begin{cases} \dfrac{\mathrm{d}\phi}{\mathrm{d}\zeta} = y(\phi^2 - y^2 + 2c), \\ \dfrac{\mathrm{d}y}{\mathrm{d}\zeta} = \phi(\phi^2 - y^2 + 2(a+c)), \end{cases} \tag{2.1}$$

where $\mathrm{d}\xi = (\phi^2 - y^2 + 2c)\mathrm{d}\zeta$, for $\phi^2 - y^2 + 2c \neq 0$.

The distribution of the equilibrium points of system (2.1) is as follows.

(1) For $c < 0$, when $a + c \geq 0$, system (2.1) has only one equilibrium point $O(0,0)$; when $a + c < 0$, system (2.1) has three equilibrium points $O(0,0)$ and $P_{\pm}(\pm\phi_1, 0)$, where $\phi_1 = \sqrt{-2(a+c)}$.

(2) For $c > 0$, when $a + c \geq 0$, system (2.1) has three equilibrium points $O(0,0)$ and $S_{\pm}(0, \pm\sqrt{2c})$; when $a + c < 0$, system (2.1) has five equilibrium points $O(0,0), P_{\pm}(\pm\phi_1, 0)$ and $S_{\pm}(0, \pm\sqrt{2c})$.

In addition, from eq. (1.8), we have

$$\begin{aligned} h_0 &= H(0,0) = 4c^2, \\ h_1 &= H(\pm\phi_1, 0) = 8a(a-c), \\ h_2 &= H(0, \pm\sqrt{2c}) = 0. \end{aligned} \tag{2.2}$$

It is known that a solitary wave solution of eq. (1.2) corresponds to a homoclinic orbit of system (2.1). Obviously, for $c < 0, a > 0, a+c < 0$, there exists a homoclinic orbit of system (2.1) homoclinic to $O(0,0)$ defined by $H(\phi, y) = h_0$. Hence, we are only interested in the case $c < 0, a > 0, a + c < 0$.

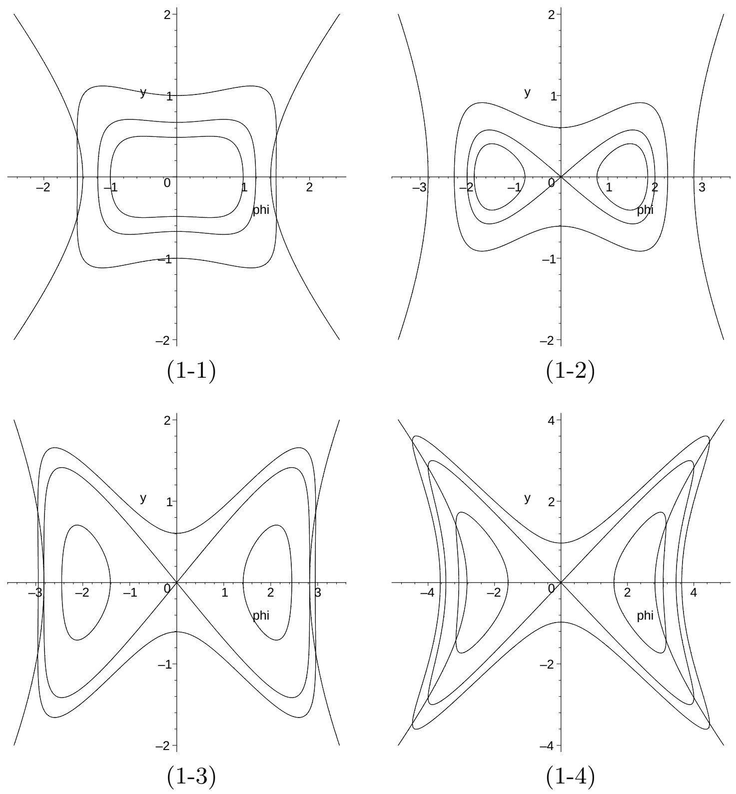


Figure 1. The change of phase portraits of system (2.1) for $c < 0, a > 0$. (**1-1**) $-a \le c < 0$, (**1-2**) $-2a < c < -a$, (**1-3**) $c = -2a$, (**1-4**) $c < -2a$.

