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Solution of an analogous Schrödinger equation for $\mathcal{PT}$-symmetric sextic potential in two dimensions

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Abstract. We investigate the quasi-exact solutions of an analogous Schrödinger wave equation for two-dimensional non-Hermitian complex Hamiltonian systems within the framework of an extended complex phase space characterized by $x = x_1 + ip_3, y = x_2 + ip_4, p_x = p_1 + ix_3, p_y = p_2 + ix_4$. Explicit expressions for the energy eigenvalues and eigenfunctions for ground and first excited states of a two-dimensional $\mathcal{PT}$-symmetric sextic potential and some of its variants are obtained. The eigenvalue spectra are found to be real within some parametric domains.

Keywords. Schrödinger equation; complex Hamiltonian; $\mathcal{PT}$ symmetry; eigenvalues and eigenfunctions.

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1. Introduction

Quantum systems characterized by non-Hermitian Hamiltonians are of great interest in several areas of theoretical physics like superconductivity, population biology, quantum cosmology, condensed matter physics, quantum field theory, and so on [1]. Therefore, in the last few years the study of complex potentials has become important for obtaining better theoretical understanding of some newly discovered phenomena in physics and chemistry, like the phenomena pertaining to resonance scattering in atomic, molecular, and nuclear physics and to some chemical reactions [2].

A complex (non-Hermitian) Hamiltonian H can provide real and bounded eigenvalues for certain domains of the underlying parameters if H is invariant under the simultaneous action of the space ($\mathcal{P}$) and time ($\mathcal{T}$) reversal [3]. Now it is possible to study complex Hamiltonians ($\mathcal{PT}$-symmetric) which were not considered earlier for not meeting the Hermiticity requirement [4–9].

There are various ways of complexifying a given Hamiltonian [10]. However, in the present work we use a scheme due to Xavier and de Aguiar [11], used to develop an algorithm for the computation of the semiclassical coherent state propagator, to transform potentials on an extended complex phase space (ECPS). In this approach the transformations for positions and momenta in two dimensions are defined as

$$x = x_1 + ip_3, \quad y = x_2 + ip_4,$$
$$p_x = p_1 + ix_3, \quad p_y = p_2 + ix_4. \tag{1}$$

The presence of variables x_3, x_4, p_3, p_4 in the above transformations may be regarded as some sort of coordinate–momentum interactions of the dynamical system [10]. This approach has also been utilized in the study of classical systems, particularly for tracing complex dynamical invariants and for obtaining the solutions of diffusion reaction equation of a number of classical dynamical systems [10,12,13]. Transformations similar to eq. (1) have also been used in the study of nonlinear evolution equations in the context of amplitude-modulated nonlinear Langmuir waves in plasma [14].

Recently, in some studies the solutions of an analogous Schrödinger wave equation (ASE) have been reported using ECPS approach [8,9]. However, such studies are confined to one-dimensional systems only. An extension of such studies in higher dimensions is desirable to explore the possibilities of finding more applications. With this motivation we have generalized ECPS approach in two dimensions and studied some interesting two-dimensional complex systems and found energy eigenvalues and eigenfunctions for ground and first two excited states [15,16]. With the same spirit, in the present work, to expand the domain of applications of ECPS approach, we investigate the solution of the ASE for a $\mathcal{PT}$-symmetric coupled complex sextic potential. Various forms of sextic potential, real as well as complex forms, are studied by many authors [9,17–21]. However, most of such studies are again confined to one-dimensional systems. The study of such potentials may be of interest in various fields, particularly in fibre optics and quantum chemistry.

The organization of the paper is as follows: in §2, we shall develop the mathematical formulation within the framework of ECPS in two dimensions, for computing eigenvalue spectra of two-dimensional complex systems. In §3, eigenvalues and eigenfunctions of a $\mathcal{PT}$-symmetric sextic potential in two dimensions for the ground and first excited states will be investigated. Finally, concluding remarks are presented in §4.

2. The method

For a two-dimensional complex Hamiltonian system $H(x, y, p_x, p_y)$, the ASE (for $\hbar = m = 1$) is written as

$$\hat{H}(x, y, p_x, p_y)\psi(x, y) = E\psi(x, y), \tag{2}$$

where

$$\hat{H}(x, y, p_x, p_y) = -\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + V(x, y). \tag{3}$$

Here we only present time-independent stationary state solutions of eq. (2) for the sake of convenience. Now, using the transformations (1), we obtain

$$\frac{\partial}{\partial x} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial p_3}\right), \quad \frac{\partial}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x_2} - i\frac{\partial}{\partial p_4}\right),$$
$$\frac{\partial}{\partial p_x} = \frac{1}{2}\left(\frac{\partial}{\partial p_1} - i\frac{\partial}{\partial x_3}\right), \quad \frac{\partial}{\partial p_y} = \frac{1}{2}\left(\frac{\partial}{\partial p_2} - i\frac{\partial}{\partial x_4}\right). \tag{4}$$

