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The solution of the Schrödinger equation for coupled quadratic and quartic potentials in two dimensions

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Abstract. We deal with the difficulties claimed by the author of [*Ann. Phys.* **206**, 90 (1991)] while solving the Schrödinger equation for the ground states of two-dimensional anharmonic potentials. It is shown that the ground state energy eigenvalues and eigenfunctions for the coupled quadratic and quartic potentials can be obtained by making some simple assumptions. Expressions for the energy eigenvalues and the eigenfunctions for the first and second excited states of these systems are also obtained.

Keywords. Schrödinger equation; ground state; excited states; eigenvalues; eigenfunctions.

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1. Introduction

The study of exactly solvable real potentials has generated a lot of interest since the early development of quantum mechanics. The exact analytic expressions for the eigenvalues, the eigenfunctions and the scattering matrices are always desirable to explore the detailed properties of dynamical systems. The exact solutions of the Schrödinger wave equation (SE) are possible only for few choices of potentials. Therefore, approximation methods are often used to obtain solutions. The solution of the SE becomes even more difficult for time-dependent potentials in higher dimensions along with higher-order anharmonicities [1,2].

There exist several methods for solving the SE for dynamical systems. However, a technique known as the eigenfunction ansatz method has been explored for solving the SE for ground and excited state energies for a variety of potentials [3–9]. The same method has also been successfully used to obtain the eigenvalue spectra of a number of non-Hermitian complex (including $\mathcal{PT}$-symmetric forms) potentials [10–12]. In [6,7], it has been pointed out that the eigenfunction ansatz method does not provide any close form solutions for the two-dimensional coupled quadratic and quartic potentials. Further, it is also claimed that these systems are only solvable with some inverse/cross terms.

In this paper, we address the issues of [6,7] and show that, with some simple assumptions, the SE for two-dimensional coupled quadratic and quartic potentials is solvable without any inverse/cross terms not only for the ground state but also for the higher excited states.

The study of such potentials may be of interest in several physical applications, particularly in the fields of fibre optics, quantum chemistry etc. [6].

The organization of the paper is as follows: In §2, a brief description of the eigenfunction ansatz method is presented. In §3, we look into the difficulties of [6,7] and obtain the expressions for the ground state energy eigenvalues and eigenfunctions for the two-dimensional coupled quadratic and quartic potentials. The first and second excited state solutions for the same systems are obtained in §§4 and 5 respectively. Finally, the concluding remarks are presented in §6.

2. The method

Here we describe the essential steps of the eigenfunction ansatz method to find the solutions of the SE for two-dimensional systems. The SE (for $\hbar = m = 1$) is written as

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + 2(E - V(x,y))\psi(x,y) = 0. \tag{1}$$

Let us make the following ansatz for the eigenfunction:

$$\psi(x,y) = \phi(x,y)\exp[g(x,y)]. \tag{2}$$

From eqs (1) and (2), we obtain

$$\begin{aligned} &g_{xx} + g_{yy} + (g_x)^2 + (g_y)^2 + 2(E - V) \\ &\quad + \frac{1}{\phi}(2\phi_x g_x + 2\phi_y g_y + \phi_{xx} + \phi_{yy}) = 0, \end{aligned} \tag{3}$$

where the subscripts to g and ϕ indicate the differentiation with respect to the variables x and y.

From the structure of the above equation it is clear that if the functional forms of g and ϕ are known for a given system then rationalization of eq. (3) would directly provide the energy eigenvalue and the corresponding eigenfunction will be provided by eq. (2). For the ground state solutions the function $\phi(x,y)$ is taken as a constant.

Therefore, for the ground state solutions, eq. (3) reduces to

$$g_{xx} + g_{yy} + (g_x)^2 + (g_y)^2 + 2(E - V) = 0. \tag{4}$$

In what follows we find the ground state solutions for the two-dimensional coupled quadratic and quartic potentials.

