PRAMANA — journal of physics © Indian Academy of Sciences Vol. 70, No. 5 May 2008 pp. 763–777

On some exact solutions of slightly variant forms of Yang's equations and their graphical representations

RUPESH KUMAR SAHA and PRANAB KRISHNA CHANDA*
Siliguri B.Ed. College, P.O. Kadamtala (Shivmandir), Dist. Darjeeling 734 011, India
*Corresponding author
E-mail: rupkumarhere@yahoo.co.in; dr_pkchanda@yahoo.com

MS received 9 July 2007; revised 1 January 2008; accepted 9 January 2008

Abstract. The equations obtained by Yang while discussing the condition of self-duality of $SU(2)$ gauge fields on Euclidean four-dimensional space have been generalized. Exact solutions and their graphical representations for the generalized equation (for some particular values of the parameters) have been reported. They represent interesting physical characteristics like waves with spreading solitary profile, spreading wave packets, waves with pulsating solitary profile (between zero and a maximum), waves with oscillatory solitary profile and chaos.

Keywords. Exact solutions with graphical representation; $SU(2)$ gauge field; self-duality; solitary wave; chaos.

PACS Nos 05.45.Yv; 11.10.Lm; 11.30.Na; 47.52.+j

1. Introduction

Yang's equations were obtained by Yang [1] himself while discussing the condition of self-duality of $SU(2)$ gauge fields on Euclidean four-dimensional space. The equations are given by

$$\Phi(\Phi_{y\bar{y}} + \Phi_{z\bar{z}}) - \Phi_y\Phi_{\bar{y}} - \Phi_z\Phi_{\bar{z}} + \rho_y\rho_{\bar{y}} + \rho_z\rho_{\bar{z}} = 0, \tag{1.1a}$$

$$\Phi(\rho_{y\bar{y}} + \rho_{z\bar{z}}) - 2\rho_y\Phi_{\bar{y}} - 2\rho_z\Phi_{\bar{z}} = 0, \tag{1.1b}$$

where an overbar denotes the complex conjugate, Φ and ρ are functions of $y, \bar{y}, z, \bar{z}, \Phi$ is real, ρ is complex and $\sqrt{2}y = x^1 + ix^2$, $\sqrt{2}z = x^3 - ix^4$ and x^1, x^2, x^3, x^4 are real.

Once one has found ρ and Φ, the corresponding R-gauge potentials are given by Yang [1] as

$$\Phi\vec{b}_y = (i\rho_y, \rho_{\bar{y}}, -i\Phi_y), \quad \Phi\vec{b}_{\bar{y}} = (-i\rho_{\bar{y}}, \bar{\rho}_{\bar{y}}, i\Phi_{\bar{y}}), \tag{1.2a, b}$$

$$\Phi\vec{b}_z = (i\rho_z, \rho_{\bar{z}}, -i\Phi_z), \quad \Phi\vec{b}_{\bar{z}} = (-i\rho_{\bar{z}}, \bar{\rho}_{\bar{z}}, i\Phi_{\bar{z}}), \tag{1.2c, d}$$

and the R-gauge field strengths $F_{\mu\nu}$ are given by

$$F_{\mu\nu} = B_{\mu,\nu} - B_{\nu,\mu} - B_\mu B_\nu + B_\nu B_\mu, \tag{1.3a}$$

$$B_\mu = b^i_\mu X_i, \tag{1.3b}$$

and

$$X_i = -(1/2)i\sigma_i, \tag{1.3c}$$

where σ_i are the 2×2 Pauli matrices.

All such solutions represent the condition of self-duality except when Φ is zero. When Φ is zero, $F_{\mu\nu}$ becomes singular and the solutions can only be treated as solutions of Yang's R-gauge equations and not self-dual solutions unless a transformation like $F'_{\mu\nu} \to U^{-1}F_{\mu\nu}U$ removes the singularity.

When written in terms of real variables the equations in (1.1) become

$$\begin{aligned}\Phi_{11} &+ \Phi_{22} + \Phi_{33} + \Phi_{44}\\ &= [(1/\Phi)(\Phi_1^2 + \Phi_2^2 + \Phi_3^2 + \Phi_4^2)\\ &\quad -(1/\Phi)(\psi_1^2 + \psi_2^2 + \psi_3^2 + \psi_4^2)\\ &\quad -(1/\Phi)(\chi_1^2 + \chi_2^2 + \chi_3^2 + \chi_4^2)\\ &\quad -(2/\Phi)(\psi_1\chi_2 - \psi_2\chi_1 + \psi_4\chi_3 - \psi_3\chi_4)],\end{aligned} \tag{1.4a}$$

$$\begin{aligned}\psi_{11} &+ \psi_{22} + \psi_{33} + \psi_{44}\\ &= [(2/\Phi)(\Phi_1\psi_1 + \Phi_2\psi_2 + \Phi_3\psi_3 + \Phi_4\psi_4)\\ &\quad +(2/\Phi)(\Phi_1\chi_2 - \Phi_2\chi_1 + \Phi_4\chi_3 - \Phi_3\chi_4)],\end{aligned} \tag{1.4b}$$

$$\begin{aligned}\chi_{11} &+ \chi_{22} + \chi_{33} + \chi_{44}\\ &= [(2/\Phi)(\Phi_1\chi_1 + \Phi_2\chi_2 + \Phi_3\chi_3 + \Phi_4\chi_4)\\ &\quad +(2/\Phi)(\Phi_2\psi_1 - \Phi_1\psi_2 + \Phi_3\psi_4 - \Phi_4\psi_3)],\end{aligned} \tag{1.4c}$$

where

$$\Phi_1 \equiv \partial\Phi/\partial x^1, \quad \Phi_{11} \equiv \partial^2\Phi/\partial(x^1)^2. \tag{1.4d}$$

Yang [1] and several other authors [2] have presented solutions to (1.1) or its equivalent (1.4). Chakraborty and Chanda [3] reported some graphical representation of one these exact solutions. It is observed from there that the solutions represent spreading wave with solitary profile and spreading wave packet. These profiles of solitary wave and wave packet tend to vanish as time tends to infinity.

