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Two-fractal overlap time series: Earthquakes and market crashes

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Abstract. We find prominent similarities in the features of the time series for the (model earthquakes or) overlap of two Cantor sets when one set moves with uniform relative velocity over the other and time series of stock prices. An anticipation method for some of the crashes have been proposed here, based on these observations.

Keywords. Cantor set; time series; earthquake; market crash.

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1. Introduction

Capturing dynamical patterns of stock prices are major challenges for financial analysts [1]. The statistical properties of their (time) variations or fluctuations [1] are now well studied and characterized (with established fractal properties), but are not very useful for studying and anticipating their dynamics in the market. Noting that a single fractal gives essentially a time-averaged picture, a minimal two-fractal overlap time series model was introduced [2–4] to capture the time series of earthquake magnitudes. We find that the same model can be used to mimic and study the essential features of the time series of stock prices.

Earthquakes occur due to dynamic stick-slip phenomena at the faults. Tectonic plates (about 12 such plates are there below the Earth's solid crust of about 30 km thickness) are in motion (about 2–3 cm/year). The crust portions on each plate try to follow that motion (creeping phenomenon; with consequent growth in elastic energy) for the 'sticking' period. After a threshold, depending on the sticking

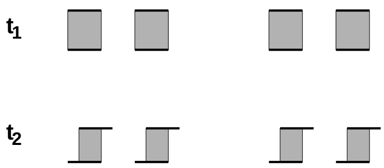

Figure 1. The overlap of two identical Cantor sets of dimension $\ln 2/\ln 3$ at generation $n = 2$ as one moves over the other with uniform velocity. The total measure O of the overlap (total shaded region) varies with time and are shown for two different time instances.

strength at that fault, the crust portion 'slips' and the acquired elastic energy is released in the form of earthquakes (stored at the fault). At the fault, between two 'rough' surfaces where continuous earth materials and the voids appear at all length scales randomly, the two-surface contact area morphology can be represented by two fractals [2–4].

We assume that the market crashes occur due to similar 'stick-slip' behaviour. The (market) price dynamics (equivalent to tectonic plate dynamics in the seismic faults) is perhaps determined by the collective dynamics of all the agents in the entire market. The agents participating in a particular stock market are like a portion of the Earth's crust, 'creeping' along the moving (tectonic plate-like) market! This creeping motion continues until the agents reach their own respective thresholds and 'slip' to hold back on a different location of the moving plate or price! The self-similar behaviour of the (nonlinear) dynamics between these two (market and the participating agents) can therefore be similarly represented by two Cantor sets in relative motions. The stock price, resulting from these competing dynamics, would similarly be given (in this model) by a quasi-random time series.

2. The two-fractal overlap model of earthquake

Let us consider first a geometric model [2–5] of the fault dynamics occurring in overlapping tectonic plates that form the Earth's lithosphere. A geological fault is created by a fracture in the Earth's rock layers followed by a displacement of one part relative to the other. The two surfaces of the fault are known to be self-similar fractals. In the model considered here [2–5], a fault is represented by a pair of overlapping identical fractals and the fault dynamics arising out of the relative motion of the associated tectonic plates is represented by sliding one of the fractals over the other; the overlap O between the two fractals represents the energy released in an earthquake whereas $\log O$ represents the magnitude of the earthquake. In the simplest form of the model, each of the two identical fractals is represented by a regular Cantor set of fractal dimension $\log 2/\log 3$ (see figure 1, where the repetition with a period 3^n occurs due to the periodic boundary condition). This is the only exactly solvable model for earthquakes known so far. The exact analysis, for a discrete version of this model [5], for a finite generation n of the Cantor sets with periodic boundary conditions showed that the probability of the overlap O (with uniform weightage for all overlap values along the time series), which assumes the values $O = 2^{n-k} (k = 0, \ldots, n)$, follows the binomial distribution F of $\log_2 O = n - k$ [6]:

$$\begin{aligned}\Pr(O = 2^{n-k}) &\equiv \Pr(\log_2 O = n - k) \\ &= \binom{n}{n-k} \left(\frac{1}{3}\right)^{n-k} \left(\frac{2}{3}\right)^{k} \equiv F(n-k).\end{aligned} \tag{1}$$