3. The ground state solutions

Case 1. Consider the following two-dimensional coupled quadratic potential

$$V(x,y) = a_{20}x^2 + a_{02}y^2 + a_{11}xy, \tag{5}$$

where the coupling parameters a_{20}, a_{02} and a_{11} are assumed to be constants. For system (5), the ansatz for function $g(x,y)$ can be taken as

$$g(x,y) = \alpha_{20}x^2 + \alpha_{02}y^2 + \alpha_{11}xy, \tag{6}$$

which is the same as in [6]. Combination of eqs (4)–(6) and rationalization of the resulting expression lead to the following set of equations:

$$E = -(\alpha_{02} + \alpha_{20}), \tag{7a}$$

$$4\alpha_{20}^2 + \alpha_{11}^2 = 2a_{20}, \tag{7b}$$

$$4\alpha_{02}^2 + \alpha_{11}^2 = 2a_{02}, \tag{7c}$$

$$2(\alpha_{20} + \alpha_{02})\alpha_{11} = a_{11}. \tag{7d}$$

The solutions of eqs (7b)–(7d) would directly provide the energy eigenvalue and the eigenfunction of the system. The solutions of these equations are not possible as mentioned in [6,7]. However, with some plausible choices of the parameters α_{02}, α_{20} and α_{11}, it is possible to solve the above equations.

Therefore, to obtain the solutions of eqs (7b)–(7d), one can make a number of choices among the parameters α_{02}, α_{20} and α_{11} and can obtain expressions for E and ψ for a given potential. However, here we choose $\alpha_{02} = \alpha_{20}$ because of the fact that under this particular choice one can easily reproduce the well-known relations of eigenvalue and eigenfunction for a two-dimensional uncoupled (for $a_{11} = 0$) simple harmonic oscillator from the general relations of the coupled system.

Thus for this particular choice, the solutions of α_{02}, α_{20} and α_{11} in terms of potential parameters are written as

$$\alpha_{20} = \alpha_{02} = -\frac{1}{2}\sqrt{a_{20} + \sqrt{a_{20}^2 - a_{11}^2/4}}, \tag{8a}$$

$$\alpha_{11} = -\sqrt{a_{20} - \sqrt{a_{20}^2 - a_{11}^2/4}}. \tag{8b}$$

Here negative signs in the above solutions are retained to keep energy eigenvalue positive.

The choice $\alpha_{20} = \alpha_{02}$ gives a constraint on the potential coupling parameters, i.e., $a_{20} = a_{02}$.

Now the energy eigenvalue is obtained from eq. (7a) as

$$E = \sqrt{a_{20} + \sqrt{a_{20}^2 - a_{11}^2/4}}, \tag{9}$$

and the corresponding eigenfunction is given as

$$\psi(x,y) = N \exp\left[-\frac{1}{2}\sqrt{a_{20} + \sqrt{a_{20}^2 - a_{11}^2/4}}\,(x^2+y^2) - \sqrt{a_{20} - \sqrt{a_{20}^2 - a_{11}^2/4}}\,xy\right]. \tag{10}$$

The normalization constant N is determined from the condition $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\psi(x,y)|^2 \mathrm{d}x\,\mathrm{d}y = 1$. For the present case, it is given by $N = \sqrt{\frac{\sqrt{A^2-B^2}}{\pi}}$, where $A = \sqrt{a_{20} + \sqrt{a_{20}^2 - a_{11}^2/4}}$ and $B = \sqrt{a_{20} - \sqrt{a_{20}^2 - a_{11}^2/4}}$. Note that the inequality $a_{11} < 2a_{20}$ must hold for a real E and real and nonzero N.

Case 2. Next, we consider a two-dimensional coupled quartic potential given by

$$\begin{aligned} V(x,y) = {} & a_{10}x + a_{01}y + a_{20}x^2 + a_{02}y^2 + a_{11}xy + a_{30}x^3 \\ & + a_{03}y^3 + a_{12}xy^2 + a_{21}x^2y + a_{22}x^2y^2 + a_{31}x^3y \\ & + a_{13}xy^3 + a_{40}x^4 + a_{04}y^4, \end{aligned} \tag{11}$$

where the parameters a_{ij} are constants.