PROPOSITION 2.1

Suppose that $c < 0, a > 0, a + c < 0$, *we have the following conclusions about critical wave speed* $c^* = -2a$:

(1) *For* $c^* < c < -a$, *the homoclinic orbit defined by* $H(\phi, y) = h_0$ *does not intersect the singular curve* $\phi^2 - y^2 + 2c = 0$. *Equation* (1.2) *has two smooth bell-shaped solitary wave solutions.*

(2) *For* $c < c^*$, *the homoclinic orbit defined by* $H(\phi, y) = h_0$ *intersects the singular curve* $\phi^2 - y^2 + 2c = 0$. *Equation* (1.2) *has a W-shaped solitary wave and an M-shaped solitary wave solutions.*

The phase portraits of system (2.1) can be shown in figure 1 for $c < 0, a > 0$.

3. Solitary wave solutions of eq. (1.2)

In this section, we will give some exact parametric representations of solitary wave solutions of eq. (1.2). We always assume that $c < 0, a > 0, a + c < 0$.

3.1 *The case* $-2a < c < -a$

In this case, the homoclinic orbit defined by $H(\phi, y) = h_0$ has no intersection point with the hyperbola $\phi^2 - y^2 + 2c = 0$. Thus, eq. (1.2) has a smooth bell-shaped solitary wave solution of valley-type and a smooth solitary wave solution of peak-type.

To find the exact explicit parametric representations of solitary wave solutions, we have the algebraic equation of homoclinic orbit

$$y^2 = \phi^2 + 2c \pm 2\sqrt{c^2 - a\phi^2}. \tag{3.1}$$

The signs before the term $2\sqrt{c^2 - a\phi^2}$ are dependent on the interval of ϕ. Under the condition $-2a < c < -a$, for $\phi \in (-2\sqrt{-(a+c)}, 2\sqrt{-(a+c)})$, we need to take $+$ before the term $2\sqrt{c^2 - a\phi^2}$. Setting $\psi^2 = c^2 - a\phi^2$, we have $y^2 = \frac{1}{a}(\psi - \psi_1)(\psi_2 - \psi)$, where $\psi_1 = -c, \psi_2 = 2a + c$. By the first equation of system (1.7), we obtain the following exact parametric representations of smooth bell-shaped solitary wave solutions of eq. (1.2)

$$\begin{aligned}
&\phi(\eta) = \pm\sqrt{\frac{c^2 - \psi^2(\eta)}{a}},\\
&\psi(\eta) = (a+c)(1 + \cosh(2\sqrt{c(a+c)}\eta)),\\
&\xi(\eta) = x - ct = -2\sqrt{c(a+c)}\left(\eta - \frac{1}{4(a+c)}\ln(\chi)\right),\\
&\chi = \frac{4c(a+c) - (a+2c)(\psi + c) - 2\sqrt{c(a+c)(\psi - c)(\psi - (2a+c))}}{-a(\psi + c)}.
\end{aligned} \tag{3.2}$$

The homoclinic orbit and profiles of bell-shaped solitary waves are shown in figure 2.