Note that the momentum operators $p_x = -i\hbar\frac{\partial}{\partial x}$ and $p_y = -i\hbar\frac{\partial}{\partial y}$ of the conventional quantum mechanics under the transformations (1) reduce to the forms: $p_1 + ix_3 = \frac{-i}{2}(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial p_3})$ and $p_2 + ix_4 = \frac{-i}{2}(\frac{\partial}{\partial x_2} - i\frac{\partial}{\partial p_4})$. These relations give $p_1 = \frac{-1}{2}\frac{\partial}{\partial p_3}$, $x_3 = \frac{-1}{2}\frac{\partial}{\partial x_1}$, $p_2 = \frac{-1}{2}\frac{\partial}{\partial p_4}$ and $x_4 = \frac{-1}{2}\frac{\partial}{\partial x_2}$. These results lead to the commutation relations namely, $[x_1, x_3] = [p_3, p_1] = [x_2, x_4] = [p_4, p_2] = 1$, $[x_i, p_j] = 0$, where $i, j = 1, 2, 3, 4$.

Also the complex coordinate transformations (1) preserve the fundamental commutation relations, $[x, p_x] = [y, p_y] = i$, which can easily be verified using eqs (1) and (4).

Now consider $V(x, y), \psi(x, y)$ and E as complex quantities

$$V = V_{\rm r} + iV_{\rm i}, \quad \psi = \psi_{\rm r} + i\psi_{\rm i}, \quad E = E_{\rm r} + iE_{\rm i},$$

where subscripts r and i denote the real and imaginary parts of the corresponding quantities and other subscripts to these quantities separated by a comma will denote the partial derivatives of the quantity concerned.

Thus, using eq. (4) in eq. (3) and using the above equations, the ASE (2), after separating real and imaginary parts, reduces to a pair of coupled partial differential equations as

$$-\frac{1}{8}(\psi_{{\rm r},x_1x_1} - \psi_{{\rm r},p_3p_3} + 2\psi_{{\rm i},x_1p_3} + \psi_{{\rm r},x_2x_2} - \psi_{{\rm r},p_4p_4} + 2\psi_{{\rm i},x_2p_4})$$
$$+V_{\rm r}\psi_{\rm r} - V_{\rm i}\psi_{\rm i} = E_{\rm r}\psi_{\rm r} - E_{\rm i}\psi_{\rm i}, \tag{5a}$$
$$-\frac{1}{8}(\psi_{{\rm i},x_1x_1} - \psi_{{\rm i},p_3p_3} - 2\psi_{{\rm r},x_1p_3} + \psi_{{\rm i},x_2x_2} - \psi_{{\rm i},p_4p_4} - 2\psi_{{\rm r},x_2p_4})$$
$$+V_{\rm r}\psi_{\rm i} + V_{\rm i}\psi_{\rm r} = E_{\rm r}\psi_{\rm i} + E_{\rm i}\psi_{\rm r}. \tag{5b}$$

The Cauchy–Riemann analyticity conditions for $\psi(x, y)$ are given as

$$\psi_{{\rm r},x_1} = \psi_{{\rm i},p_3}, \quad \psi_{{\rm r},p_3} = -\psi_{{\rm i},x_1}, \quad \psi_{{\rm r},x_2} = \psi_{{\rm i},p_4}, \quad \psi_{{\rm r},p_4} = -\psi_{{\rm i},x_2}. \tag{6}$$

Other higher-order conditions which $\psi_{\rm r}$ and $\psi_{\rm i}$ have to satisfy are derived from eq. (6) as

$$\frac{\partial^2\psi_m}{\partial x_1^2} + \frac{\partial^2\psi_m}{\partial p_3^2} = 0, \quad \frac{\partial^2\psi_m}{\partial x_2^2} + \frac{\partial^2\psi_m}{\partial p_4^2} = 0,$$
$$\frac{\partial^2\psi_m}{\partial x_1 x_2} + \frac{\partial^2\psi_m}{\partial p_3 p_4} = 0, \quad \frac{\partial^2\psi_m}{\partial x_1 p_4} - \frac{\partial^2\psi_m}{\partial x_2 p_3} = 0, \tag{7}$$

where $m =$ i, r. Note that the analyticity conditions (6) and the other conditions listed in eq. (7) on eigenfunctions greatly simplifies the underlying computation in determining the nature of the eigenvalue spectra.

Hence, in view of eqs (6), eqs (5a) and (5b) are written as

$$-\frac{1}{2}(\psi_{\mathrm{r},x_1x_1}+\psi_{\mathrm{r},x_2x_2})+V_\mathrm{r}\psi_\mathrm{r}-V_\mathrm{i}\psi_\mathrm{i}=E_\mathrm{r}\psi_\mathrm{r}-E_\mathrm{i}\psi_\mathrm{i}, \tag{8a}$$

$$-\frac{1}{2}(\psi_{\mathrm{i},x_1x_1}+\psi_{\mathrm{i},x_2x_2})+V_\mathrm{r}\psi_\mathrm{i}+V_\mathrm{i}\psi_\mathrm{r}=E_\mathrm{r}\psi_\mathrm{i}+E_\mathrm{i}\psi_\mathrm{r}. \tag{8b}$$

