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Locally-rotationally-symmetric Bianchi type-V cosmology with heat flow

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Abstract. In this paper we present a spatially homogeneous locally-rotationally-symmetric (LRS) Bianchi type-V cosmological model with perfect fluid and heat flow. A general approach is introduced to solve Einstein's field equations using a law of variation for the mean Hubble parameter, which is related to average scale factor of the model that yields a constant value for the deceleration parameter. Exact solutions that correspond to singular and non-singular models are found with heat flow. The physical constraints on the solution and, in particular, the thermodynamical laws that govern such solutions are discussed in some detail.

Keywords. Cosmology; Bianchi models; inflationary phase; Hubble parameter; deceleration parameter.

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1. Model and field equations

We consider a locally-rotationally-symmetric (LRS) Bianchi type-V space-time with metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + A^2(t)\mathrm{d}x^2 + \mathrm{e}^{2x}B^2(t)\big(\mathrm{d}y^2 + \mathrm{d}z^2\big), \tag{1}$$

where $A(t)$ and $B(t)$ are the cosmic scale functions.

The energy–momentum tensor of a perfect fluid with heat conduction has the form [1]:

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu + p g_{\mu\nu} + Q_\mu u_\nu + Q_\nu u_\mu, \tag{2}$$

where p is the thermodynamic pressure, ρ is the energy density, u_μ is the four-velocity of the fluid and Q_μ is the heat flow vector satisfying $Q_\mu Q^\mu > 0$ and $Q_\mu u^\mu = 0$. The generalized mean Hubble parameter H is given by

$$H = \frac{\dot{a}}{a} = \frac{1}{3}\left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right), \tag{3}$$

where $a = \left(AB^2\right)^{1/3}$ is the average scale factor. The dot denotes a derivative with respect to cosmic time t.

In a co-moving coordinate system $u^\mu = \delta_0^\mu$ and $Q^\mu = \delta_1^\mu Q^1$, the existing components of the Einstein's field equations, in the system of units $8\pi G = c = 1$, and in view of eq. (2), for the Bianchi type-V space-time (1), explicitly give the following set of equations:

$$2\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} - \frac{1}{A^2} = -p \tag{4}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}}{A}\frac{\dot{B}}{B} - \frac{1}{A^2} = -p \tag{5}$$

$$2\frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\dot{B}^2}{B^2} - \frac{3}{A^2} = \rho \tag{6}$$

$$2\left(\frac{\dot{B}}{B} - \frac{\dot{A}}{A}\right) = Q_1. \tag{7}$$

From eqs (4) and (6), we obtain the Raychaudhuri equation [2]:

$$\dot{\theta} = -\frac{1}{3}\theta^2 - 2\sigma^2 - \frac{1}{2}\left(\rho + 3p\right). \tag{8}$$

Finally, the law of energy conservation equation $T^{\mu\nu}_{;\nu} = 0$ gives

$$\dot{\rho} + (\rho + p)\left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right) = -\frac{2}{A^2}Q_1. \tag{9}$$

2. Solution of field equations

In order to solve completely, we need one extra relation among the variables. As were observed earlier by Berman [3] and Berman and Gomide [4], the constant deceleration parameter models yield laws for the scale factor that stand adequately for our present view of the different phases of the Universe. So we shall consider here the case in LRS Bianchi type-V perfect fluid model with heat conduction. Recently, Singh [5] has extended the work to LRS Bianchi type-V cosmological models and obtained solutions of the field equations in general relativity.

According to the law, the variation for the mean Hubble parameter is given by

$$H = la^{-n} = l(AB^2)^{-n/3}, \tag{10}$$

where $l(>0)$ and $n(\geq 0)$ are constants, and

$$n = q + 1, \tag{11}$$

where H is defined as in eq. (3) and q, the deceleration parameter, is defined as

$$q = -\frac{\ddot{a}a}{\dot{a}^2}. \tag{12}$$

Using eqs (10) and (11), the solution of eq. (12) gives the law of variation of the average scale factor of the form

$$a = (nlt)^{1/n} \tag{13}$$

for $n \neq 0$ and

$$a = c\exp(lt) \tag{14}$$

for $n = 0$, where c is the constant of integration. Here, in eq. (13), we have assumed that for $t = 0$ the value $a = 0$ so that the constant of integration turns out to be zero.

