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All-order results for soft and collinear gluons

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Abstract. Some general features and some recent developments concerning the resummation of long-distance singularities in QCD and in more general non-Abelian gauge theories are reviewed. The field-theoretical tools of the trade are emphasized, with the focus mostly on the exponentiation of infra-red and collinear divergences in amplitudes, which underlies the resummation of large logarithms in the corresponding cross-sections. Some recent results concerning the conformal limit, notably the case of $\mathcal{N} = 4$ superymmetric Yang–Mills theory are also described.

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1. Introduction

Massless gauge field theories, which are classically conformal invariant, are characterized by the fact that all length scales enter in the calculation of any given physical process. Consider for example the perturbative calculation of a scattering amplitude $\mathcal{A}(p_1, \ldots, p_n)$ with momentum invariants characterized by a common scale Q. Just like in any quantum theory, when we compute $\mathcal{A}$ beyond the leading perturbative order, we must allow for exchanges of virtual quanta of arbitrarily high energy, $E \gg Q$, corresponding to processes happening at arbitrarily small length scales. These exchanges are responsible for the ultraviolet (UV) problems we encounter in perturbative calculations, and must be dealt with, when possible, with renormalization. If our theory is massless (or if the masses are negligible with respect to the scale Q), a similar problem arises at the other end of the spectrum: we must allow for exchanges of very low-energy quanta, $E \ll Q$, which happen at very large distances. Such processes may or may not, in a general field theory, endanger our calculational framework, but they certainly do so in the case of gauge theories, even Abelian ones. This is the origin of the infra-red (IR) problems of perturbative calculations, which are usually dealt with using factorization.

In either case, it is worth recalling that the appearance of infinities in our perturbative calculations is due to the fact that we have stretched our approximations beyond their limits of applicability. In the UV regime, we have tacitly assumed that our theory should be applicable at extremely short distances, where in fact we do not even know what the relevant degrees of freedom might be. In the case of renormalizable theories, our arrogance is forgiven, since we can show that physics at very short distances effectively decouples from the calculation of our amplitude at the scale Q: the contributions of high-energy quanta factorize, and can be absorbed into rescalings of the local couplings of our original theory. In the IR regime, for gauge theories, our mistake is more subtle: massless gauge bosons mediate long-range interactions, which cannot be switched off even at asymptotically large distances; hence, it is not correct to formulate our perturbative expansion in a Hilbert space of Fock states built with creation and annihilation operators of free fields. The true asymptotic states of the theory are coherent states containing an infinite number of massless quanta, and the price of stretching our approximation is that the S matrix actually does not exist in our Hilbert space: all scattering amplitudes diverge because of virtual IR exchanges.

In QED, and to a more limited extent in QCD, IR problems are alleviated by the KLN theorem [1,2]. When we compensate for our inadequate choice of Hilbert space by constructing physically measurable probabilities, which are obtained by summing over all Fock states that are degenerate in energy, all IR divergences must cancel. This is sufficient to solve the IR problem in QED, where actually a sum over final-state degeneracies is enough to achieve the required cancellation. In an unbroken non-Abelian gauge theory like QCD, things are considerably more complicated: because of confinement, the relationship between partonic Fock states and the true non-perturbative asymptotic states of the theory (which are colour-singlet hadrons) is highly non-trivial, and beyond the range of our techniques; this is reflected at the perturbative level in the fact that a sum over initial-state degeneracies is both necessary to cancel divergences, and impossible to perform in practice, since initial-state partons are far from free at large distances.

In order to rescue the applicability of perturbative methods, one must resort once again to factorization, coupled with asymptotic freedom. One exploits the quantum-mechanical incoherence of processes happening at different distance scales to show that high-energy inclusive cross-sections can be written as convolutions of short-distance finite partonic cross-sections, which can be computed in perturbation theory thanks to asymptotic freedom, with long-distance factors (parton distributions or fragmentation functions), which are non-perturbative but universally associated with hadronic wave functions.

Proofs of factorization are highly non-trivial in perturbation theory [3], but they pay big dividends. First of all, they underpin essentially all perturbative QCD predictions for high-energy cross-sections, from deep inelastic scattering, to Drell–Yan production of electroweak vector bosons and Higgs bosons, to general jet cross-sections. There are, furthermore, several other applications, important for both theory and phenomenology, some of which will be reviewed below.

- Factorization leads to evolution equations. All factorization theorems introduce intermediate arbitrary scales separating the momentum space regions one wishes to disentangle. Demanding the physical observables to be independent of these arbitrary scales leads to evolution equations for individual contributions to the factorized observable. A prime example is of course Altarelli–Parisi [4] evolution of parton distributions.