Jimbo *et al* [4] adopted the algorithm of Weiss *et al* [5] and showed that eqs (1.1) pass the Painlevé test for integrability. Using the same algorithm, Chakraborty and Chanda [6] have found that the real form of eqs (1.1), i.e. eqs (1.4), pass the Painlevé test for integrability and admit truncation of series leading to non-trivial exact solutions obtained previously and auto-Backlund transformation between two pairs of these solutions (see, for example, the work of Larsen [7] and Roychowdhury [8]). An important aspect of the work of Chakraborty and Chanda [6] was that they had analyzed the equation keeping the singularity manifold completely general, whereas Jimbo *et al* [4] analyzed the same equation with a restricted nature of singularity manifold.

With this background and success, here we generalize eqs (1.4). The generalized form is given by eqs (2) of the next section.

2. The generalized Yang equations under study

$$\begin{aligned}&\Phi_{11}+\Phi_{22}+\Phi_{33}+\varepsilon\Phi_{44}\\&\quad=k'[(1/\Phi)(\Phi_1^2+\Phi_2^2+\Phi_3^2+\varepsilon\Phi_4^2)\\&\qquad-(1/\Phi)(\psi_1^2+\psi_2^2+\psi_3^2+\varepsilon\psi_4^2)\\&\qquad-(1/\Phi)(\chi_1^2+\chi_2^2+\chi_3^2+\varepsilon\chi_4^2)\\&\qquad-(2/\Phi)(\psi_1\chi_2-\psi_2\chi_1+\psi_4\chi_3-\psi_3\chi_4)],\end{aligned}\tag{2.1a}$$

$$\begin{aligned}&\psi_{11}+\psi_{22}+\psi_{33}+\varepsilon\psi_{44}\\&\quad=k'[(2/\Phi)(\Phi_1\psi_1+\Phi_2\psi_2+\Phi_3\psi_3+\varepsilon\Phi_4\psi_4)\\&\qquad+(2/\Phi)(\Phi_1\chi_2-\Phi_2\chi_1+\Phi_4\chi_3-\Phi_3\chi_4)],\end{aligned}\tag{2.1b}$$

$$\begin{aligned}&\chi_{11}+\chi_{22}+\chi_{33}+\varepsilon\chi_{44}\\&\quad=k'[(2/\Phi)(\Phi_1\chi_1+\Phi_2\chi_2+\Phi_3\chi_3+\varepsilon\Phi_4\chi_4)\\&\qquad+(2/\Phi)(\Phi_2\psi_1-\Phi_1\psi_2+\Phi_3\psi_4-\Phi_4\psi_3)],\end{aligned}\tag{2.1c}$$

where $\varepsilon=\pm1$; k' are arbitrary constants.

Equations (2) are being termed as the *generalized Yang equations*. However, one should distinguish these equations from the extended Yang equations as discussed by Chakraborty and Chanda in [3] where $\varepsilon=1$, $k'\neq0$, $k''\neq0$ and there are more terms (obtainable from Charap's equations for pion dynamics [9]) other than those indicated here.

In this paper we have presented exact solutions along with their graphical representations for (2) with (I) $\varepsilon=1$, $k'=1$, (II) $\varepsilon=1$, $k'=1/2$, (III) $\varepsilon=-1$, $k'=1$, (IV) $\varepsilon=-1$, $k'=1/2$.

It may be noted that, for Case I, i.e. with $\varepsilon=1$, $k'=1$, eqs (2) reduce to Yang's equations (1.4) which is again equivalent to (1.1). Here we report the work done previously by De and Ray [2] and Chakraborty and Chanda [3].

The motivation for taking $\varepsilon = 1$ or -1 is that in such cases Φ, ψ, χ can ultimately be represented in terms of ζ which satisfy the standard equations

$$\zeta_{11} + \zeta_{22} + \zeta_{33} + \zeta_{44} = 0, \quad \text{for} \quad \varepsilon = 1$$

and

$$\zeta_{11} + \zeta_{22} + \zeta_{33} - \zeta_{44} = 0, \quad \text{for} \quad \varepsilon = -1.$$

The motivation for the generalization of eqs (1.4) to eqs (2) is the identification of model differential equations that may be useful for the representation of physical reality. Here we have demonstrated that the variant forms of the celebrated equations (1.4) offer diverse types of physical solutions ranging from waves with solitary profile to chaos.

On the other hand, the motivation for the choice of $k' = 1$, $k' = 1/2$ will be revealed later in the subsequent section where it will be shown that for such values of k' the integration becomes straightforward [10].

We have used the same ansatz as was used by Ray [11] and De and Ray [12] and rediscovered by Chakraborty and Chanda [3]. Actually, the ansatz used by De and Ray [12] was $\psi = \psi(\Phi)$ and $\chi = \chi(\Phi)$. However, if we write $\Phi = \Phi(u)$, where u is an unspecified function of x^1, x^2, x^3, x^4, then we finally get

$$\Phi = \Phi(u), \quad \psi = \psi(u), \quad \chi = \chi(u).$$

With this we actually propose a regular space curve solution of the equation with Φ, ψ, χ parametrized being of the form $\Phi(u)$, $\psi(u)$ and $\chi(u)$. However, unless a regular parametric curve is considered, it cannot have allowable change of parameters. We use a regular parametric curve denoted by two parameters u and $v(u)$. Unless u and v are related by the condition of allowed change of parameters of regular curves, one may face the problem of invert ability between u and v. In other words, local inverses between these functions must exist. Fortunately this property of regular curve has been satisfied in the calculations presented here [13].