We now make an ansatz for the wave function $\psi(x,y)$ as

$$\psi(x,y)=\phi(x,y)\ \exp[g(x,y)], \tag{9}$$

where $\phi(x,y)$ and $g(x,y)$ are complex functions and are expressed as

$$\phi=\phi_\mathrm{r}+i\phi_\mathrm{i},\quad g=g_\mathrm{r}+ig_\mathrm{i}. \tag{10}$$

Substituting eq. (10) in eq. (9), the real and imaginary parts of $\psi(x,y)$ become

$$\psi_\mathrm{r}=\mathrm{e}^{g_\mathrm{r}}(\phi_\mathrm{r}\cos g_\mathrm{i}-\phi_\mathrm{i}\sin g_\mathrm{i}),\quad \psi_\mathrm{i}=\mathrm{e}^{g_\mathrm{r}}(\phi_\mathrm{i}\cos g_\mathrm{i}+\phi_\mathrm{r}\sin g_\mathrm{i}). \tag{11}$$

Equations (8a) and (8b), with the help of eq. (11), are written as

$$\begin{aligned}
&g_{\mathrm{r},x_1x_1}+g_{\mathrm{r},x_2x_2}+(g_{\mathrm{r},x_1})^2+(g_{\mathrm{r},x_2})^2-(g_{\mathrm{i},x_1})^2-(g_{\mathrm{i},x_2})^2\\
&+\frac{1}{(\phi_\mathrm{r}^2+\phi_\mathrm{i}^2)}[\phi_\mathrm{r}(\phi_{\mathrm{r},x_1x_1}+\phi_{\mathrm{r},x_2x_2}+2\phi_{\mathrm{r},x_1}g_{\mathrm{r},x_1}+2\phi_{\mathrm{r},x_2}g_{\mathrm{r},x_2}\\
&-2\phi_{\mathrm{i},x_1}g_{\mathrm{i},x_1}-2\phi_{\mathrm{i},x_2}g_{\mathrm{i},x_2})+\phi_\mathrm{i}(\phi_{\mathrm{i},x_1x_1}+\phi_{\mathrm{i},x_2x_2}+2\phi_{\mathrm{r},x_1}g_{\mathrm{i},x_1}\\
&+2\phi_{\mathrm{r},x_2}g_{\mathrm{i},x_2}+2\phi_{\mathrm{i},x_1}g_{\mathrm{r},x_1}+2\phi_{\mathrm{i},x_2}g_{\mathrm{r},x_2})]+2(E_\mathrm{r}-V_\mathrm{r})=0,
\end{aligned} \tag{12a}$$

$$\begin{aligned}
&g_{\mathrm{i},x_1x_1}+g_{\mathrm{i},x_2x_2}+2g_{\mathrm{r},x_1}g_{\mathrm{i},x_1}+2g_{\mathrm{r},x_2}g_{\mathrm{i},x_2}+\frac{1}{(\phi_\mathrm{r}^2+\phi_\mathrm{i}^2)}\\
&\times[\phi_\mathrm{r}(\phi_{\mathrm{i},x_1x_1}+\phi_{\mathrm{i},x_2x_2}+2\phi_{\mathrm{r},x_1}g_{\mathrm{i},x_1}+2\phi_{\mathrm{r},x_2}g_{\mathrm{i},x_2}+2\phi_{\mathrm{i},x_1}g_{\mathrm{i},x_1}\\
&+2\phi_{\mathrm{i},x_2}g_{\mathrm{i},x_2})+\phi_\mathrm{i}(-\phi_{\mathrm{r},x_1x_1}-\phi_{\mathrm{r},x_2x_2}+2\phi_{\mathrm{i},x_1}g_{\mathrm{i},x_1}+2\phi_{\mathrm{i},x_2}g_{\mathrm{i},x_2}\\
&-2\phi_{\mathrm{r},x_1}g_{\mathrm{i},x_1}-2\phi_{\mathrm{r},x_2}g_{\mathrm{i},x_2})]+2(E_\mathrm{i}-V_\mathrm{i})=0.
\end{aligned} \tag{12b}$$

Note that, for given forms of $\phi(x,y)$ and $g(x,y)$, the rationalization of eqs (12a) and (12b) yield the real and imaginary parts of the eigenvalue spectra for a given system.

However, the ground state solutions for complex systems can be obtained by choosing $\phi(x,y)$ as constant in eqs (12a) and (12b). Thus for ground state solutions eqs (12a) and (12b) reduce to

$$\begin{aligned}
&g_{\mathrm{r},x_1x_1}+g_{\mathrm{r},x_2x_2}+(g_{\mathrm{r},x_1})^2+(g_{\mathrm{r},x_2})^2-(g_{\mathrm{i},x_1})^2\\
&\qquad-(g_{\mathrm{i},x_2})^2+2(E_\mathrm{r}-V_\mathrm{r})=0,
\end{aligned} \tag{13a}$$

$$g_{\mathrm{i},x_1x_1}+g_{\mathrm{i},x_2x_2}+2g_{\mathrm{r},x_1}g_{\mathrm{i},x_1}+2g_{\mathrm{r},x_2}g_{\mathrm{i},x_2}+2(E_\mathrm{i}-V_\mathrm{i})=0. \tag{13b}$$

Equations (13a) and (13b) now can be rationalized to obtain ground state eigenvalues for a given potential.