Since the index of the central term (i.e., the term for the most probable event) of the above distribution is $n/3 + \delta$, $-2/3 < \delta < 1/3$, for large values of n, eq. (1) may be written as

$$F\left(\frac{n}{3} \pm r\right) \approx \binom{n}{n \pm r} \left(\frac{1}{3}\right)^{\frac{n}{3} \pm r} \left(\frac{2}{3}\right)^{\frac{2n}{3} \mp r} \tag{2}$$

by replacing $n - k$ with $n/3 \pm r$. For $r \ll n$, we can write the normal approximation to the above binomial distribution as

$$F\left(\frac{n}{3} \pm r\right) \sim \frac{3}{\sqrt{2\pi n}} \exp\left(-\frac{9r^2}{2n}\right). \tag{3}$$

Since $\log_2 O = n - k = \frac{n}{3} \pm r$, we have

$$F\left(\log_2 O\right) \sim \frac{1}{\sqrt{n}} \exp\left[-\frac{(\log_2 O)^2}{n}\right], \tag{4}$$

not mentioning the factors that do not depend on O. Now

$$F(\log_2 O)\mathrm{d}(\log_2 O) \equiv G(O)\mathrm{d}O, \tag{5}$$

where

$$G(O) \sim \frac{1}{O} \exp\left[-\frac{(\log_2 O)^2}{n}\right] \tag{6}$$

is the log-normal distribution of O. As the generation index $n \to \infty$, the normal factor spreads indefinitely (since its width is proportional to $\sqrt{n}$) and becomes a very weak function of O so that it may be considered to be almost constant; thus $G(O)$ asymptotically assumes the form of a simple power law with an exponent that is independent of the fractal dimension of the overlapping Cantor sets [6]:

$$G(O) \sim \frac{1}{O} \quad \text{for } n \to \infty. \tag{7}$$

3. The Cantor set overlap time series

We now consider the time series $O(t)$ of the overlap set (of two identical fractals [4,5]), as one slides over the other with uniform velocity. Let us again consider

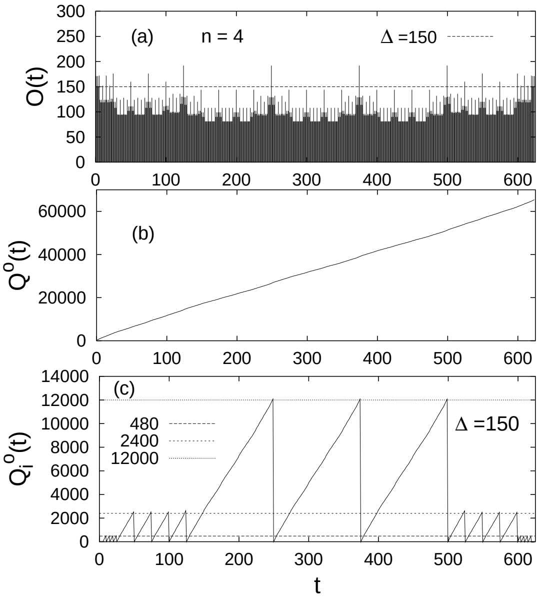

Figure 2. (**a**) The time series data of overlap size $O(t)$ for a regular Cantor set of dimension $\ln 4/\ln 5$ at generation $n = 4$. (**b**) Cumulative overlap $Q^{\rm o}(t)$ and (**c**) the variation of the cumulative overlap $Q_i^{\rm o}(t)$ for the same series, where Q is reset to zero after any big event of size greater than $\Delta = 150$.