Furthermore, the regular space curve considered in this calculation should at least be class C^2, because, u: $R^4 \to R$ and v: $R^4 \to R$. The choice of ψ as a function of v, for example, satisfies this requirement [13].

The procedure adopted by Ray [11] for obtaining the solutions of Charap's equations for pion dynamics [9] has been used for obtaining all the solutions mentioned in this paper.

3. Solutions

We start with the ansatz given by

$$\Phi = \Phi(u), \quad \psi = \psi(u), \quad \chi = \chi(u), \tag{3.1}$$

where u is an unspecified function of x^1, x^2, x^3 and x^4.

At first we proceed keeping ε and k' in (2) to be unspecified constants.

After the use of (3.1), eqs (2) reduce to

$$(u_{11}+u_{22}+u_{33}+\varepsilon u_{44})+A(u_1^2+u_2^2+u_3^2+\varepsilon u_4^2)=0, \tag{3.2a}$$

$$(u_{11}+u_{22}+u_{33}+\varepsilon u_{44})+D(u_1^2+u_2^2+u_3^2+\varepsilon u_4^2)=0, \tag{3.2b}$$

$$(u_{11}+u_{22}+u_{33}+\varepsilon u_{44})+E(u_1^2+u_2^2+u_3^2+\varepsilon u_4^2)=0, \tag{3.2c}$$

$$A=(\Phi_{uu}/\Phi_u)-k'\{(\Phi_u^2-\psi_u^2-\chi_u^2)/(\Phi\Phi_u)\}, \tag{3.2d}$$

$$D=(\psi_{uu}/\psi_u)-k'(2\Phi_u/\Phi), \tag{3.2e}$$

$$E=(\chi_{uu}/\chi_u)-k'(2\Phi_u/\Phi), \tag{3.2f}$$

so that either

$$A=D=E \tag{3.3}$$

or

$$u_1^2+u_2^2+u_3^2+\varepsilon u_4^2=0 \tag{3.4a}$$

and

$$u_{11}+u_{22}+u_{33}+\varepsilon u_{44}=0. \tag{3.4b}$$

Equations (3.4) have simple solutions and are given in the work of Ray [11] and Ghosh *et al* [14].

Considering (3.2e,f) one arrives at

$$\chi=k_1\psi+k_2, \tag{3.5}$$

where k_1 and k_2 are arbitrary constants.

Let us define

$$\psi=k_3v+k_4, \tag{3.6}$$

where (i) v is some unspecified function of u and (ii) k_3, k_4 are arbitrary constants.

Since u is an unspecified function of x^1, x^2, x^3, x^4 (as defined in (3.1)) one can conclude till now that v is an unspecified function of x^1, x^2, x^3 and x^4.

Putting (3.6) in (3.5) we get

$$\chi=k_5v+k_6, \tag{3.7}$$

where $k_1k_3=k_5$ and $k_1k_4+k_2=k_6$.

Now, from (3.1) we have $\psi=\psi(u)$, $\chi=\chi(u)$, where ψ and χ are unspecified functions of u. Comparing this with (3.5) and (3.6) we have $v=v(u)$, where v is an unspecified function of u. Since from (3.1) we have $\Phi=\Phi(u)$, where Φ is an unspecified function of u one can now say without any loss of generality that

$$\Phi = \Phi(v). \tag{3.8}$$

The use of (3.6), (3.7) and (3.8) in (2) leads to

$$(v_{11} + v_{22} + v_{33} + \varepsilon v_{44}) + A'(v_1^2 + v_2^2 + v_3^2 + \varepsilon v_4^2) = 0 \tag{3.9a}$$

and

$$(v_{11} + v_{22} + v_{33} + \varepsilon v_{44}) + D'(v_1^2 + v_2^2 + v_3^2 + \varepsilon v_4^2) = 0, \tag{3.9b}$$

where

$$A' = (\Phi_{vv}/\Phi_v) - k'\{(\Phi_v^2 - k_3^2 - k_5^2)/(\Phi\Phi_v)\}, \tag{3.9c}$$

and

$$D' = -2k'(\Phi_v/\Phi). \tag{3.9d}$$

Just as in the above the possibility other than

$$v_{11} + v_{22} + v_{33} + \varepsilon v_{44} = 0, \tag{3.10a}$$

$$v_1^2 + v_2^2 + v_3^2 + \varepsilon v_4^2 = 0, \tag{3.10b}$$

requires that

$$A' = D'. \tag{3.11}$$

From (3.11) one gets

$$\Phi\Phi_{vv} + k'\Phi_v^2 + k'(k_3^2 + k_5^2) = 0 \tag{3.12}$$

which may be integrated once to give

$$\Phi_v = (1/\Phi^{k'})\surd\{k_7 - (k_3^2 + k_5^2)\Phi^{2k'}\} = 0, \tag{3.13}$$

where k_7 is another arbitrary constant of integration and $k_7 > 0$.

Simultaneously (3.9b) and (3.9d) can be rewritten as

$$\zeta_{11} + \zeta_{22} + \zeta_{33} + \varepsilon\zeta_{44} = 0, \tag{3.14a}$$

where

$$\zeta = \int \left[\exp\left\{\int D'\mathrm{d}v\right\}\right] \mathrm{d}v \tag{3.14b}$$

and D' is given by (3.9d).