- Solving the evolution equations dictated by factorization leads to the resummation of classes of logarithmic contributions to all orders in perturbation theory. Renormalization group evolution and collinear evolution of parton distributions are examples, but also Sudakov resummation, both for threshold and transverse momentum logarithms, can be derived in this way (see [5] and, for a review, [6]). Such resummations are important, in some cases essential, for the phenomenological success of perturbative QCD. Alternatively, having established a factorization theorem, one can apply effective field theory methods to derive the same results in a different systematical way (see [7]).
- Resummation, in turn, probes the all-order structure of perturbation theory and helps identify leading non-perturbative corrections to many cross-sections. The study of power-suppressed corrections to QCD factorization theorems has in fact developed into an active subfield of research, with many phenomenological applications and new interesting theoretical ideas. The literature is vast, involving renormalon techniques as well as resummations; for reviews and further references on some of these ideas, see [8,9].
- At the amplitude level, resummation of IR singularities displays universal features of gauge theories which find application in both theory and phenomenology. Understanding the structure of infra-red and collinear poles at high orders is instrumental to construct subtraction algorithms to compute efficiently multiparticle cross-sections at colliders. On the other hand, remarkably, the all-order structure of infra-red poles, uncovered in QCD, has recently found application in the context of supersymmetric gauge theories, and most notably for $\mathcal{N} = 4$ super-Yang–Mills (SYM) theory, which is conjectured to be equivalent to string theory on the background of an $\mathrm{AdS}_5 \times S^5$ space-time. Gluon amplitudes can now be computed in weak-coupling $\mathcal{N} = 4$ SYM to very high order, and can also, in some cases, be computed at strong ('t Hooft) coupling using the AdS-CFT correspondence directly [10]. The universal structure of infra-red and collinear singularities of these amplitudes places powerful constraints on their structure, and helps identify the relevant anomalous dimensions. This is a very active and fast-developing field, recently reviewed in [11] and [12].

The following sections will describe some of the field theory tools that are used to study the universal structure of gauge theories at long distances. In §2, the connection between factorization, evolution and resummation will be discussed qualitatively; then, in §3, some technical tools that are needed to make precise all-order statements will be introduced. Section 4 will focus on the simplest example of non-Abelian exponentiation of infra-red singularities, the form factor of a massless particle in a gauge theory. Finally, §5 will describe some applications of these results to conformal gauge theories. Several new results recently obtained in [13] will also be included.

2. Factorization leads to resummation

A factorization theorem is a sufficient (though not necessary) condition to perform a resummation of perturbation theory, typically through the solution of an appropriate evolution equation. Clearly, the difficult work is proving factorization to all orders in perturbation theory, which requires a detailed diagrammatic analysis and often a delicate implementation of the symmetry properties of the theory. Once factorization is established, evolution equations automatically follow, and their solutions entail an all-order organization of certain perturbative contributions. Let us look at three classical examples, along the lines of ref. [5].

- The prototypical perturbative factorization is the renormalization of UV divergences. In this case the difficult step is to prove, to all orders, that the dependence of a generic Green function of the theory on the cut-off scale can be associated with a finite number of universal multiplicative factors. If that is the case, then one may write, for an n-point bare correlator $G_0^{(n)}$,

$$G_0^{(n)}\left(p_i, \Lambda, g_0\right) = \prod_{i=1}^{n} Z_i^{1/2}\left(\Lambda/\mu, g(\mu)\right) G_R^{(n)}\left(p_i, \mu, g(\mu)\right), \tag{1}$$

  where g_0 is the bare coupling and Λ the UV cut-off, which can be interpreted as the scale of 'new physics', or the scale at which the effective low-energy theory under consideration breaks down. In order to perform the factorization in eq. (1) it has been necessary to introduce an intermediate, arbitrary scale μ. Bare Green functions, however, do not depend on μ, so that we can write

$$\frac{\mathrm{d}G_0^{(n)}}{\mathrm{d}\mu} = 0 \quad \to \quad \frac{\mathrm{d}\log G_R^{(n)}}{\mathrm{d}\log\mu} = -\sum_{i=1}^{n} \gamma_i\left(g(\mu)\right), \tag{2}$$