Now, from eqs (4) and (5), we get

$$\frac{\ddot{B}}{B} - \frac{\ddot{A}}{A} + \frac{\dot{B}^2}{B^2} - \frac{\dot{A}}{A}\frac{\dot{B}}{B} = 0. \tag{15}$$

Integrating eq. (15) and using $a = \left(AB^2\right)^{1/3}$, the quadrature form of the metric functions A and B are given by

$$A(t) = (d_1)^{-2/3}\, a\exp\left(-\frac{2k_1}{3}\int a^{-3}\mathrm{d}t\right), \tag{16}$$

$$B(t) = (d_1)^{1/3}\, a\exp\left(\frac{k_1}{3}\int a^{-3}\mathrm{d}t\right), \tag{17}$$

where k_1 and d_1 are the constants of integration.

2.1 *Solutions with $n \neq 0$*

In this case, using eq. (13) into eqs (16) and (17), the solution of the metric functions is given by

$$A(t) = (d_1)^{-2/3}(nlt)^{1/n}\exp\left[-\frac{2k_1}{3l(n-3)}(nlt)^{(n-3)/n}\right], \tag{18}$$

$$B(t) = (d_1)^{1/3} (nlt)^{1/n} \exp\left[\frac{k_1}{3l(n-3)} (nlt)^{(n-3)/n}\right], \tag{19}$$

where $n \neq 3$. The solution for heat flow is given by

$$Q_1 = 2k_1 (nlt)^{-3/n} . \tag{20}$$

The energy density and pressure are, respectively, given by

$$\rho = 3 (nt)^{-2} - \frac{k_1^2}{3} (nlt)^{-6/n} - 3 (d_1)^{4/3} (nlt)^{-2/n} \times \exp\left[\frac{4k_1}{3l(n-3)} (nlt)^{(n-3)/n}\right], \tag{21}$$

$$p = (2n-3) (nt)^{-2} - \frac{k_1^2}{3} (nlt)^{-6/n} + (d_1)^{4/3} (nlt)^{-2/n} \times \exp\left[\frac{4k_1}{3l(n-3)} (nlt)^{(n-3)/n}\right]. \tag{22}$$

We find that the above solutions satisfy the Raychaudhuri equation (8) and conservation equation (9) identically. We find that $\theta \propto (nt)^{-1}$, $A_p \propto (nlt)^{(2n-6)/n}$ and $\sigma^2 \propto (nlt)^{-6/n}$. From the above set of solutions we observe that the spatial volume is zero at $t = 0$. The metric functions $A(t)$ and $B(t)$ are also zero at this initial epoch. The energy density and pressure become infinite at $t = 0$. The physical parameters θ, A_p and σ^2 are all infinite at this initial point. The model has a point singularity at $t = 0$. The heat conduction is a decreasing function of time and is maximum at the initial epoch. As $t \to \infty$, both the scale factors $A(t)$ and $B(t)$ tend to infinity whereas p and ρ tend to zero. The physical parameters such as θ, A_p and σ^2 tend to zero as $t \to \infty$, which indicates that the Universe is expanding with cosmic time but the rate of expansion is decreasing and the model becomes isotropic at large times. The heat flow diminishes as $t \to \infty$. Also, we find that $\lim_{t\to\infty}\sigma/\theta^2 = 0$ for $n < 3$ and $\lim_{t\to\infty}\rho/\theta^2 =$const., which indicate that the models approach isotropy for large t. The flow of heat conduction along the x-direction was maximum early on, and it diminishes as $t \to \infty$. We also observe that $\sigma^2 = Q_1^2/12$, which implies that the shear scalar is directly proportional to heat conduction throughout the evolution.