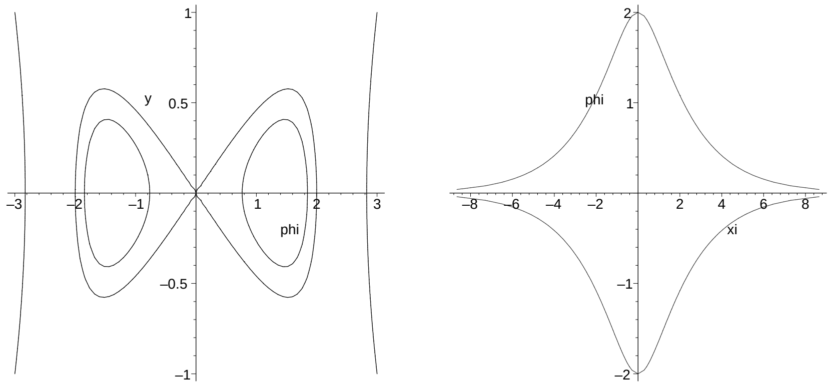
3.2 *The case* $c = -2a$

In this case, we have algebraic equation of homoclinic orbit

$$y^2 = \phi^2 - 4a + 2\sqrt{4a^2 - a\phi^2}. \tag{3.3}$$

Setting $\psi^2 = 4a^2 - a\phi^2$, we have $y^2 = \frac{1}{a}\psi(\psi_1 - \psi)$, where $\psi_1 = 2a$. By the first equation of system (1.7), we obtain the following exact parametric representations of smooth solitary wave solutions of eq. (1.2)

$$\begin{aligned}
&\phi(\eta) = \pm\sqrt{\frac{4a^2 - \psi^2(\eta)}{a}},\\
&\psi(\eta) = -a(1 + \cosh(2\sqrt{2}a\eta)),\\
&\xi(\eta) = x - ct = -2\sqrt{2}a\left(\eta + \frac{1}{4a}\ln\left(\frac{2a + 3\psi - 2\sqrt{2\psi(\psi + 2a)}}{2a - \psi}\right)\right).
\end{aligned} \tag{3.4}$$



(2-1) (2-2)

Figure 2. The homoclinic orbit (**2-1**) and the profile of a bell-shaped solitary wave (**2-2**) for $-2a < c < -a$.

3.3 *The case* $c < -2a$

The hyperbola $\phi^2 - y^2 + 2c = 0$ intersects the homoclinic orbit defined by $H(\phi, y) = h_0$ at four points $Q_1^{\pm}(-\phi^*, \pm y^*)$ and $Q_2^{\pm}(\phi^*, \pm y^*)$, where $\phi^* = -c\sqrt{a}/a, y^* = \sqrt{c(2a+c)/a}$. We have $y^2 = \phi^2 + 2c + 2\sqrt{c^2 - a\phi^2}$ in the interval between negative and positive half branches of the hyperbola $\phi^2 - y^2 + 2c = 0$. While in the left-hand side of the negative half branch and right-hand side of the positive half branch of the hyperbola $\phi^2 - y^2 + 2c = 0$, we have $y^2 = \phi^2 + 2c - 2\sqrt{c^2 - a\phi^2}$. Therefore, we can respectively write that

$$\begin{aligned} y^2 &= \frac{1}{a}(-\psi^2 + 2a\psi + c^2 + 2ac) = \frac{1}{a}(\psi - \psi_1)(\psi_2 - \psi), \\ y^2 &= \frac{1}{a}(-\psi^2 - 2a\psi + c^2 + 2ac) = -\frac{1}{a}(\psi + \psi_1)(\psi_2 + \psi), \end{aligned} \tag{3.5}$$

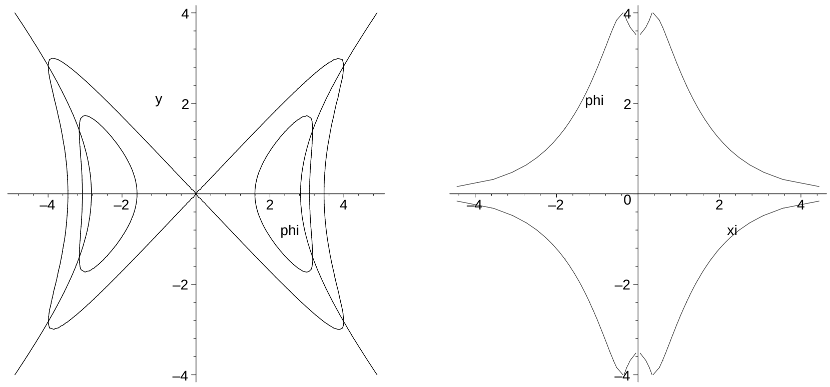
where $\psi_1 = -c, \psi_2 = 2a + c$. We next define a value η^* by satisfying

$$\phi(\eta^*) = \sqrt{\frac{c^2 - \psi^2(\eta)}{a}} = -\frac{c\sqrt{a}}{a} = \phi^*. \tag{3.6}$$