In what follows, we use the derivations made in the present section to solve the ASE for two-dimensional complex sextic potentials.

3. Sextic potential

Here, we consider a two-dimensional complex sextic potential and obtain the eigenvalue spectrum for this by solving the ASE. A general even powered sextic potential is written as

$$V(x,y) = a_{20}x^2 + a_{02}y^2 + a_{11}xy + a_{40}x^4 + a_{04}y^4 + a_{22}x^2y^2 + a_{13}xy^3 \\ +a_{31}x^3y + a_{24}x^2y^4 + a_{42}x^4y^2 + a_{60}x^6 + a_{06}y^6, \tag{14}$$

where a_{ij} are real coupling parameters.

The $\mathcal{PT}$-symmetric form of the potential (14) is obtained by applying the transformations (1) along with the condition

$$(x_1, p_3, x_2, p_4, p_1, x_3, p_2, x_4; i) \to (-x_1, p_3, -x_2, p_4, p_1, -x_3, p_2, -x_4; -i).$$

The real and imaginary parts of the $\mathcal{PT}$-symmetric potential (14) are given by

$$\begin{aligned} V_{\mathrm{r}} &= a_{20}(x_1^2 - p_3^2) + a_{02}(x_2^2 - p_4^2) + a_{11}(x_1x_2 - p_3p_4) \\ &+a_{40}(x_1^4 + p_3^4 - 6x_1^2p_3^2) + a_{04}(x_2^4 + p_4^4 - 6x_2^2p_4^2) \\ &+a_{22}(x_1^2x_2^2 + p_3^2p_4^2 - x_1^2p_4^2 - x_2^2p_3^2 - 4x_1x_2p_3p_4) \\ &+a_{31}(x_1^3x_2 + p_3^3p_4 - 3x_1x_2p_3^2 - 3x_1^2p_3p_4) \\ &+a_{13}(x_1x_2^3 + p_3p_4^3 - 3x_1x_2p_4^2 - 3x_2^2p_3p_4) \\ &+a_{60}(x_1^6 - p_3^6 - 15x_1^4p_3^2 + 15x_1^2p_3^4) \\ &+a_{06}(x_2^6 - p_4^6 - 15x_2^4p_4^2 + 15x_2^2p_4^4) \\ &+a_{42}(-x_1^4p_4^2 - 8x_1^3x_2p_3p_4 + 8x_1x_2p_3^3p_4 - p_3^4p_4^2 \\ &-6x_1^2x_2^2p_3^2 + x_2^2p_3^4 + x_1^4x_2^2 + 6x_1^2p_3^2p_4^2) \\ &+a_{24}(-x_2^4p_3^2 - 8x_1x_2^3p_3p_4 + 8x_1x_2p_3p_4^3 \\ &-p_3^2p_4^4 - 6x_1^2x_2^2p_4^2 + x_1^2p_4^4 + x_1^2x_2^4 + 6x_2^2p_3^2p_4^2), \end{aligned} \tag{15a}$$

$$\begin{aligned} V_{\mathrm{i}} &= 2a_{20}x_1p_3 + 2a_{02}x_2p_4 + a_{11}(x_1p_4 + x_2p_3) \\ &+a_{13}(3x_1x_2^2p_4 - x_1p_4^3 + x_2^3p_3 - 3x_2p_3p_4^2) \\ &+a_{31}(3x_1^2x_2p_3 - x_2p_3^3 + x_1^3p_4 - 3x_1p_3^2p_4) \\ &+2a_{22}(x_1^2x_2p_4 - x_2p_3^2p_4 + x_1x_2^2p_3 - x_1p_3p_4^2) \\ &+4a_{40}(x_1^3p_3 - x_1p_3^3) + 4a_{04}(x_2^3p_4 - x_2p_4^3) \\ &+a_{60}(6x_1^5p_3 + 6x_1p_3^5 - 20x_1^3p_3^3) \\ &+a_{06}(6x_2^5p_4 + 6x_2p_4^5 - 20x_2^3p_4^3) \\ &+2a_{42}(2x_1^3x_2^2p_3 + x_1^4x_2p_4 + x_2p_3^4p_4 \\ &-2x_1^3p_3p_4^2 - 2x_1x_2^2p_3^2 + 2x_1p_3^3p_4^2 - 6x_1^2x_2p_3^2p_4) \\ &+2a_{24}(2x_2p_3^2p_4^3 + 2x_1^2x_2^3p_4 - 6x_1x_2^2p_3p_4^2 + x_1p_3p_4^4 \\ &-2x_2^3p_3^2p_4 - 2x_1^2x_2p_4^3 + x_1x_2^4p_3). \end{aligned} \tag{15b}$$