  where $\gamma_i(g) \equiv \mathrm{d}\log Z_i/\mathrm{d}\log\mu^2$. Solving the renormalization group equation, eq. (2), resums the logarithmic dependence on the renormalization scale μ. Furthermore, since renormalized Green functions, up to their overall engineering dimension, depend on μ only through ratios such as $p_i \cdot p_j/\mu^2$, the same equation can be used to extract information about the dependence of $G_R^{(n)}$ on external momenta. Notice that, in order to enforce the cancellation in eq. (2), the anomalous dimensions $\gamma_i(g)$ can only depend on arguments that are common to Z_i and to $G_R^{(n)}$, the only one in this case being the renormalized coupling g.
- Another familiar example is collinear factorization in high-energy QCD cross-sections, most topically DIS. In this case the difficult task is to show that collinear singularities in the cross-section can be absorbed into universal factors associated with the wave functions of initial state hadrons. For DIS structure functions, say $F_2(x, Q^2)$, this is true in the form of a convolution, which becomes an ordinary product upon taking a Mellin transform. One writes then

$$\widetilde{F}_2\left(N, \frac{Q^2}{m^2}, \alpha_s\right) = \widetilde{C}\left(N, \frac{Q^2}{\mu_F^2}, \alpha_s\right) \widetilde{f}\left(N, \frac{\mu_F^2}{m^2}, \alpha_s\right). \tag{3}$$

  Here m is a label for a collinear regulator, say a light quark mass; $\widetilde{C}$ is a perturbatively computable coefficient function, free of collinear sensitivity, while $\widetilde{f}$ is a universal (but non-computable) parton distribution. Again, to perform factorization it has been necessary to introduce an arbitrary scale μ_F, and one can exploit the fact that the structure function $\widetilde{F}_2$ does not depend in the choice of μ_F. One derives

$$\frac{\mathrm{d}\widetilde{F}_2}{\mathrm{d}\mu_F} = 0 \quad \rightarrow \quad \frac{\mathrm{d}\log \widetilde{f}}{\mathrm{d}\log \mu_F} = \gamma_N\left(\alpha_s\right), \tag{4}$$

where $\gamma_N(\alpha_s) \equiv -\mathrm{d}\log \widetilde{C}/\mathrm{d}\log \mu_F$ are the Mellin moments of the appropriate Altarelli–Parisi splitting function. The anomalous dimension γ_N can only depend on arguments common to $\widetilde{C}$ and $\widetilde{f}$, in this case N and α_s. Solving eq. (4) resums logarithms of the factorization scale, and allows the evolution of parton distributions to the scales appropriate for applications to other high-energy cross-sections.

- Let us finally turn to the most significant (and difficult) case of Sudakov resummation. In the previous two cases one was dealing with the resummation of single logarithms, arising from single (UV or collinear) poles of the corresponding amplitudes. Sudakov resummation involves double (IR and collinear) poles, which requires in principle a more elaborate factorization (see, however, the arguments in [14]). Since our focus below will be on amplitudes rather than cross-sections, let us begin by considering the simplest scattering amplitude which is affected by such double poles, the form factor of any massless particle minimally coupled to a massless gauge boson. Using a massless quark as an example one can define

$$\begin{aligned}\Gamma_\mu(p_1, p_2; \mu^2, \epsilon) &\equiv \langle p_1, p_2 | J_\mu(0) | 0 \rangle \\ &= \bar{u}(p_1)\gamma_\mu v(p_2) \Gamma\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right),\end{aligned} \tag{5}$$

corresponding to pair creation of a $q\bar{q}$ pair out of the QCD vacuum by means of a source (an off-shell photon) of mass Q. It can be shown (as reviewed in [15]) that this amplitude factorizes into the product of different functions, each one responsible for a specific set of singularities. In dimensional regularization, the precise form of this factorization can be written as [13]

$$\begin{aligned}\Gamma\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) &= C\left(\frac{Q^2}{\mu^2}, \frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \epsilon\right) \\ &\quad \times \mathcal{S}\left(\beta_1 \cdot \beta_2, \alpha_s(\mu^2), \epsilon\right) \\ &\quad \times \prod_{i=1}^{2}\left[\frac{J\left(\frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \epsilon\right)}{\mathcal{J}\left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon\right)}\right].\end{aligned} \tag{6}$$

Here β_i are four velocities associated with the quark and the antiquark (so that $p_i^\nu \propto Q\beta_i^\nu$), while n_i are auxiliary vectors associated with Wilson lines (to be described below), which are introduced in order to factorize wide-angle soft radiation from the collinear one. The function $\mathcal{S}$ is an eikonal function responsible for the radiation of soft gluons, while the jet function J is associated with radiation collinear to either the quark or the antiquark, and C is finite as $\epsilon \to 0$. Including both $\mathcal{S}$ and J double counts the soft-collinear region, which is compensated for by introducing eikonal versions of the jets, $\mathcal{J}$. In order to derive evolution equations, one can now exploit the renormalization group invariance of Γ, which must not depend on the scale μ, as well as the manifest independence on the auxiliary vectors n_i. Depending on how the calculation is performed, the independence of Γ on n_i can be understood either as gauge invariance (if working in an axial gauge), or as Lorentz invariance (if working in Feynman gauge, as we will do below).