For this system the ansatz for the function $g(x,y)$ is considered as

$$\begin{aligned} g(x,y) = {} & \alpha_{20}x^2 + \alpha_{02}y^2 + \alpha_{11}xy + \alpha_{12}xy^2 + \alpha_{21}x^2y \\ & + \alpha_{30}x^3 + \alpha_{03}y^3. \end{aligned} \tag{12}$$

For the sake of simplicity, we have not considered linear terms like $\alpha_{10}x + \alpha_{01}y$ in $g(x,y)$ which makes a difference between the present form of $g(x,y)$ and that given in ref. [6]. Inserting eqs (11) and (12) in eq. (4) and then rationalizing, we get the following set of equations:

$$E = -(\alpha_{02} + \alpha_{20}), \tag{13a}$$

$$3\alpha_{30} + \alpha_{12} = a_{10}, \tag{13b}$$

$$3\alpha_{03} + \alpha_{21} = a_{01}, \tag{13c}$$

$$4\alpha_{20}^2 + \alpha_{11}^2 = 2a_{20}, \tag{13d}$$

$$4\alpha_{02}^2 + \alpha_{11}^2 = 2a_{02}, \tag{13e}$$

$$2(\alpha_{20}\alpha_{11} + \alpha_{02}\alpha_{11}) = a_{11}, \tag{13f}$$

$$4\alpha_{20}\alpha_{21} + 3\alpha_{11}\alpha_{30} + 2\alpha_{02}\alpha_{21} + 2\alpha_{11}\alpha_{12} = a_{21}, \tag{13g}$$

$$4\alpha_{02}\alpha_{12} + 3\alpha_{11}\alpha_{03} + 2\alpha_{20}\alpha_{12} + 2\alpha_{11}\alpha_{21} = a_{12}, \tag{13h}$$

$$6\alpha_{20}\alpha_{30} + 2\alpha_{11}\alpha_{21} = a_{30}, \tag{13i}$$

$$6\alpha_{02}\alpha_{03} + 2\alpha_{11}\alpha_{12} = a_{03}, \tag{13j}$$

$$6\alpha_{21}\alpha_{30} + 2\alpha_{12}\alpha_{21} = a_{31}, \tag{13k}$$

$$6\alpha_{12}\alpha_{03} + 2\alpha_{12}\alpha_{21} = a_{13}, \tag{13l}$$

$$2\alpha_{21}^2 + 2\alpha_{12}^2 + 3\alpha_{12}\alpha_{30} + 3\alpha_{21}\alpha_{03} = a_{22}, \tag{13m}$$

$$9\alpha_{30}^2 + \alpha_{21}^2 = 2a_{40}, \tag{13n}$$

$$9\alpha_{03}^2 + \alpha_{12}^2 = 2a_{04}. \tag{13o}$$

Again, the solutions for seven unknown parameters α_{ij} are not easy. However, some simple assumptions for the parameters α_{ij} make this task possible.

First, we choose $\alpha_{20} = \alpha_{02}$ as earlier. For this choice, eqs (13d)–(13f) immediately lead to

$$\alpha_{20} = \alpha_{02} = -\frac{1}{2}\sqrt{a_{20} + \sqrt{a_{20}^2 - a_{11}^2/4}}, \tag{14a}$$

$$\alpha_{11} = -\sqrt{a_{20} - \sqrt{a_{20}^2 - a_{11}^2/4}}, \tag{14b}$$

and a constraining relation $a_{02} = a_{20}$.

Secondly, choosing $\alpha_{21} = 3\alpha_{30}$ and $\alpha_{12} = 3\alpha_{03}$, we obtain the following solutions of eqs (13n) and (13o):

$$\alpha_{30} = -\frac{1}{3}\sqrt{a_{40}}, \tag{15a}$$

$$\alpha_{21} = -\sqrt{a_{40}}, \tag{15b}$$

$$\alpha_{03} = -\frac{1}{3}\sqrt{a_{04}}, \tag{15c}$$

$$\alpha_{12} = -\sqrt{a_{04}}. \tag{15d}$$

Since only five equations are utilized for the solutions of α_{ij}, the remaining equations, (13b), (13c) and (13g)–(13m) will provide nine constraining relations among potential parameters a_{ij}.