It is easy to see that for $\eta \in (-\infty, -\eta^*)$ and $\eta \in (\eta^*, +\infty)$, we have the same parametric representations of solitary wave solution as eq. (3.2). For $\eta \in (-\eta^*, \eta^*)$, we have

$$\frac{\psi \, \mathrm{d}\psi}{\sqrt{(\psi + \psi_1)^2(\psi - \psi_1)(\psi + \psi_2)}} = -\mathrm{d}\xi. \tag{3.7}$$

Integrating (3.7), we obtain the following exact parametric representations of solitary wave solutions of eq. (1.2)



(3-1) (3-2)

Figure 3. The homoclinic orbit (**3-1**) and the profile of W/M-shaped solitary wave (**3-2**) for $c < -2a$.

$$
\begin{aligned}
&\phi(\eta) = \pm\sqrt{\frac{c^2 - \psi^2(\eta)}{a}},\\
&\psi(\eta) = -(a+c) - a\cosh(2\sqrt{c(a+c)}\eta),\\
&\xi(\eta) = x - ct = -2\sqrt{c(a+c)}\left(\eta + \frac{1}{4(a+c)}\ln(\chi)\right),\\
&\chi = \frac{4c(a+c) + (a+2c)(\psi - c) - 2\sqrt{c(a+c)(\psi+c)(\psi+2a+c)}}{a(\psi - c)}.
\end{aligned}
\tag{3.8}
$$

The homoclinic orbit and profiles of W/M-shaped solitary waves are shown in figure 3.

4. Conclusion

In this paper,we have studied the dynamical behaviour and solitary wave solutions of an integrable nonlinear wave equation using bifurcation theory of dynamical systems. It is pointed out that the solitary waves of this equation evolve from bell-shaped solitary waves to W/M-shaped solitary waves when wave speed passes certain critical wave speed. Equation (1.2) naturally has a physical meaning since it is derived from the two-dimensional Euler equation. It can be cast into the Newton equation, ${u'}^2 = P(u) - P(A)$, of a particle with a new potential $P(u) = u^2 \pm \sqrt{(A^2+2c)^2 + 4a(A^2 - u^2)}$, where $A = \lim_{\xi\to\pm\infty} u$. In this paper, we successfully solve this Newton equation with W/M-shaped solitary wave solutions. These solitary wave solutions may be applied to neuroscience for providing a mathematical model and explaining electrophysiological responses of visceral nociceptive neurons and sensitization of dorsal root reflexes [10]. The mathematical results we have obtained about the singular travelling wave equation provide a deep insight into the nonlinear wave model and will be useful for physicists to comprehend the dynamical behaviour of nonlinear wave models.

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References

[1] D J Korteweg and G de Vries, *Philos. Mag.* **39**, 422 (1895)
[2] J S Russell, On waves, in: *Report of 14th Meeting of the British Association for the Advancement of Science*, York (1844) 311–390
[3] R Camassa and D Holm, *Phys. Rev. Lett.* **71**, 1661 (1993)
[4] P Rosenau, *Phys. Lett.* **A230**, 305 (1997)
[5] Z Qiao, *J. Math. Phys.* **47**, 112701 (2006)
[6] J B Li and Y Zhang, *Nonlin. Anal.* **10**, 1797 (2009)
[7] J B Li and Z R Liu, *Appl. Math. Modell.* **25**, 41 (2000)
[8] D H Feng and J B Li, *Pramana – J. Phys.* **68**, 863 (2007)
[9] A Y Chen and Z J Ma, *Pramana – J. Phys.* **71**, 57 (2008)
[10] J H Chen, H R Weng and P M Dougherty, *Neuroscience* **126**, 743 (2004)