In order to move forward and be more precise we need at this point to step back and introduce some technical tools. Specifically, in order to give precise operator expressions for the functions entering eq. (6), we will need to introduce Wilson line operators, which are also instrumental in the mapping to strong coupling which becomes possible for $\mathcal{N} = 4$ SYM. Furthermore, in order to perform a consistent resummation in dimensional regularization, we need to define the running coupling in $d = 4 - 2\epsilon$ dimensions. We briefly turn to these issues in the next section.

3. Tools of the trade

3.1 *Dimensional regularization for the strong coupling*

Dimensional regularization (DR), in its various flavours, is a unique tool for the study of non-Abelian gauge theories. It can be used both for UV and IR divergences, it preserves gauge invariance, it is by far the simplest scheme to use from the computational point of view. In the context of all-order calculations, it has further virtues. In this case, one starts with the renormalized theory, and regulates long-distance singularities by taking $d = 4 - 2\epsilon$, with $\epsilon < 0$. One must then recall that RG equations acquire ϵ dependence in $d \neq 4$. The coupling, for example, runs according to

$$\mu \frac{\partial \bar{\alpha}}{\partial \mu} \equiv \beta(\epsilon, \bar{\alpha}) = -2\epsilon \bar{\alpha} + \hat{\beta}(\bar{\alpha}), \quad \hat{\beta}(\bar{\alpha}) = -\frac{\bar{\alpha}^2}{2\pi} \sum_{n=0}^{\infty} b_n \left(\frac{\bar{\alpha}}{\pi} \right)^n . \tag{7}$$

The ϵ dependence of the β function is a consequence of the engineering dimension of the bare coupling, and it implies that the running coupling behaves like a power of its scale, $\alpha_s(\mu^2)/\alpha_s(\mu_0^2) \sim (\mu^2/\mu_0^2)^{-\epsilon}$: in fact, in $d > 4$, the β function has an IR-free fixed point at $\alpha_s = 0$, where it vanishes with a positive derivative. As a consequence, $\alpha_s(\mu^2 = 0) = 0$. The RG equation, eq. (7), is easily solved at one loop, yielding

$$\bar{\alpha}\left(\mu^2, \epsilon\right) = \alpha_s(\mu_0^2) \left[\left(\frac{\mu^2}{\mu_0^2} \right)^{\epsilon} - \frac{1}{\epsilon} \left(1 - \left(\frac{\mu^2}{\mu_0^2} \right)^{\epsilon} \right) \frac{b_0}{4\pi} \alpha_s(\mu_0^2) \right]^{-1} . \tag{8}$$

Note that $\bar{\alpha}$ depends only on the scale μ^2 and on ϵ, but not on the chosen initial condition μ_0^2. At higher orders an explicit analytic solution such as eq. (8) is not available, but one may still expand $\bar{\alpha}$ in powers of the coupling at a fixed reference scale, as

$$\bar{\alpha}(\xi^2, \epsilon) = \alpha_s \xi^{-2\epsilon} + \alpha_s^2 \xi^{-4\epsilon} \frac{b_0}{4\pi\epsilon}(1 - \xi^{2\epsilon}) + \alpha_s^3 \xi^{-6\epsilon} \frac{1}{8\pi^2 \epsilon} \left[\frac{b_0^2}{2\epsilon} \left(1 - \xi^{2\epsilon}\right)^2 + b_1 \left(1 - \xi^{4\epsilon}\right) \right] + \mathcal{O}\left(\alpha_s^4\right). \qquad (9)$$

The key advantage of eq. (7) is that it provides a simple initial condition for the solution of evolution equations for amplitudes, basically stating that all radiative corrections vanish when the scale vanishes. This fact was first exploited to give an explicit exponentiated expression for the Sudakov form factor in [16]. A further advantage of eq. (7) is that the Landau pole for the running coupling acquires a non-vanishing imaginary part when $\epsilon < -b_0 \alpha_s(\mu_0^2)/(4\pi)$, a fact that can be exploited to evaluate resummed amplitudes explicitly as analytic functions of the coupling and of ϵ [17].