Putting D' from (3.9d) in (3.14b) one gets

$$\zeta = k_8 \int (1/\Phi^{2k'})\mathrm{d}v + k_9, \tag{3.15}$$

where k_8 is an arbitrary constant of integration with a restriction $k_8 > 0$ and k_9 is an arbitrary constant of integration.

Equation (3.15) can further be rewritten using (3.13) which is given as follows:

$$\zeta = k_8 \int \{1/f(\Phi)\}\mathrm{d}\Phi + k_9, \tag{3.16a}$$

where

$$f(\Phi) = \Phi^{k'}\sqrt{\{k_7 - (k_3^2 + k_5^2)\Phi^{2k'}\}}. \tag{3.16b}$$

From (3.16) we get Φ in terms of ζ. Going back to (3.13) one can write that

$$v = \int [\Phi^{k'}/\sqrt{\{k_7 - (k_3^2 + k_5^2)\Phi^{2k'}\}}]\mathrm{d}\Phi \tag{3.17}$$

which may be rewritten as

$$v = [(k_7)^{1/2k'}/\{k'(k_3^2 + k_5^2)^{(1+k')/2k'}\}] \int (\sin\alpha)^{1/k'}\mathrm{d}\alpha + k_{10}, \tag{3.18a}$$

where

$$\alpha = \sin^{-1}[\{(k_3^2 + k_5^2)/k_7\}^{1/2}\Phi^{k'}] \tag{3.18b}$$

and k_{10} is an arbitrary constant of integration.

Thus (3.18) gives v in terms of Φ whereas (3.16) gives Φ in terms of ζ. So, we can say that (3.16) and (3.18) give v in terms of ζ. Finally from (3.6) and (3.7) we get ψ and χ in terms of ζ.

Here is the motivation for the choice $k' = 1$ or $1/2$. From (3.16) and (3.18) we see that for such value of k' integration becomes straightforward.

In the following we represent the solution for particular forms of (2) defined by particular values of ε and k'.

Case I: $\varepsilon = 1$, $k' = 1$

This case is exactly same as (1.4) which is the real form of (1.1) obtained by Yang at the time of discussing self-duality of $SU(2)$ gauge fields on Euclidean four-dimensional space. Exact solutions for this case were reported by De and Ray [2] and were rediscovered by Chakraborty and Chanda [3]. From (3.16), (3.18), (3.6), (3.7), $\varepsilon = 1, k' = 1$ and for some particular values of integration constants one can have

$$\Phi = \{\sqrt{k_7}/\sqrt{(k_3^2 + k_5^2)}\}\mathrm{sech}(\zeta\sqrt{k_7}/k_8), \tag{3.19a}$$

$$\psi = k_3\{\sqrt{k_7}/(k_3^2 + k_5^2)\}\tanh(\zeta\sqrt{k_7}/k_8), \tag{3.19b}$$

$$\chi = k_5\{\sqrt{k_7}/(k_3^2 + k_5^2)\}\tanh(\zeta\sqrt{k_7}/k_8), \tag{3.19c}$$

where ζ satisfies

$$\zeta_{11} + \zeta_{22} + \zeta_{33} + \zeta_{44} = 0, \tag{3.19d}$$

and a particular example of ζ satisfying (3.19d) is given in [3] and is represented by

$$\zeta = [(\sin\tau)/\tau]\cosh t, \quad t = x^4, \tag{3.19e}$$

where

$$\tau^2 = (x^1)^2 + (x^2)^2 + (x^3)^2, \quad k_i \text{ are constants.} \tag{3.19f}$$

It may be noted that eqs (3.19) are exactly same as those reported by Chakraborty and Chanda [3] subject to some minor scaling transformations. Therefore, we have avoided reporting detailed calculations required for arriving at (3.19). However, the calculations proceed in the same way as reported in the case of $\varepsilon = 1$, $k' = 1/2$.

From the graphical representations of (3.19) reported in [3] it was observed that the solutions represent wave with a spreading solitary profile for Φ and spreading wave packet for ψ and χ. These profiles of solitary wave and wave packet tend to vanish as time tends to infinity. The graphical representations express the dependence of Φ, ψ and χ on x^1 (keeping $x^2 = 0$ and $x^3 = 0$) at different values of x^4 (i.e. time). This does not lead to much loss of generality as the solutions have exact symmetrical dependence on x^1, x^2, x^3.

The graphical representations mentioned above are given in figures 1.1–1.9 and figures 2.1–2.9 by Chakraborty and Chanda [3].

Case II: $\varepsilon = 1$, $k' = 1/2$

Without much loss of generality we can choose $k_9 = 0$ when from (3.16) one can write

$$\zeta = k_8 \int [1/\Phi^{1/2}\surd\{k_7 - (k_3^2 + k_5^2)\Phi\}]\mathrm{d}\Phi + k_9. \tag{3.20}$$

With the transformation

$$\Phi^{1/2} = \{\surd k_7/\surd(k_3^2 + k_5^2)\}\sin\theta, \tag{3.21}$$

eq. (3.20) reduces to

$$\zeta = [2k_8/\surd(k_3^2 + k_5^2)]\theta \tag{3.22}$$

which means

$$\Phi = \{k_7/(k_3^2 + k_5^2)\}\sin^2[\{\zeta\surd(k_3^2 + k_5^2)\}/2k_8]. \tag{3.23}$$

Without much loss of generality we can choose $k_{10} = 0$. Then from (3.18a)

$$v = [2k_7/\{k_3^2 + k_5^2)^{3/2}\}] \int \sin^2\alpha\,\mathrm{d}\alpha, \tag{3.24a}$$

where

$$\alpha = \sin^{-1}[\{(k_3^2 + k_5^2)/k_7\}^{1/2}\Phi^{1/2}]. \tag{3.24b}$$

Equations (3.24) can be integrated as

$$v = [k_7/(k_3^2 + k_5^2)^{3/2}] \sin^{-1}[\{(k_3^2 + k_5^2)/k_7\}\Phi^{1/2}] \\ -[\sqrt{k_7}/(k_3^2 + k_5^2)]\sqrt{[\Phi - \{(k_3^2 + k_5^2)/k_7\}\Phi^2]}. \tag{3.25}$$