3.1 *Ground state solution*

For obtaining the ground state solution for the sextic potential, we choose $\phi(x_1, p_3, x_2, p_4) = 1$ and the ansatz for $g_{\rm r}(x_1, p_3, x_2, p_4)$ and $g_{\rm i}(x_1, p_3, x_2, p_4)$ are considered as

$$\begin{aligned} g_{\rm r} = & \frac{1}{2}\alpha_{20}(x_1^2 - p_3^2) + \frac{1}{2}\alpha_{02}(x_2^2 - p_4^2) + \alpha_{11}(x_1x_2 - p_3p_4) \\ & + \frac{1}{4}\alpha_{40}(x_1^4 + p_3^4 - 6x_1^2p_3^2) + \frac{1}{4}\alpha_{04}(x_2^4 + p_4^4 - 6x_2^2p_4^2) \\ & + \frac{1}{2}\alpha_{22}(x_1^2x_2^2 + p_3^2p_4^2 - x_1^2p_4^2 - x_2^2p_3^2 - 4x_1x_2p_3p_4), \end{aligned} \tag{16}$$

$$\begin{aligned} g_{\rm i} = & \alpha_{20}x_1p_3 + \alpha_{02}x_2p_4 + \alpha_{11}(x_1p_4 + x_2p_3) \\ & + \alpha_{40}(x_1^3p_3 - x_1p_3^3) + \alpha_{04}(x_2^3p_4 - x_2p_4^3) \\ & + \alpha_{22}(x_1x_2^2p_3 - x_1p_3p_4^2 + x_1^2x_2p_4 - x_2p_3^2p_4). \end{aligned} \tag{17}$$

Now, on rationalization of eqs (13a) and (13b), after substituting equations (15a)–(17), we obtain

$$E_{\rm r} = -\frac{1}{2}(\alpha_{02} + \alpha_{20}), \quad E_{\rm i} = 0, \tag{18a}$$

$$\alpha_{22} + 3\alpha_{40} + \alpha_{11}^2 + \alpha_{20}^2 = 2a_{20}, \tag{18b}$$

$$\alpha_{22} + 3\alpha_{04} + \alpha_{11}^2 + \alpha_{02}^2 = 2a_{02}, \tag{18c}$$

$$\alpha_{20}\alpha_{11} + \alpha_{02}\alpha_{11} = a_{11}, \tag{18d}$$

$$\alpha_{20}\alpha_{22} + \alpha_{02}\alpha_{22} = a_{22}, \tag{18e}$$

$$\alpha_{11}\alpha_{22} + \alpha_{11}\alpha_{04} = 2a_{13}, \tag{18f}$$

$$\alpha_{11}\alpha_{22} + \alpha_{11}\alpha_{40} = 2a_{31}, \tag{18g}$$

$$\alpha_{22}^2 + 2\alpha_{22}\alpha_{04} = 2a_{24}, \tag{18h}$$

$$\alpha_{22}^2 + 2\alpha_{22}\alpha_{40} = 2a_{42}, \tag{18i}$$

$$\alpha_{20}\alpha_{40} = a_{40}, \tag{18j}$$

$$\alpha_{02}\alpha_{04} = a_{04}, \tag{18k}$$

$$\alpha_{40}^2 = 2a_{60}, \tag{18l}$$

$$\alpha_{04}^2 = 2a_{06}. \tag{18m}$$

In order to obtain the solutions of various wave function parameters α_{ij} in terms of the potential parameters a_{ij}, we make some plausible choices among the wave function parameters α_{ij}, i.e.

$$\alpha_{20} = \alpha_{02}, \quad \alpha_{22} = -3\alpha_{40} = -3\alpha_{04}. \tag{19}$$

Thus from eqs (18b)–(18d) we obtain

$$\alpha_{20} = \alpha_{02} = -\sqrt{a_{20} + \sqrt{a_{20}^2 - (a_{11}/2)^2}}, \tag{20a}$$

$$\alpha_{11} = -\sqrt{a_{20} - \sqrt{a_{20}^2 - (a_{11}/2)^2}}, \tag{20b}$$

and eq. (18l) (or 18m) provides

$$\alpha_{40} = \alpha_{04} = -\sqrt{2a_{60}}, \quad \alpha_{22} = \sqrt{18a_{60}}. \tag{21}$$

It is to be noted that the restrictions, eq. (19), render the potential parameters $a_{02} = a_{20},\ a_{13} = a_{31},\ a_{24} = a_{42},\ a_{04} = a_{40}$ and $a_{06} = a_{60}$.

The remaining equations give four constraining relations among potential parameters as

$$\sqrt{2a_{60}(a_{20} + \sqrt{a_{20}^2 - (a_{11}/2)^2})} - a_{40} = 0, \tag{22a}$$

$$\sqrt{2a_{60}(a_{20} + \sqrt{a_{20}^2 - (a_{11}/2)^2})} + a_{13} = 0, \tag{22b}$$

$$\sqrt{72a_{60}(a_{20} + \sqrt{a_{20}^2 - (a_{11}/2)^2})} + a_{22} = 0, \tag{22c}$$

$$3a_{60} - a_{24} = 0. \tag{22d}$$

Although the presence of constraining relations, eqs (22a)–(22d), make the problem quasi-solvable, such relations can be helpful in defining an appropriate sub-domain in complex parametric space in which a given complex potential will provide real spectra.