3.2 *Wilson lines and the eikonal approximation*

An important feature of eq. (6) is the fact that all singular factors comprising the form factor have well-defined operator expressions. This is especially significant when one is trying to make a connection to non-perturbative features of the theory, as is the case for $\mathcal{N} = 4$ SYM. It is well-known, and easily verified, that in the soft approximation, relevant for the calculation of infra-red poles, gluon interactions with other hard partons can be completely expressed in terms of correlators of Wilson lines: energetic partons do not recoil against soft radiation, so that the only effect of interactions with soft gluons is the buildup of an eikonal phase on the parton field; soft gluons, in turn, are only sensitive to the direction and colour representation of the hard parton, but not to its spin and energy. The situation with collinear gluons is not as simple: it can be shown that they couple eikonally to hard partons moving in different light-cone directions, but they retain to some extent the spin and energy dependence of the coupling to partons belonging to their own jet. In either case, eikonal lines play a major role in factorization formulas such as eq. (6). Defining the Wilson line operator as

$$\Phi_n(\lambda_2, \lambda_1) = P \exp\left[\mathrm{i}g \int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda n \cdot A(\lambda n) \right], \qquad (10)$$

we can give explicit operator expressions for all the functions appearing in eq. (6). The soft function $\mathcal{S}$ is just the eikonal approximation of the full form factor

$$\mathcal{S}\left(\beta_1 \cdot \beta_2, \alpha_s(\mu^2), \epsilon\right) = \langle 0 | \Phi_{\beta_2}(\infty, 0)\, \Phi_{\beta_1}(0, -\infty)\, | 0 \rangle; \qquad (11)$$

the jet functions J, on the other hand, couple a hard parton to an eikonal line off the light cone, along an arbitrary space-like direction n^μ,

$$J\left(\frac{(p \cdot n)^2}{n^2 \mu^2}, \alpha_s(\mu^2), \epsilon \right) u(p) = \langle 0 | \Phi_n(\infty, 0) \psi(0) | p \rangle. \qquad (12)$$

Eikonal jets $\mathcal{J}$, finally, represent the soft approximation of the partonic jets J, so that the parton field ψ is replaced by its own Wilson line,

$$\mathcal{J}\left(\frac{(\beta\cdot n)^2}{n^2},\alpha_s(\mu^2),\epsilon\right)=\langle 0|\Phi_n(\infty,0)\,\Phi_\beta(0,-\infty)\,|0\rangle. \tag{13}$$

As we will briefly summarize below, while the evolution equations (2) and (4) were stemming from the invariance of the observable before factorization with respect to variations of a mass scale, in the case of eq. (6) one derives evolution by demanding invariance with respect to the choice of the 'factorization vectors' n_i^μ.

4. Resummation for the form factor

The derivation of the evolution equation for the form factor can be understood by considering n_i^μ dependence in eq. (6). Clearly, all such dependence is through the dimensionless ratio $x_i \equiv (\beta_i\cdot n_i)^2/n_i^2$. Demanding that $\partial\log\Gamma/\partial\log x_i = 0$, and noting that Γ depends on x_i only through the jet functions and the finite coefficient function C, one derives

$$\begin{aligned} x_i\frac{\partial}{\partial x_i}\log J_i &= -x_i\frac{\partial}{\partial x_i}\log C + x_i\frac{\partial}{\partial x_i}\log\mathcal{J}_i \\ &\equiv \frac{1}{2}[\mathcal{G}_i(x_i,\alpha_s(\mu^2),\epsilon)+\mathcal{K}(\alpha_s(\mu^2),\epsilon)], \end{aligned} \tag{14}$$

where the second line defines the functions $\mathcal{G}_i$ and $\mathcal{K}_i$. The key feature of eq. (14) is that the n_i^μ dependence of the partonic jets has been organized in a function $\mathcal{G}_i$, which carries all the kinematic dependence, but is finite as $\epsilon\to 0$ (because C is finite), plus a function $\mathcal{K}_i$, which on the contrary is a pure counterterm (because $\mathcal{J}$ is), but carries no kinematic dependence. Note also that, while $\mathcal{J}_i$ has a double pole, its derivative with respect to x_i must have only a single pole, since the double pole is independent of kinematics. Integrating eq. (14) thus leads to one of the key features of resummation: double pole observables exponentiate, and their logarithms contain only single poles.