Finally the energy eigenvalue is written from eq. (13a) as

$$E = \sqrt{a_{20} + \sqrt{a_{20}^2 - a_{11}^2/4}}, \tag{16}$$

and the eigenfunction is given by

$$\begin{aligned}\phi(x,y) = N\exp\Bigg[&-\frac{1}{2}\sqrt{a_{20} + \sqrt{a_{20}^2 - a_{11}^2/4}}(x^2+y^2) \\ &-\sqrt{a_{20} - \sqrt{a_{20}^2 - a_{11}^2/4}}xy \\ &-\frac{1}{3}\sqrt{a_{40}}(x^3+3x^2y) - \frac{1}{3}\sqrt{a_{04}}(y^3+3xy^2)\Bigg].\end{aligned} \tag{17}$$

4. The first excited state solution

In this section, we solve the SE to obtain the energy eigenvalues and eigenfunctions for the first excited state of the two-dimensional quadratic and quartic potentials.

Case 1. For the quadratic potential, eq. (5), we assume the same form of $g(x,y)$ as given in eq. (6). However the form of $\phi(x,y)$ is assumed as

$$\phi(x,y) = \alpha_1 x + \beta_1 y + \gamma_1, \tag{18}$$

where α_1, β_1 and γ_1 are considered as real constants.

Now using eqs (5), (6) and (18) in eq. (3) and choosing $\alpha_1 = -\beta_1$ and $\gamma_1 = 0$, we get a set of four equations, out of which three are same as eqs (7b)–(7d) and the fourth equation is given as

$$E^1 = -4\alpha_{20} + \alpha_{11}. \tag{19}$$

The solutions for the wave function parameters α_{20}, α_{02} and α_{11} are the same as obtained in eqs (8a) and (8b).

Hence the energy eigenvalue for the first excited state is computed from eq. (19) as

$$E^1 = 2\sqrt{a_{20} + \sqrt{a_{20}^2 - a_{11}^2/4}} - \sqrt{a_{20} - \sqrt{a_{20}^2 - a_{11}^2/4}}, \tag{20}$$

and the corresponding eigenfunction becomes

$$\psi^1(x,y) = N\alpha_1(x-y)\exp\left[-\frac{1}{2}\sqrt{a_{20} + \sqrt{a_{20}^2 - a_{11}^2/4}}(x^2+y^2) - \sqrt{a_{20} - \sqrt{a_{20}^2 - a_{11}^2/4}}xy\right]. \tag{21}$$

The normalization constant is given as $N = \sqrt{\frac{(A-B)\sqrt{A^2-B^2}}{\alpha_1^2\pi}}$.

Case 2. For the quartic potential, eq. (11), we again take the same ansatz for $\phi(x,y)$ and the same conditions on α_1, β_1 and γ_1 as taken in the previous case. Therefore, using eqs (11), (12) and (18) in eq. (3) and then rationalizing we get the following equation in addition to eqs (13b)–(13o):

$$E^1 = -4\alpha_{20} + \alpha_{11}. \tag{22}$$

Here, in order to get the solutions of these equations, we follow the same prescription as adopted in Case 2 of §3 and obtain the same solutions as given there. Hence, the energy eigenvalue for the first excited state is written as

$$E^1 = 2\sqrt{a_{20} + \sqrt{a_{20}^2 - a_{11}^2/4}} - \sqrt{a_{20} - \sqrt{a_{20}^2 - a_{11}^2/4}}, \tag{23}$$

and the corresponding eigenfunction is given as

$$\psi^1(x,y) = N\alpha_1(x-y)\exp\left[-\frac{1}{2}\sqrt{a_{20} + \sqrt{a_{20}^2 - a_{11}^2/4}}(x^2+y^2) - \sqrt{a_{20} - \sqrt{a_{20}^2 - a_{11}^2/4}}xy - \frac{1}{3}\sqrt{a_{40}}(x^3+3x^2y) - \frac{1}{3}\sqrt{a_{04}}(y^3+3xy^2)\right]. \tag{24}$$

5. The second excited state solutions

For obtaining the energy eigenvalues of the second excited state of the systems considered in this work, we assume the following form of $\phi(x, y)$:

$$\phi(x, y) = \alpha_2 x^2 + \beta_2 y^2 + \gamma_2, \tag{25}$$

where α_2, β_2 and γ_2 are considered as real constants.