Finally, the eigenvalue for the ground state is given by

$$E_{\rm r}^0 = \sqrt{a_{20} + \sqrt{a_{20}^2 - (a_{11}/2)^2}}, \tag{23}$$

and the eigenfunction is given by

$$\begin{aligned}\psi^0(x,y) = \exp\Bigg[&-\frac{1}{2}\sqrt{a_{20} + \sqrt{a_{20}^2 - (a_{11}/2)^2}}(x^2+y^2) \\ &-\sqrt{a_{20} - \sqrt{a_{20}^2 - (a_{11}/2)^2}}xy \\ &-\frac{1}{4}\sqrt{2a_{60}}(x^4 + y^4 - 6x^2y^2)\Bigg].\end{aligned} \tag{24}$$

It is clear from eq. (23) that $E_{\rm r}^0$ is real if $a_{20} > 0$ and $a_{20} \geq a_{11}/2$. Otherwise, it is complex.

Note that the ground state solution of a real two-dimensional harmonic oscillator can easily be obtained from the general results, eqs (23) and (24), by taking potential parameters $a_{20} = a_{02}$ and remaining parameters as zero in eq. (14), as

$$E_{\rm r} = \sqrt{2a_{20}}, \quad \psi(x,y) = \exp\left[-\sqrt{\frac{a_{20}}{2}}\ (x^2+y^2)\right]. \tag{25}$$

We can also obtain the ground state eigenvalue of the real form of the potential (14) from eqs (23) and (24) by setting the variables x_3, x_4, p_3, p_4 zero.

3.2 *First excited state solution*

For obtaining the first excited state solution for the sextic potential (14), take the function $\phi(x_1, x_2, p_3, p_4)$ as

$$\phi(x_1, x_2, p_3, p_4) = \alpha x + \beta y + \gamma, \tag{26}$$

or equivalently, using eq. (1), we have

$$\phi_{\rm r}(x_1, p_3, x_2, p_4) = \alpha x_1 + \beta x_2 + \gamma, \quad \phi_{\rm i}(x_1, p_3, x_2, p_4) = \alpha p_3 + \beta p_4. \tag{27}$$

Here α, β and γ are considered as real constants. The functions $g_{\rm r}$ and $g_{\rm i}$ are the same as considered in ground state solutions.

On substituting eqs (15a)–(17) and (27) in eqs (12a) and (12b), we obtain again a set of 14 equations. The solutions of these equations can be obtained by assuming $\alpha_{20} = \alpha_{02}$, $\alpha_{22} = -5\alpha_{40} = -5\alpha_{04}$, $\alpha = -\beta$ and $\gamma = 0$. The solutions are written as

$$\alpha_{20} = \alpha_{02} = -\sqrt{a_{20} + \sqrt{a_{20}^2 - (a_{11} + 6\sqrt{2a_{60}})^2/4}}, \tag{28a}$$

$$\alpha_{11} = -\sqrt{a_{20} - \sqrt{a_{20}^2 - (a_{11} + 6\sqrt{2a_{60}})^2/4}}, \tag{28b}$$

$$\alpha_{40} = \alpha_{04} = -\sqrt{2a_{60}}, \quad \alpha_{22} = \sqrt{50a_{60}}. \tag{28c}$$

Finally, the eigenvalue and eigenfunction for the first excited state are given as

$$\begin{aligned} E_{\rm r}^1 = {} & 2\sqrt{a_{20} + \sqrt{a_{20}^2 - (a_{11} + 6\sqrt{2a_{60}})^2/4}} \\ & -\sqrt{a_{20} - \sqrt{a_{20}^2 - (a_{11} + 6\sqrt{2a_{60}})^2/4}}, \end{aligned} \tag{29}$$

$$\begin{aligned} \psi^1(x, y) = {} & \alpha(x - y) \\ & \times \exp\Bigg[-\frac{1}{2}\sqrt{a_{20} + \sqrt{a_{20}^2 - (a_{11} + 6\sqrt{2a_{60}})^2/4}}\ (x^2 + y^2) \\ & -\sqrt{a_{20} - \sqrt{a_{20}^2 - (a_{11} + 6\sqrt{2a_{60}})^2/4}}\ \ xy \\ & -\frac{1}{4}\sqrt{2a_{60}}(x^4 + y^4 - 10x^2y^2)\Bigg]. \end{aligned} \tag{30}$$

The eigenvalue is again real and discrete for $a_{20} > 0$ and $2a_{20} \geq (a_{11} + 6\sqrt{2a_{60}})$.

3.3 *Sextic potential with inverse/cross terms*

Here we study some other forms of $\mathcal{PT}$-symmetric sextic potential with inverse harmonic (centrifugal barrier) terms and some other cross terms. Some implications of such inverse/cross terms in a potential are discussed in [22] in reference to the solution of the Schrödinger equation for a real coupled quartic potential in two dimensions.

Case 1. In the first case, consider a $\mathcal{PT}$-symmetric sextic potential with inverse harmonic terms as

$$V(x,y) = V_{13} + \frac{A}{x^2} + \frac{B}{y^2}, \tag{31}$$

where the term V_{13} is the sextic potential given in eq. (14) and A and B are real constants.