It is not difficult to generalize the argument leading to eq. (14) to the full form factor [15]. One finds an equation of the same form

$$\begin{aligned} Q^2\frac{\partial}{\partial Q^2}\log\left[\Gamma\left(\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)\right] \\ =\frac{1}{2}\left[K\left(\epsilon,\alpha_s(\mu^2)\right)+G\left(\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)\right], \end{aligned} \tag{15}$$

where G is finite as $\epsilon\to 0$ and carries the full Q^2 dependence, while K is a Q^2-independent pure counterterm. In order to solve eq. (15), we still need three ingredients.

- Renormalization group invariance of the form factor requires

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(\epsilon,\alpha_s)\frac{\partial}{\partial\alpha_s}\right)G\left(\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)$$
$$=-\beta(\epsilon,\alpha_s)\frac{\partial}{\partial\alpha_s}K\left(\epsilon,\alpha_s(\mu^2)\right)\equiv\gamma_K\left(\alpha_s(\mu^2)\right), \tag{16}$$

where in the second equation we have used the fact that K is a pure counterterm, and thus has no explicit scale dependence. Equation (16) defines the anomalous dimension γ_K, and allows one to solve for the μ dependence of G in terms of an initial condition, say at $\mu = Q$.

- The infra-red freedom of the theory for $\epsilon < 0$ provides us with a simple initial condition for eq. (15),

$$\bar{\alpha}(\mu^2=0,\epsilon<0)=0\rightarrow\Gamma(0,\alpha_s(\mu^2),\epsilon)=\Gamma\left(1,\bar{\alpha}(0,\epsilon),\epsilon\right)=1\ . \tag{17}$$

- By the same token, the counterterm K can be expressed directly as an integral of the anomalous dimension γ_K, using eq. (16) and the vanishing of the coupling at $\mu^2 = 0$. One verifies that

$$K(\epsilon,\alpha_s(\mu^2))=-\frac{1}{2}\int_0^{\mu^2}\frac{\mathrm{d}\lambda^2}{\lambda^2}\gamma_K(\bar{\alpha}(\lambda^2,\epsilon)). \tag{18}$$

Putting these ingredients together, one can express the solution to eq. (15) in a simplified form, displaying the fact that infra-red and collinear poles to all orders are generated by just two functions of the coupling, G and γ_K. One finds [13]

$$\Gamma(Q^2,\epsilon)=\exp\left\{\frac{1}{2}\int_0^{-Q^2}\frac{\mathrm{d}\xi^2}{\xi^2}\left[G(-1,\bar{\alpha}(\xi^2,\epsilon),\epsilon)\right.\right.$$
$$\left.\left.-\frac{1}{2}\,\gamma_K(\bar{\alpha}(\xi^2,\epsilon))\,\log\left(\frac{-Q^2}{\xi^2}\right)\right]\right\}. \tag{19}$$

In light of recent developments, both in QCD and in $\mathcal{N} = 4$ SYM, it is worth emphasizing that the form factors play an important role also in the much more general case of fixed-angle scattering amplitudes with any number of external legs, for massless gauge theories. Such amplitudes also factorize in a manner similar to eq. (6), albeit with a more complicated colour structure. Indeed, an amplitude with m external coloured legs, $\mathcal{M}_{\{a_i\}}$, $i = 1,\ldots,m$, can be written as a vector in the space of available colour configuration, with components $\mathcal{M}_L^{(m)}$ in a suitable basis of colour tensors $c^L_{\{a_i\}}$. One may then write [18]

$$\mathcal{M}_L^{[m]}\left(\beta_j,\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)=\prod_{i=1}^{m}J_i\left(\frac{Q'^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)S_{LK}^{[m]}$$
$$\times\left(\beta_j,\frac{Q'^2}{\mu^2},\frac{Q'^2}{Q^2},\alpha_s(\mu^2),\epsilon\right)$$
$$\times\ H_K^{[m]}\left(\beta_j,\frac{Q^2}{\mu^2},\frac{Q'^2}{Q^2},\alpha_s(\mu^2)\right). \tag{20}$$

Equation (20) is expressed in terms of velocity four-vectors β_i for each external leg, and the restriction to fixed-angle scattering has been exploited to extract from particle momenta a common hard scale Q; the scale Q', on the other hand, plays the role of a factorization scale separating infra-red and collinear momenta; collinear singularities are organized into the m 'jet' functions J_i, each characterized only by the properties of the originating parton; soft gluons, on the other hand, can mix the colour components of the hard scattering and thus are organized into a matrix $S_{LK}^{[m]}$, acting on a vector of finite coefficient functions $H_K^{[m]}$. One may now exploit the fact that the jets J_i collect the same collinear and infra-red-collinear singular regions as the form factors Γ_i for the same parton species: soft wide-angle radiation would be different, but one can make use of the fact that the soft matrix S is defined up to multiplication times a multiple of the identity matrix in order to reconstruct the appropriate soft emission structure. In other words, there exists a factorization scheme such that one can define

$$J_i\left(\frac{Q'^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \left[\Gamma_i\left(\frac{Q'^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right)\right]^{1/2}. \tag{21}$$