With the use of (3.23) in (3.25) one finally gets

$$v = [k_7/\{2(k_3^2 + k_5^2)\}][(\zeta/k_8) - \{1/\sqrt{(k_3^2 + k_5^2)}\} \sin\{\zeta\sqrt{(k_3^2 + k_5^2)}/k_8\}]. \tag{3.26}$$

So, finally from (3.23), (3.26), (3.6) and (3.7) we get (with $k_4 = 0$, $k_6 = 0$)

$$\Phi = [k_7/\{2(k_3^2 + k_5^2)\}][1 - \cos\{\zeta\sqrt{(k_3^2 + k_5^2)}/k_8\}], \tag{3.27a}$$

$$\psi = [k_3k_7/\{2(k_3^2 + k_5^2)\}][(\zeta/k_8) - \{1/\sqrt{(k_3^2 + k_5^2)}\} \sin\{\zeta\sqrt{(k_3^2 + k_5^2)}/k_8\}], \tag{3.27b}$$

$$\chi = [k_5k_7/\{2(k_3^2 + k_5^2)\}][(\zeta/k_8) - \{1/\sqrt{(k_3^2 + k_5^2)}\} \sin\{\zeta\sqrt{(k_3^2 + k_5^2)}/k_8\}]. \tag{3.27c}$$

Here ζ satisfies

$$\zeta_{11} + \zeta_{22} + \zeta_{33} + \zeta_{44} = 0, \tag{3.27d}$$

and a particular example of ζ satisfying (3.27d) is given in [3] and is represented by

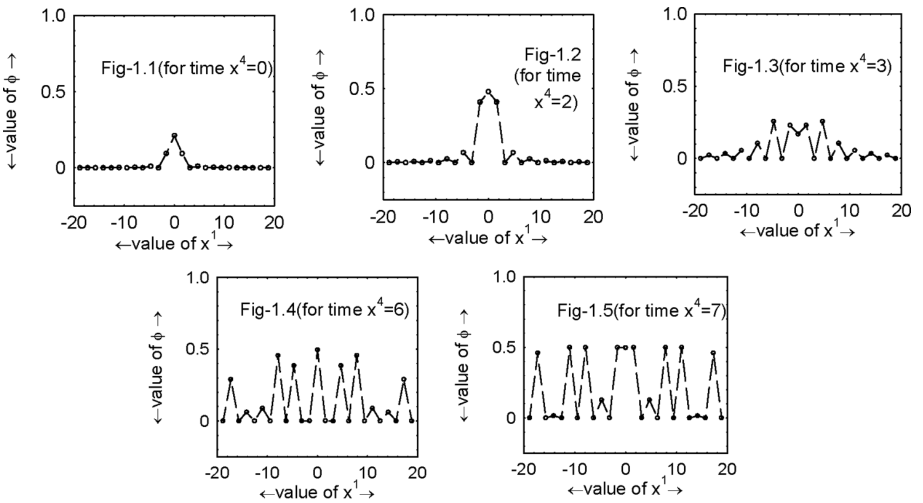
$$\zeta = [(\sin\tau)/\tau]\cosh t, \quad t = x^4, \tag{3.27e}$$

where

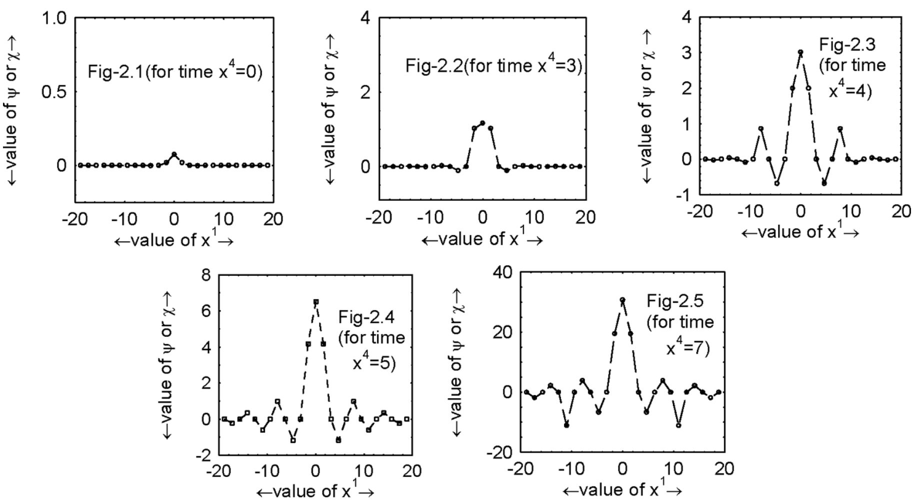
$$\tau^2 = (x^1)^2 + (x^2)^2 + (x^3)^2, \quad k_i \text{ are constants.} \tag{3.27f}$$

Using the same procedure as stated in the case of (3.19) we can represent solutions (3.27) graphically.