Note that such one-dimensional sextic potentials are studied by many authors [9,19–21]. Levai and Arias [20] showed that a sextic potential with inverse square term has the properties of the Bohr Hamiltonian which describes collective motion in nuclei in terms of shape variables.

The real and complex components of the potential (31), using the transformations (1), are written as

$$V_{\rm r} = V_1 + \frac{A(x_1^2 - p_3^2)}{(x_1^2+p_3^2)^2} + \frac{B(x_2^2 - p_4^2)}{(x_2^2+p_4^2)^2}, \tag{32a}$$

$$V_{\rm i} = V_2 + \frac{2Ax_1p_3}{(x_1^2+p_3^2)^2} + \frac{2Bx_2p_4}{(x_2^2+p_4^2)^2}, \tag{32b}$$

where the forms of V_1 and V_2 are the same as given in eqs (15a) and (15b) respectively.

The forms of $g_{\rm r}$ and $g_{\rm i}$ for the present case are considered as

$$g_{\rm r} = g_1 - \frac{1}{2}\gamma_1 \log(x_1^2+p_3^2) - \frac{1}{2}\gamma_2 \log(x_2^2+p_4^2), \tag{33a}$$

$$g_{\rm i} = g_2 + \frac{1}{2}\gamma_1 \tan^{-1}\left(\frac{x_1}{p_3}\right) + \frac{1}{2}\gamma_2 \tan^{-1}\left(\frac{x_2}{p_4}\right), \tag{33b}$$

where the functions g_1 and g_2 are the same as given in eqs (16) and (17) respectively.

After substituting eqs (32a) and (33b) in eqs (13a) and (13b), we get the following equations:

$$E_{\rm r} = -\frac{1}{2}(\alpha_{02}+\alpha_{20}) + \gamma_2\alpha_{02} + \gamma_1\alpha_{20}, \quad E_{\rm i} = 0, \tag{34a}$$

$$\alpha_{22}(1-2\gamma_2) + \alpha_{40}(3-2\gamma_1) + \alpha_{11}^2 + \alpha_{20}^2 = 2a_{20}, \tag{34b}$$

$$\alpha_{22}(1-2\gamma_1) + \alpha_{04}(3-2\gamma_2) + \alpha_{11}^2 + \alpha_{02}^2 = 2a_{02}, \tag{34c}$$

$$\alpha_{20}\alpha_{11} + \alpha_{02}\alpha_{11} = a_{11}, \tag{34d}$$

$$\alpha_{20}\alpha_{22} + \alpha_{02}\alpha_{22} = a_{22}, \tag{34e}$$

$$\alpha_{11}\alpha_{22} + \alpha_{11}\alpha_{04} = 2a_{13}, \tag{34f}$$

$$\alpha_{11}\alpha_{22} + \alpha_{11}\alpha_{40} = 2a_{31}, \tag{34g}$$
$$\alpha_{22}^2 + 2\alpha_{22}\alpha_{04} = 2a_{24}, \tag{34h}$$
$$\alpha_{22}^2 + 2\alpha_{22}\alpha_{40} = 2a_{42}, \tag{34i}$$
$$\alpha_{20}\alpha_{40} = a_{40}, \tag{34j}$$
$$\alpha_{02}\alpha_{04} = a_{04}, \tag{34k}$$
$$\alpha_{40}^2 = 2a_{60}, \tag{34l}$$
$$\alpha_{04}^2 = 2a_{06}, \tag{34m}$$
$$\gamma_1^2 + \gamma_1 = 2A, \tag{34n}$$
$$\gamma_2^2 + \gamma_2 = 2B, \tag{34o}$$
$$\alpha_{11}\gamma_1 = 0, \tag{34p}$$
$$\alpha_{11}\gamma_2 = 0. \tag{34q}$$

For obtaining the solutions of the above equations, we again make the following choices among the wave function parameters α_{ij}:

$$\alpha_{20} = \alpha_{02}, \quad \alpha_{22} = -3\alpha_{40} = -3\alpha_{04}, \quad \gamma_1 = \gamma_2. \tag{35}$$

These choices lead to $a_{02} = a_{20},\ a_{24} = a_{42},\ a_{04} = a_{40}$ and $a_{06} = a_{60}$, $A = B$ and $a_{13} = a_{31} = a_{11} = 0$.

The solutions of various wave function parameters are given as

$$\alpha_{20} = \alpha_{02} = -\sqrt{2a_{20} - 2\sqrt{8a_{60}}(1+\sqrt{1+8A})}, \tag{36a}$$
$$\alpha_{40} = \alpha_{04} = -\sqrt{2a_{60}}, \quad \alpha_{22} = \sqrt{18a_{60}}, \quad \alpha_{11} = 0, \tag{36b}$$
$$\gamma_1 = \gamma_2 = -\frac{1}{2}(1+\sqrt{1+8A}). \tag{36c}$$