This factorization is especially useful because it teaches us that the structure of collinear singularities of fixed-angle multi-leg scattering amplitudes is completely captured by partonic form factors. Furthermore, in this factorization scheme, the matrix S becomes proportional to the identity matrix in the planar, $N_c \to \infty$ limit. This feature simplifies considerably the analysis in the interesting case of planar $\mathcal{N} = 4$ SYM, where a continuation of the amplitude to strong coupling has, in some cases, becomes possible.

5. Beyond QCD: Conformal gauge theories

One striking feature of eq. (19) is the fact that the logarithm of the form factor is expressed in terms of two *finite* functions of the coupling, G and γ_K. All infra-red and collinear poles are generated by the explicit integration over the scale of the running coupling. In QCD, and for $\epsilon < 0$, the scale dependence of the coupling (see for example eq. (9)) is such that poles up to $1/\epsilon^{p+1}$ are generated at order α_s^p. By contrast, in a conformal gauge theory such as $\mathcal{N} = 4$ SYM, regularized by dimensional continuation, the coupling runs simply according to its engineering dimension in $d = 4 - 2\epsilon$; as a consequence, the integration in eq. (19) yields at most double poles. Expanding γ_K and G in powers of α_s/π, and denoting their perturbative coefficients by $\gamma_K^{(n)}$ and $G^{(n)}(\epsilon)$ respectively, one easily finds [19]

$$\begin{aligned}
&\log\left[\Gamma\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right)\right] \\
&\quad = -\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^n \left(\frac{\mu^2}{-Q^2}\right)^{n\epsilon}\left[\frac{\gamma_K^{(n)}}{2n^2\epsilon^2} + \frac{G^{(n)}(\epsilon)}{n\epsilon}\right] \\
&\quad = -\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{\alpha_s(Q^2)}{\pi}\right)^n \mathrm{e}^{-\mathrm{i}\pi n\epsilon}\left[\frac{\gamma_K^{(n)}}{2n^2\epsilon^2} + \frac{G^{(n)}(\epsilon)}{n\epsilon}\right],
\end{aligned} \tag{22}$$

where in the second line it is noted that the logarithm of the form factor displays exact renormalization group invariance, as expected. Using eq. (22), it is possible to study the analytic continuation of the form factor from time-like to space-like kinematics, in the conformal case. This continuation is of practical interest in QCD: in fact, as shown to all orders in [16], the modulus of the ratio of the time-like to the space-like form factor is finite in $d = 4$; furthermore, this ratio is closely related to physically observable cross-sections: for example, it resums a class of large constant contributions to the Drell–Yan cross-section in the DIS factorization scheme [20–22]. In the present case, having constructed a finite quantity, one may take $\epsilon \to 0$ and compute the ratio in the four-dimensional theory with exact conformal invariance. One finds [13]

$$\left| \frac{\Gamma(Q^2)}{\Gamma(-Q^2)} \right|^2 = \exp\left[\frac{\pi^2}{4} \gamma_K \left(\alpha_s(Q^2) \right) \right]. \tag{23}$$

Equation (23) resums perturbation theory for finite quantities which admit a non-perturbative definition in terms of operator matrix elements. Thus, it can be conjectured to be an exact result. It would be of great interest if a strong-coupling analogue could be derived.

Combining eqs (20)–(22) places strong constraints on the all-order structure of scattering amplitudes in dimensionally-regularized $\mathcal{N} = 4$ SYM. In fact, after a re-analysis of one- and two-loop results for the four-point planar MHV amplitude [23], Bern, Dixon and Smirnov (BDS) performed the highly non-trivial calculation of the same amplitude at three loops [19], and found an intriguing pattern of exponentiation, consistent with eq. (22), but extending to non-singular, ϵ-independent terms. In order to illustrate this pattern, note that in the planar limit the all-order factorized matrix element in eq. (20) has colour structure proportional to the tree-level amplitude; in this case the soft matrix S can be taken to be diagonal. Defining then, in shorthand notation, a reduced matrix element $\widetilde{\mathcal{M}}^{[m]}(\epsilon)$, by dividing out the tree-level result, one can show that the non-Abelian exponentiation following from eqs (20) and (21) leads to the expression [19]

$$\widetilde{\mathcal{M}}^{[m]}(\epsilon) = \exp\left[\sum_{p=1}^{\infty} \left(\frac{\lambda}{8\pi^2} \right)^p \left(f^{(p)}(\epsilon)\, M_1^{[m]}(p\,\epsilon) + h_p^{[m]}(k_i) + \mathcal{O}(\epsilon) \right) \right]. \tag{24}$$