Case 1. For the quadratic potential, using eqs (5), (6) and (25) in eq. (3) and rationalizing we get the following equation in addition to eqs (7b)–(7d) as

$$E^2 = -6\alpha_{20} + \alpha_{11}, \tag{26}$$

under the condition $\alpha_2 = -\beta_2$ and $\gamma_2 = 0$.

The solutions for α_{20}, α_{02} and α_{11} are the same as given in eqs (8a) and (8b). Hence from eq. (26), the eigenvalue is written as

$$E^2 = 3\sqrt{a_{20} + \sqrt{a_{20}^2 - a_{11}^2/4}} - \sqrt{a_{20} - \sqrt{a_{20}^2 - a_{11}^2/4}}, \tag{27}$$

and the eigenfunction is given by

$$\psi^2(x, y) = \alpha_2(x^2 - y^2) \exp\left[-\frac{1}{2}\sqrt{a_{20} + \sqrt{a_{20}^2 - a_{11}^2/4}}(x^2 + y^2) - \sqrt{a_{20} - \sqrt{a_{20}^2 - a_{11}^2/4}}xy\right]. \tag{28}$$

The normalization constant for this case is computed as $N = \sqrt{\frac{(A^2-B^2)\sqrt{A^2-B^2}}{\alpha_2^2 \pi}}$.

Case 2. For the quartic potential, we use eqs (11), (12) and (25) in eq. (3) and set $\alpha_2 = -\beta_2$ and $\gamma_2 = 0$, so that the following equations are obtained in addition to eqs (13d)–(13o) as

$$E^2 = -6\alpha_{20}, \tag{29a}$$

$$9\alpha_{30} + \alpha_{12} = a_{10}, \tag{29b}$$

$$9\alpha_{03} + \alpha_{21} = a_{01}. \tag{29c}$$

As far as solutions of various α_{ij} are concerned, these are the same as obtained in Case 2 of §3. However, a few constraining relations are different.

Hence the eigenvalue is written as

$$E^2 = 3\sqrt{a_{20} + \sqrt{a_{20}^2 - a_{11}^2/4}}, \tag{30}$$

and the eigenfunction is given as

$$\psi^2(x,y) = \alpha_2(x^2-y^2)\exp\left[-\frac{1}{2}\sqrt{a_{20}+\sqrt{a_{20}^2-a_{11}^2/4}}(x^2+y^2)\right.$$
$$-\sqrt{a_{20}-\sqrt{a_{20}^2-a_{11}^2/4}}xy$$
$$\left.-\frac{1}{3}\sqrt{a_{40}}(x^3+3x^2y)-\frac{1}{3}\sqrt{a_{04}}(y^3+3xy^2)\right]. \tag{31}$$

6. Conclusions

In the present work, we have overcome the difficulties posed in [6,7] pertaining to the solutions of the SE for the coupled quadratic and quartic potentials in two dimensions. Within the framework of the eigenfunction ansatz method, with some simple restrictions, we could obtain the ground state solutions for such systems and found explicit expressions for the ground state energies and the associated eigenfunctions without adding any cross/inverse terms. We further extended the scope of the eigenfunction ansatz method for higher excited states and found energy eigenvalues and eigenfunctions for the first and second excited states of the quadratic and quartic potentials. The solutions found in this study are quasi-exact with certain constraints on the potential parameters. The role of these constraints is very crucial not only in deciding the ground and higher excited states but also in obtaining the bound states for the system [7]. The number of constraints increases further when some restrictions on the choices of coefficients of the functions $g(x,y)$ and $\phi(x,y)$ are imposed in order to solve an overdetermined set of equations for various α_{ij}'s.

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