two regular Cantor sets at finite generation n. As one set slides over the other, the overlap set changes. The total overlap $O(t)$ at any instant t changes with time (see figure 2a). In figure 2b we show the behaviour of the cumulative overlap [4] $Q^{\rm o}(t) = \int_0^t O(\tilde{t}){\rm d}\tilde{t}$. This curve, for sets with generation $n = 4$, is approximately a straight line [4] with slope $(16/5)^4$. In general, this curve approaches a strict straight line in the limit $a \to \infty$, asymptotically, where the overlap set comes from the Cantor sets formed of $a - 1$ blocks, taking away the central block, giving dimension of the Cantor sets equal to $\ln(a-1)/\ln a$. The cumulative curve is then almost a straight line and has then a slope $[(a-1)^2/a]^n$ for sets of generation n. If one defines a 'crash' occurring at time t_i when $O(t_i) - O(t_{i+1}) \geq \Delta$ (a pre-assigned large value) and one redefines the zero of the scale at each t_i, then the behaviour of the cumulative overlap $Q_i^{\rm o}(t) = \int_{t_{i-1}}^t O(\tilde{t}){\rm d}\tilde{t}$, $\tilde{t} \leq t_i$, has got the peak value (geometric series) 'steps' as shown in figure 2c. The reason is obvious (comes from the fact that the overlap can take only discrete values 2^{n-k}). This justifies the simple thumb rule: one can simply count the cumulative $Q_i^{\rm o}(t)$ of the overlaps since the last 'crash' or 'shock' at t_{i-1} and if the value exceeds the minimum value (q_0; suitably defined for the time series), one can safely extrapolate linearly and expect growth up to αq_0 here and face a 'crash' or overlap greater than Δ (= 150 in figure 2). If nothing happens there, one can again wait up to a time until which the cumulative grows up to $\alpha^2 q_0$ and feel a 'crash' and so on ($\alpha = 5$ in the set considered in figure 2).

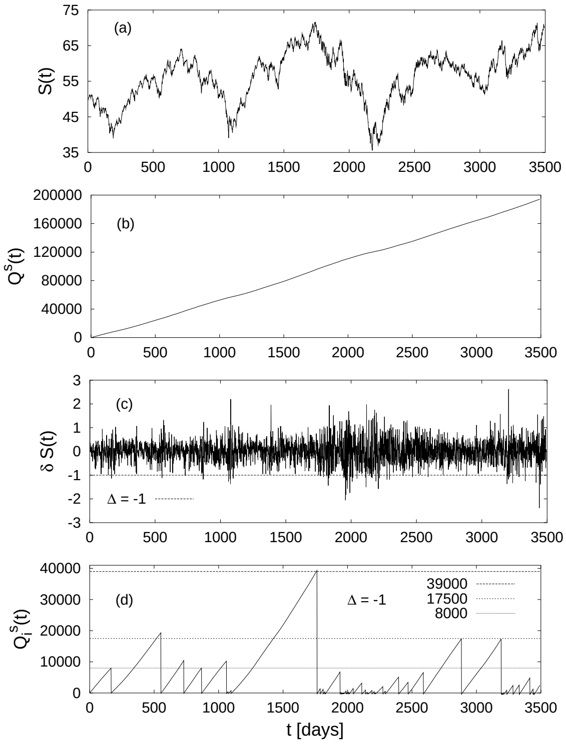

Figure 3. Data from New York Stock Exchange from January 1966 to December 1979: industrial index [7]: (**a**) Daily closing index $S(t)$, (**b**) integrated $Q^{\mathrm{s}}(t)$, (**c**) daily changes $\delta S(t)$ of the index $S(t)$ defined as $\delta S(t) = S(t+1) - S(t)$, and (**d**) behaviour of $Q_i^{\mathrm{s}}(t)$ where $\delta S(t_i) > \Delta$. Here, $\Delta = -1.0$ as shown in (c) by the dotted line (from [8]).