Here λ is the 't Hooft coupling, $\lambda = g^2 N_c$; $M_1^{[m]}$ is the one-loop amplitude, which however is evaluated with a rescaled value of ϵ (a feature clearly visible in eq. (22)); $f^{(p)}(\epsilon)$ is a quadratic polynomial in ϵ, with constant and linear terms determined by eq. (22),

$$f^{(p)}(\epsilon) = \sum_{n=0}^{2} f_n^{(p)} \epsilon^n, \quad f_0^{(p)} = \frac{\gamma_K^{(p)}}{4}, \quad f_1^{(p)} = \frac{p}{2}\, G^{(p)}(0), \tag{25}$$

while $f_2^{(p)}$ can be determined by consistency, considering the case in which subsets of external momenta become collinear; finally, $h_p^{[m]}(k_i)$ is a finite remainder, which in a general gauge theory depends both on the number of particles m and on their momenta k_i. BDS [19] observed that the finite remainder $h_p^{(4)}(k_i)$ is a constant, independent of kinematics, for $p \leq 3$; using also results on collinear limits derived in [23], they conjectured that this property might remain true to all orders in the 't Hooft coupling and for any number m of particles. On the other hand, Alday and Maldacena [10], computing the four-point amplitude at strong 't Hooft coupling by means of the AdS-CFT correspondence, found a structure closely matching eq. (24). These results have lead to a sustained effort by several groups, employing rather different theoretical tools, to study the structure of amplitudes in $\mathcal{N} = 4$ SYM and related theories, and to constrain and compute the anomalous dimensions γ_K and G that govern their singularities (for references, see the reviews in [11,12]). To summarize very briefly the status of these efforts to date, the BDS conjecture is now expected to hold for the four- and five-point amplitudes, while it is known to break down for the six-point amplitude, starting at two loops [24,25]; an ansatz exists [26] for the function $\gamma_K(\lambda)$, which reproduces all available perturbative results, both at weak and strong couplings [27,28]; the function G has also been analysed in detail [13], in the general case of an arbitrary massless gauge theory, expressing it in terms of anomalous dimensions of operators involving Wilson lines and fundamental fields, plus running coupling contributions. For a conformal gauge theory, one finds the very simple result

$$G(1, \alpha_s, \epsilon = 0) = 2B_\delta(\alpha_s) + G_{\rm eik}(\alpha_s), \tag{26}$$

where $G_{\rm eik}$ is a subleading anomalous dimension associated with Wilson lines, and thus in principle amenable, like γ_K, to studies with non-perturbative techniques, while B_δ is the virtual contribution to the Altarelli–Parisi splitting kernel. B_δ involves matrix elements of local fields as well as Wilson lines, so it would be quite interesting to to see how an equation of the form of eq. (26) might arise at strong coupling.

6. Conclusion

The study of long-distance singularities of gauge theories began seventy years ago [29], yet it remains an active and fertile field of research. In QCD, all-order results for soft and collinear gluons are instrumental for phenomenology, providing non-trivial tests of finite order calculations, and forming the basis for the resummation of several classes of large logarithms that would otherwise hinder the applicability of perturbation theory. From a theoretical standpoint, studying long-distance effects to all orders in perturbation theory opens a window on non-perturbative effects, which are suppressed by powers of the hard scale but may still be very relevant for high-energy cross-sections in certain kinematical regimes. When tools are available for a quantitative study of a gauge theory at strong coupling, as is the case for maximally supersymmetric Yang–Mills theory, soft and collinear singularities of amplitudes still provide a bridge between weak coupling and non-perturbative regimes. Recent progress in the study of $\mathcal{N} = 4$ SYM has been especially remarkable; bringing together tools from perturbation theory, string theory and integrable models, it has been possible to reach results that point towards a very ambitious goal: a full and detailed understanding of a non-trivial four-dimensional gauge theory. We may indeed look forward to new developments and applications, both on the practical side of collider phenomenology, and on our way to a deeper understanding of quantum field theory.

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