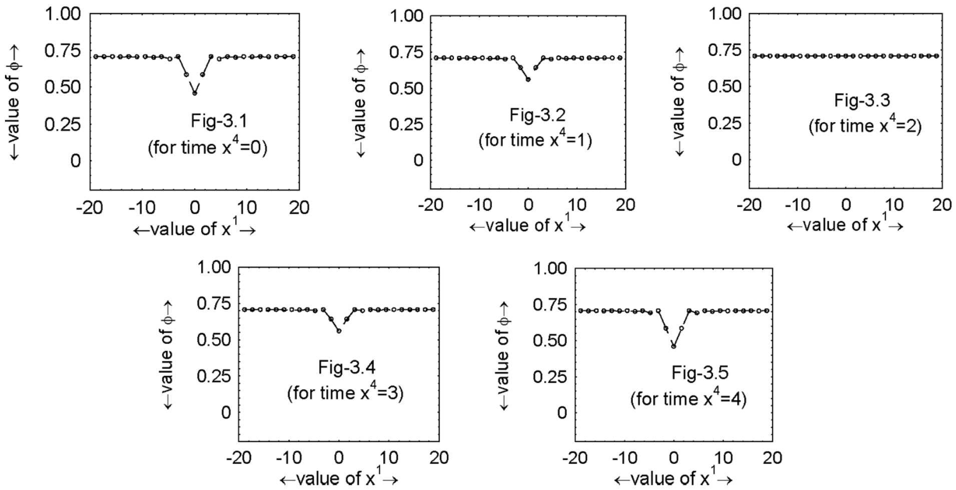
The graphical representations for (3.27) are given in figures 1.1–1.5 for Φ and figures 2.1–2.5 for ψ or χ. It is interesting to see that the simple-looking solutions given by (3.27) actually represent something like spatio-temporal chaos. Initially the solution for Φ, ψ and χ represent waves with solitary profile, a regular one, which gradually leads with time to irregular shapes over space. However, one should note that nonintegrability is required for obtaining chaos [13]. One way to ascertain



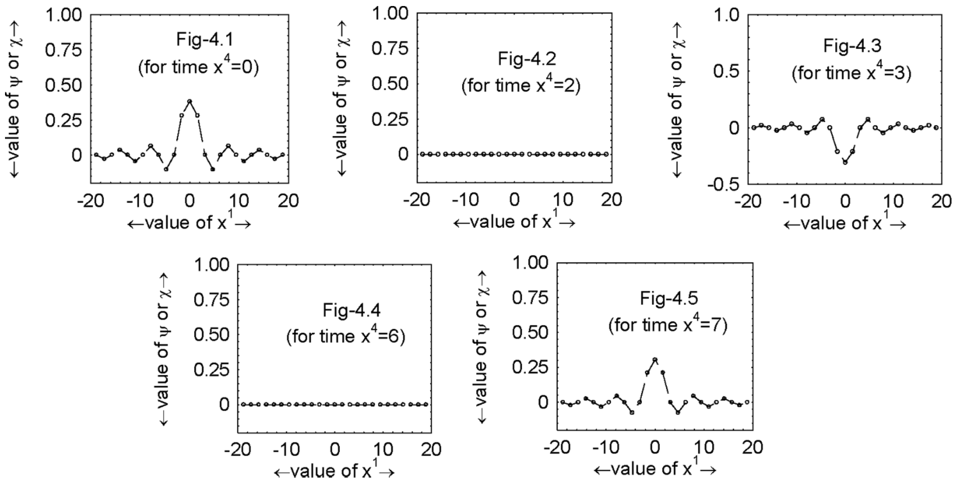
Figures 1. 1–1.5. Solutions for Φ in generalized Yang equations with $k' = 1/2$, $\varepsilon = 1$. Initially Φ represents a wave with solitary profile, a regular one, which gradually leads with time to irregular shape over space. The situation may be correlated with spatio-temporal chaos. However non-integrability is required for obtaining chaos. (Values of x^1 in x-axis are in multiples of $\pi/2$ and values of time, x^4 are in multiples of $\pi/4$.)



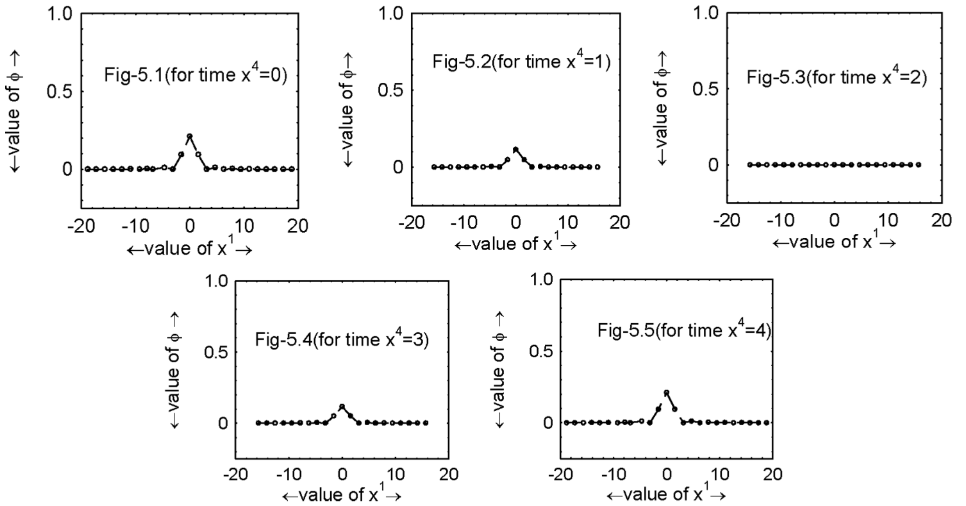
Figures 2.1–2.5. Solutions for ψ and χ in generalized Yang equations with $k' = 1/2$, $\varepsilon = 1$. Initially ψ and χ represent waves with solitary profile, a regular one, which gradually leads with time to irregular shape over space. The situation may be correlated with spatio-temporal chaos. However non-integrability is required for obtaining chaos. (Values of x^1 in x-axis are in multiples of $\pi/2$ and values of time, x^4 are in multiples of $\pi/4$.)