4. The stock price time series

We now consider some typical stock price time series data, available in the Internet. The data analysed here are for the New York Stock Exchange (NYSE) Indices [7]. In figure 3a, we show the daily stock price $S(t)$ variations for about 10 years (daily closing price of the 'industrial index') from January 1966 to December 1979 (3505 trading days). The cumulative $Q^{\mathrm{s}}(t) = \int_0^t S(t)\mathrm{d}t$ has again a straight line variation with time t (figure 3b). Similar to the Cantor set analogy, we then define the major shock by identifying those variations when $\delta S(t)$ of the prices in successive days exceeded a pre-assigned value Δ (figure 3c). The variation of $Q_i^{\mathrm{s}}(t) = \int_{t_{i-1}}^{t} S(\tilde{t})\mathrm{d}\tilde{t}$ where t_i are the times when $\delta S(t_i) \leq -1$ show similar geometric series like peak values (see figure 3d); see [8,9].

We observed striking similarity between the 'crash' patterns in the Cantor set overlap model and that derived from the dataset of the stock market index. For both cases, the magnitude of crashes follow a similar pattern – the crashes occur in a geometric series.

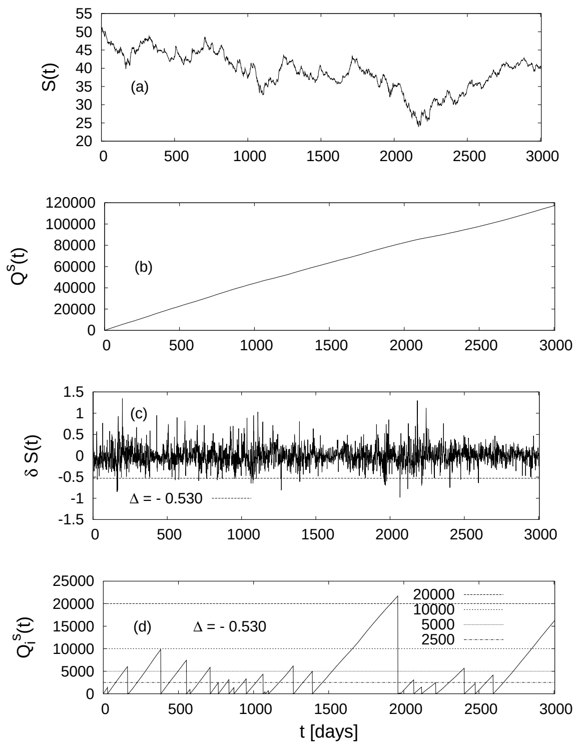

Figure 4. Data from New York Stock Exchange from January 1966 to December 1979: utility index [7]: (**a**) Daily closing index $S(t)$, (**b**) integrated $Q^{\mathrm{s}}(t)$, (**c**) daily changes $\delta S(t)$ of the index $S(t)$ defined as $\delta S(t) = S(t+1) - S(t)$, and (d) behaviour of $Q_i^{\mathrm{s}}(t)$ where $\delta S(t_i) > \Delta$. Here, $\Delta = -0.530$ as shown in (c) by the dotted line.

A simple 'anticipation strategy' for some of the crashes may be as follows: If the cumulative $Q_i^{\mathrm{s}}(t)$ since the last crash has grown beyond $q_0 \simeq 8000$ here, wait until it grows (linearly with time) until about 17,500 ($\simeq 2.2q_0$) and expect a crash there. If nothing happens, then wait until $Q_i^{\mathrm{s}}(t)$ grows (again linearly with time) to a value of the order of 39,000 ($\simeq (2.2)^2 q_0$) and expect a crash, and so on.

The same kind of analysis for the NYSE 'utility index', for the same period, is shown in figure 4.