From the above solutions, the energy eigenvalue and the corresponding eigenfunction are written as

$$E_{\mathrm{r}}^0 = \sqrt{2a_{20} - \sqrt{8a_{60}}(1+\sqrt{1+8A})}(2+\sqrt{1+8A}), \tag{37}$$

$$\begin{aligned}\psi^0(x,y) = {} & \sqrt{xy}(xy)^{-\sqrt{1+8A}/2} \\ & \times \exp\Bigg[-\frac{1}{2}\sqrt{2a_{20} - \sqrt{8a_{60}}(1+\sqrt{1+8A})}(x^2+y^2) \\ & -\frac{1}{4}\sqrt{2a_{60}}(x^4+y^4-6x^2y^2) + \frac{i}{2}(-1+\sqrt{1+8A}) \\ & \times \left(\tan^{-1}\left(\frac{x_1}{p_3}\right) + \tan^{-1}\left(\frac{x_2}{p_4}\right)\right)\Bigg].\end{aligned} \tag{38}$$

Again E_{r}^0 is real for positive values of A, a_{20} and a_{60}, and $a_{20} \geq \sqrt{8a_{60}}(1+\sqrt{1+8A})$.

Case 2. In this case, we consider again the $\mathcal{PT}$-symmetric sextic potential with inverse harmonic and some other cross terms as

$$V(x,y) = V_{13} + \frac{A_1}{x^2} + \frac{B_1}{y^2} + A_2\frac{x}{y} + B_2\frac{y}{x}, \tag{39}$$

where the parameters A_1, A_2, B_1 and B_2 are real.

The ground state eigenvalue spectrum of this case can straightforwardly be obtained by following the same route as in the previous cases under the same ansatz (33a) and (33b) for g_{r} and g_{i} and within the same parametric restrictions.

The energy eigenvalue and the corresponding eigenfunction are written as

$$E_{\mathrm{r}}^0 = \sqrt{a_{20} + \sqrt{2a_{60}}(\sqrt{1+8A_1} - 1) + \sqrt{a_{20} + \sqrt{2a_{60}}(\sqrt{1+8A_1} - 1)^2 - a_{11}/4}} \\ \times(\sqrt{1+8A_1} - 2), \tag{40}$$

$$\begin{aligned} \psi^0(x,y) = {} & \sqrt{xy}(xy)^{-\sqrt{1+8A_1}/2} \\ & \times \exp\left[\alpha_{20}(x^2+y^2) + \alpha_{11}xy - \frac{1}{4}\sqrt{2a_{60}}(x^4+y^4 \right. \\ & -6x^2y^2) + \frac{i}{2}(-1+\sqrt{1+8A}) \\ & \left. \times\left(\tan^{-1}\left(\frac{x_1}{p_3}\right) + \tan^{-1}\left(\frac{x_2}{p_4}\right)\right)\right], \end{aligned} \tag{41}$$

where α_{20} and α_{11} are given as

$$\alpha_{20} = \sqrt{a_{20} + \sqrt{2a_{60}}(\sqrt{1+8A_1} - 1) + \sqrt{a_{20} + \sqrt{2a_{60}}(\sqrt{1+8A_1} - 1)^2 - a_{11}/4}},$$

$$\alpha_{11} = \sqrt{a_{20} + \sqrt{2a_{60}}(\sqrt{1+8A_1} - 1) - \sqrt{a_{20} + \sqrt{2a_{60}}(\sqrt{1+8A_1} - 1)^2 - a_{11}/4}}.$$

Again, within some parametric domain the energy eigenvalue will be real and positive.

4. Conclusion

With a view to explore more nontrivial applications of the ECPS method, in the present work, we investigated the quasi-exact solutions of the ASE under a suitable ansatz for the eigenfunction. The ground and excited state energies and the corresponding eigenfunctions are found for a two-dimensional $\mathcal{PT}$-symmetric complex sextic potential. We also found the ground state solutions of the ASE of some variants of the sextic potential. In all the systems considered here, potential coupling parameters are taken as real and the complexities are generated through transformations (1).

In the ECPS method, along with coordinate complexities, parametric complexification can be introduced. These complex potential parameters in turn yield complex eigenvalues. The imaginary parts $E_{\rm i}$ of eigenvalues E can be made zero by considering suitable choices of the potential parameters, which will then give real eigenvalues within some parametric domain. This is an interesting feature of the method as it provides us additional flexibility for obtaining real eigenvalue spectra of non-Hermitian Hamiltonian systems.

Although in the present study, we have not explicitly computed the normalization constants for wave functions, these can be obtained by generalizing the condition of [9] for two-dimensional systems, i.e.

$$N^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi^2(x_1, p_3, x_2, p_4)\,\mathrm{d}x_1\,\mathrm{d}p_3\,\mathrm{d}x_2\,\mathrm{d}p_4 = 1.$$

It is also mentioned that from the general expressions of the eigenvalues and the eigenfunctions found in the present work, the eigenvalues and the eigenfunctions of analogous real systems can directly be obtained by setting x_3, x_4, p_3, p_4 as zero.

The ECPS approach in two dimensions can be utilized to study more nontrivial two-dimensional potentials. However, for more involved complex systems, particularly in higher dimensions, studies may become a bit tedious due to the expansion of the algebra and difficulty in choosing appropriate forms of ϕ, $g_{\rm r}$ and $g_{\rm i}$ of the eigenfunction.

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