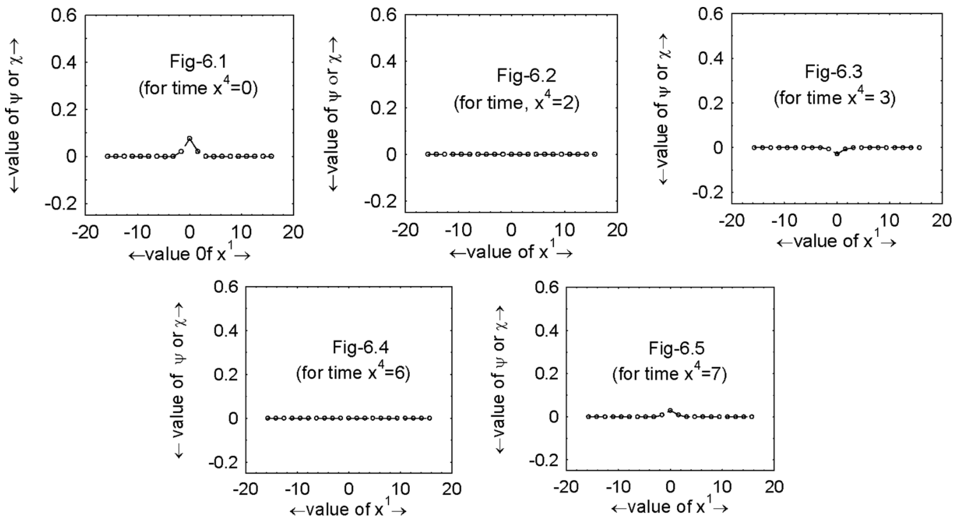
Figures 3.1–3.5. Solutions for Φ in generalized Yang equations with $k' = 1$, $\varepsilon = -1$. Φ represents a wave with pulsating (between zero and a maximum) solitary profile. Φ in figures 3.1–3.5 and that in figures 5.1–5.5 seem to have a phase difference of 180°. (Values of x^1 in x-axis are in multiples of $\pi/2$ and values of time, x^4 are in multiples of $\pi/4$.)



Figures 4.1–4.5. Solutions for Ψ and χ in generalized Yang equations with $k' = 1$, $\varepsilon = -1$. ψ and χ represent solitary waves with oscillatory solitary profile. (Values of x^1 in x-axis are in multiples of $\pi/2$ and values of time, x^4 are in multiples of $\pi/4$.)



Figures 5. 1–5.5. Solutions for Φ in generalized Yang equations with $k' = 1/2$, $\varepsilon = -1$. Φ represents a wave with pulsating (between zero and a maximum) solitary profile. Φ in figures 3.1–3.5 and that in figures 5.1–5.5 seem to have a phase difference of 180°. (Values of x^1 in x-axis are in multiples of $\pi/2$ and values of time, x^4 are in multiples of $\pi/4$.)



Figures 6.1–6.5. Solutions for ψ and χ in generalized Yang equations with $k' = 1/2$, $\varepsilon = -1$. ψ and χ represent solitary waves with oscillatory profile. (Values of x^1 in x-axis are in multiples of $\pi/2$ and values of time, x^4 are in multiples of $\pi/4$.)

nonintegrability is Painlevé test for integrability [5]. Work in that direction is in progress.

Case III: $\varepsilon = -1$, $k' = 1$

Here the solutions have the same form as (3.19a,b,c) with the difference that here ζ satisfies a separate equation. Starting again from (3.16) and (3.18) for $\varepsilon = -1$ and $k' = 1$ and with (3.6) and (3.7) we arrive (for some particular values of integration constants) at

$$\Phi = \{\sqrt{k_7}/\sqrt{(k_3^2 + k_5^2)}\}\mathrm{sech}(\zeta\sqrt{k_7}/k_8), \tag{3.28a}$$

$$\psi = k_3\{\sqrt{k_7}/(k_3^2 + k_5^2)\}\tanh(\zeta\sqrt{k_7}/k_8), \tag{3.28b}$$

$$\chi = k_5\{\sqrt{k_7}/(k_3^2 + k_5^2)\}\tanh(\zeta\sqrt{k_7}/k_8), \tag{3.28c}$$

where ζ satisfies

$$\zeta_{11} + \zeta_{22} + \zeta_{33} - \zeta_{44} = 0, \tag{3.28d}$$

and a particular example of ζ satisfying (3.28d) is given in [3] and is represented by

$$\zeta = [(\sin\tau)/\tau]\cos t, \quad t = x^4, \tag{3.28e}$$

where

$$\tau^2 = (x^1)^2 + (x^2)^2 + (x^3)^2, \quad k_i \text{ are constants.} \tag{3.28f}$$

Using again the same procedure as stated in (3.19) we observe from the graphical representations (figures 3.1–3.5 and figures 4.1–4.5) of (3.28) that Φ represents a wave with pulsating (between zero and a maximum) solitary profile, while both ψ and χ represent waves with oscillatory solitary profile.