5. Earthquake magnitude time series

Unlike in the case of stock price time series where accurate data are easily available, the time series for earthquake magnitudes $M(t)$ at any fault involves considerably coordinated measurements and comparable accuracies are not easily achievable. Still, from the available data, as in the case of stock market (where the integrated stock price $Q^{\mathrm{s}}(t)$ shows clear linear variations with time and this fits well with

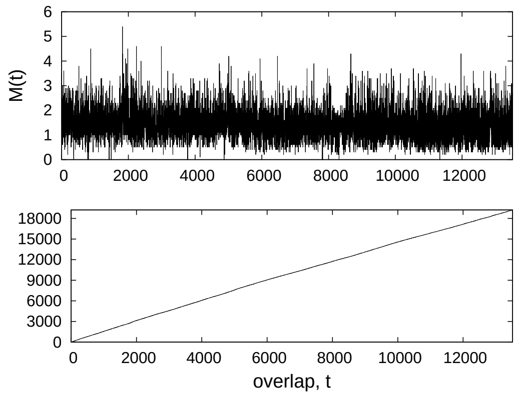

Figure 5. Local earthquake data from US Geological Survey Southern California catalogs [11]: Top: Time series of successive quakes $M(t)$; Bottom: integrated $Q^{\mathrm{m}}(t)$. The dataset is a record of quakes between 1 January 2003 to 31 March 2004, between depths 0 and 700 km, between latitudes 32°N and 37°N and longitudes −122°W and −114°W.

that for the cumulative overlap $Q^{\circ}(t)$ for the fractal overlap model; see also [10]), the integrated earthquake magnitude $Q^{\mathrm{m}}(t) = \int_0^t M(t)\mathrm{d}t$ of the aftershocks does also show such prominent linear variations (see figure 5). We believe, the slopes of these linear $Q^{\mathrm{m}}(t)$ vs. t curves for different faults would give us the signature of the corresponding fractal structure of the underlying fault. It may be noted in this context, in our model, the slope becomes $[(a-1)^2/a]^n$ for an nth generation Cantor set, formed out of the remaining $a-1$ blocks having the central block removed.

6. Summary

Based on the formal similarity between the two-fractal overlap model of earthquake time series and of the stock market, we considered here a detailed comparison. We find, the features of the time series for the overlap of two Cantor sets when one set moves with uniform relative velocity over the other looks somewhat similar to the time series of stock prices. We analyse both and explore the possibilities of anticipating a large (change in Cantor set) overlap or a large change in stock price. An anticipation method for some of the crashes has been proposed here, based on these observations.

We modeled these 'quakes' as accumulated overlaps between two fractals (here Cantor sets) having uniform relative velocity. The quake magnitudes are given by the measure of the overlap region between two sets as one moves uniformly below the other (see the upper panel $O(t)$ in figure 2). Although the overlap has a quasi-random time variation as shown in this figure, the integrated overlap $Q^{\mathrm{o}}(t)$ shown in the second panel in figure 2, has a very simple linear time variation (exact linearity appears in the true fractal limit; generation number n going to infinity). One of our observations, which we consider to be important, is that the slope of that linear variation gives a characteristic signature of the fault (or the fractal) and hence of the earthquake (seismic) time series for that epicentre (see the second panel in figure 2, for our model, and figure 5, for the real earthquake data). Over the years, this slope is seen to remain the same, giving a 'fingerprint' of the fault.

As argued and indicated in figure 2 (upper panel), the model stock price time series will have a similar behaviour. That means, the cumulative would again have a linear time variation – as indeed observed (see figures 3 and 4)! Like in the case of earthquakes, this slope would give similar 'fingerprint' of the market dynamics and, as seen, this slope differs from market to market (see second panels in figures 3 and 4). Maintaining such characteristic slopes in the time behaviour of the cumulative of the (stock) market price for over more than a decade (14 years in figures 3 and 4), the markets clearly show quite non-Markovian or non-random features and well-structured correlated fluctuations: Like the time variations of $O(t)$ in figure 2a of our Cantor set model, where $O(t)$ is not at all random, neither as correlated as periodic, the stock prices $S(t)$ have got some strongly correlated, yet quasi-random, time variations, keeping its cumulative slope with time t preserved.

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