2.2 *Solutions with $n = 0$*

In this case, using eq. (14) in eqs (16) and (17), the solution for the scale factors is given by

$$A(t) = (d_1)^{-2/3} c\exp\left[lt + \frac{2k_1}{3lc^3}\exp(-3lt)\right], \tag{23}$$

$$B(t) = (d_1)^{1/3} c\exp\left[lt - \frac{k_1}{3lc^3} \exp(-3lt)\right]. \tag{24}$$

The solution for heat conduction is given by

$$Q_1 = \frac{2k_1}{c^3}\exp(-3lt). \tag{25}$$

The energy density and pressure are respectively given by

$$\begin{aligned}\rho = {} & 3l^2 - \frac{1}{3}\frac{k_1^2}{c^6}\exp(-6lt) \\ & -3\,(d_1)^{4/3}\,c^{-2}\exp\left[-2\left\{lt + \frac{2k_1}{3lc^3}\exp(-3lt)\right\}\right],\end{aligned} \tag{26}$$

$$\begin{aligned}p = {} & -3l^2 - \frac{1}{3}\frac{k_1^2}{c^6}\exp(-6lt) \\ & +(d_1)^{4/3}\,c^{-2}\exp\left[-2\left\{lt + \frac{2k_1}{3lc^3}\exp(-3lt)\right\}\right].\end{aligned} \tag{27}$$

The above solutions identically satisfy the Raychaudhuri equation (8) and conservation equation (9), respectively. We find that $\theta = 3l$, $A_p \propto \exp(-6lt)$ and $\sigma^2 \propto \exp(-6lt)$. Thus all the geometrical and physical parameters such as $A(t)$, $B(t)$, ρ, p, θ, A_p, σ^2 and the heat flow are constant at $t = 0$. Thus, the Universe starts evolving with constant physical and geometrical parameters. The rate of expansion is uniform throughout the evolution. As $t \to \infty$, $A(t)$ and $B(t)$ tend to infinity whereas ρ and p are related by the equation of state $p = -\rho$.

3. Thermodynamical relations

3.1 *Baryon conservation law*

In standard cosmology, conservation of total particle number gives

$$N^{\mu}_{;\mu} = 0, \tag{28}$$

where $N^{\mu} = \chi u^{\mu}$ is the particle flux and χ is the particle number density, which is given by

$$\frac{\mathrm{d}\chi}{\mathrm{d}t} = -\chi\theta. \tag{29}$$

For the power-law solutions, the particle number density is

$$\chi = b_1\,(nlt)^{-3/n}, \tag{30}$$

and for the exponential solutions ($n = 0$), we get

$$\chi = \frac{b_2}{c^3}\exp(-3lt), \tag{31}$$

where b_1 and b_2 are constants of integration. Using eqs (20) and (25), eqs (30) and (31) yield $\chi \propto Q_1$. We observe that the particle number density is large at $t = 0$ in the case of $n \neq 0$, and so the heat conduction has a greater influence during the early stages of evolution. When $n = 0$, the particle density is constant at $t = 0$, which implies that heat conduction is constant.

3.2 *Temperature gradient law*

The usual expression for heat conduction [6] is

$$Q^{\mu} = -\kappa(g^{\mu\nu} + u^{\mu}u^{\nu})(T_{,\nu} + Tu_{\nu;\alpha}u^{\alpha}), \tag{32}$$

where $\kappa \geq 0$ is the heat conduction coefficient, i.e., thermal conductivity, T is the temperature and $u_{\nu;\alpha}u^{\alpha}$ is the acceleration. Since in our case only the x-component of heat flux is retained, from the above equation we obtain $Q_1 = \kappa T_{;1}$.

The temperature distribution for $n \neq 0$ and $n = 0$ respectively are given in the forms:

$$T = \frac{k_1(nlt)^{-3/n}}{\kappa(t)}x + \eta_1(t), \tag{33}$$

$$T = \frac{k_1}{c^3}\frac{\exp(-3lt)}{\kappa(t)}x + \eta_2(t), \tag{34}$$

where $\eta_1(t)$ and $\eta_2(t)$ appear as integration functions, which may be either an arbitrary function of time or constants. We observe that T diverges at the initial epoch as long as the coefficient of thermal conductivity remains finite. At the final stage of expansion $t \to \infty$, we have $T \to \eta_1(t)$ when $n \neq 0$ and $T \to \eta_2(t)$ when $n = 0$, which implies that the Universe will be in thermal equilibrium at the final stage of evolution.

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