Case IV: $\varepsilon = -1, k' = 1/2$

Here the solutions have the same form as (3.27a,b,c) with the difference that here ζ satisfies a separate equation. Starting again from (3.16) and (3.18) for $\varepsilon = -1$ and $k' = 1/2$ and with (3.6) and (3.7) we arrive (for some particular values of integration constants) at

$$\Phi = [k_7/\{2(k_3^2 + k_5^2)\}][1 - \cos\{\zeta\sqrt{(k_3^2 + k_5^2)}/k_8\}], \tag{3.29a}$$

$$\psi = [k_3k_7/\{2(k_3^2 + k_5^2)\}][(\zeta/k_8) - \{1/\sqrt{(k_3^2 + k_5^2)}\}\sin\{\zeta\sqrt{(k_3^2 + k_5^2)}/k_8\}], \tag{3.29b}$$

$$\chi = [k_5k_7/\{2(k_3^2 + k_5^2)\}][(\zeta/k_8) - \{1/\sqrt{(k_3^2 + k_5^2)}\}\sin\{\zeta\sqrt{(k_3^2 + k_5^2)}/k_8\}]. \tag{3.29c}$$

Here again ζ satisfies

$$\zeta_{11} + \zeta_{22} + \zeta_{33} - \zeta_{44} = 0, \tag{3.29d}$$

and a particular example of ζ satisfying (3.29d) is given in [3] and is represented by

$$\zeta = [(\sin\tau)/\tau]\cos t, \quad t = x^4, \tag{3.29e}$$

where

$$\tau^2 = (x^1)^2 + (x^2)^2 + (x^3)^2, \quad k_i \text{ are constants.} \tag{3.29f}$$

Here also we use the same procedure as has been used in case of (3.19) for obtaining the graphical representations (figures 5.1–5.5 and figures 6.1–6.5) of (3.29). Same types of solutions as in (3.28) are observable for (3.29) as well. Φ represents a wave with pulsating (between zero and a maximum) solitary profile, while both ψ and χ represent waves with oscillatory solitary profile. However the Φ in (3.28) and in (3.29) seem to have a phase difference of 180°.

4. Summary

The study reveals that eqs (2), called here as the *generalized Yang equations*, have interesting physical solutions.

For $\varepsilon = 1$, the dependence on k' seems to be sensitive. For $\varepsilon = 1$, $k' = 1$ (Case I of this paper and ref. [3]) the solutions are relatively regular. The solutions represent spreading solitary profile (in case of Φ) and spreading wave packet (in case of ψ and χ). These profiles of solitary wave and wave packet tend to vanish as time tends to infinity. The situation becomes worse with $\varepsilon = 1$, $k' = 1/2$ (Case II). Here initially the solutions are, as usual, waves with solitary profile. But as time proceeds the solutions become chaotic (the transition being faster for Φ).

For $\varepsilon = -1$, the dependence on k' do not seem to be sensitive. The solutions are also much more regular than that in the case of $\varepsilon = 1$. For Case III ($\varepsilon = -1, k' = 1$) and Case IV ($\varepsilon = -1, k' = 1/2$), Φ is a wave with pulsating (between zero and a maximum) solitary profile, while both ψ and χ represent waves with oscillatory solitary profile. However, Φ for Case III ($\varepsilon = -1, k' = 1$) and Φ for Case IV ($\varepsilon = -1, k' = 1/2$) seems to have a phase difference of 180°.

One may note that solutions for $\varepsilon = -1$ are obtainable from the solutions for $\varepsilon = 1$ (for the same value of k') by performing Wick's rotation on the variable x^4, i.e. $x^4 \to ix^4$. Any kind of moving solutions in the category '$\varepsilon = 1$' will transform into breather-type solutions in the category '$\varepsilon = -1$'. Breathers are spatially localized time periodic solutions [13].

It may be noted that Case I ($\varepsilon = 1, k' = 1$) represents the celebrated equations given by (1.1) which was obtained by Yang [1] while discussing the condition of self-duality of $SU(2)$ gauge fields on Euclidean four-dimensional space. One can comment that the slightly variant forms of these equations and their physically significant solutions form a class of mathematical models usable (may be in future) in different physical situations.

Acknowledgement

One of the authors (PKC) thanks the University Grants Commission, India, for financial assistance. Both the authors thank Susanto Chakraborty of Central Drug Research Laboratory, Kolkata, India and Amritendu Chaterjee, Department of Chemistry, University of North Bengal, India for helpful discussions. The authors are grateful to the Department of Mathematics, University of North Bengal, India for cooperation. Finally, the authors thank the referee for some very useful and helpful comments.

References

[1] C N Yang, *Phys. Rev. Lett.* **33(7)**, 445 (1974); **38(24)**, 1377 (1977)
[2] D Ray, *J. Phys.* **A11(5)**, 995 (1978)
D Ray, *Phys. Lett.* **A70**, 274 (1980)
U K De and D Ray, *Phys. Lett.* **B101(5)**, 335 (1981)
P K Chanda and D Ray, *Phys. Rev.* **D31(12)**, 3183 (1985)
[3] S Chakraborty and P K Chanda, *Pramana – J. Phys.* **52(6)**, 579 (1999)
63(5), 1039 (2004)
[4] M Jimbo, M D Kruskal and T Miwa, *Phys. Lett.* **A92**, 59 (1982)
[5] J Weiss, M Tabor and G Carnevale, *J. Math. Phys.* **24(3)**, 522 (1983)
[6] S Chakraborty and P K Chanda, *Pramana – J. Phys.* **68(4)**, 535 (2007)
[7] A L Larsen, *Phys. Lett.* **A179**, 284 (1993)
[8] S Roychowdhury, *Phys. Lett.* **A159**, 311 (1991)
[9] J M Charap, *J. Phys.* **A9**, 1331 (1973)
[10] K C Sarkar, Govt. Training College, Malda 732 101, India, private communication
[11] D Ray, *J. Phys.* **A11(5)**, 995 (1978)
[12] U K De and D Ray, *Phys. Lett.* **B101(5)**, 335 (1981)
[13] Reviewer: Comments and suggestion
[14] K K Ghosh, D Ray and P Chanda, *Int. J. Theor. Phys.* **28(1)**, 111 (1989)