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Symmetries and conservation laws of the damped harmonic oscillator

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Abstract. We work with a formulation of Noether-symmetry analysis which uses the properties of infinitesimal point transformations in the space-time variables to establish the association between symmetries and conservation laws of a dynamical system. Here symmetries are expressed in the form of generators. We have studied the variational or Noether symmetries of the damped harmonic oscillator representing it by an explicitly time-dependent Lagrangian and found that a five-parameter group of transformations leaves the action integral invariant. Amongst the associated conserved quantities only two are found to be functionally independent. These two conserved quantities determine the solution of the problem and correspond to a two-parameter Abelian subgroup.

Keywords. Damped harmonic oscillator; explicitly time-dependent Lagrangian representation; Noether symmetries; conservation laws.

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1. Introduction

It is well-known that the formal description for the connection between symmetry properties and conserved quantities of a dynamical system is provided by Noether's theorem [1]. This theorem asserts that if a given differential equation representing the time evolution of some physical system follows from the variational principle, then a continuous symmetry transformation (point, contact or higher-order) that leaves the action functional invariant yields a conservation law. Thus studies in symmetries and conservation laws of a physical system using this theorem require the associated equation of motion to follow from the action principle [2].

The object of the present work is to apply Noether's theorem on the equation

$$\ddot{x} + \lambda\dot{x} + \omega^2 x = 0, \quad x = x(t) \tag{1}$$

and thereby envisages a study for the connection between symmetries and conservation laws of the system represented by it. Equation (1) describes the motion of a harmonic oscillator of natural frequency ω embedded in a viscous medium characterized by the frictional coefficient λ. Lanczos [3] observed that the forces of friction are outside the realm of variational principle although the Newtonian scheme has no difficulty to accommodate them. This observation tend to present one of the main difficulties in applying Noether's theorem on the damped harmonic oscillator.

Being nonself-adjoint (1) does not satisfy the Helmholtz criterion [4] to have a Lagrangian representation. However, multiplying (1) by $\mathrm{e}^{\lambda t}$ we can convert it to the self-adjoint form such that

$$L = \mathrm{e}^{\lambda t}\left(\frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2\right) \tag{2}$$

provides an admissible Lagrangian [5] for the damped harmonic oscillator.

The first derivative term in (1) can formally be eliminated changing the dependent variables, by $x(t) = z(t)\mathrm{e}^{-\frac{1}{2}\lambda t}$. Under this point transformation the form of Lagrange's equations is invariant [2]. We note that the relation between x and z noted here is a special instance of the transformation [6] $y(x) = z(x)\exp[-\frac{1}{2}\int^x P(t)\mathrm{d}t]$ used to recast the general linear second-order ordinary differential equation $y'' + P(x)y' + Q(x)y = 0$ into the canonical form $z'' + h(x)z = 0$ with $h(x) = Q(x) - \frac{1}{2}P'(x) - \frac{1}{4}P^2(x)$. On a number of occasions, Kaushal [7] used the transformation of ref. [6] to study Ermakov systems with particular emphasis on the derivation of dynamical invariants for time-dependent damped systems. The Lagrangian in (2) is explicitly time-dependent. Recently, Chandrasekar *et al* [8] used a modified Prelle-Singer approach to construct explicitly time-independent Lagrangian for the damped harmonic oscillator employing the first integral of (1), which are also explicitly time-independent. Although the approach followed in ref. [8] appears to be mathematically elegant, the results obtained are not completely new. For example, more than a decade ago, while investigating the geometrical origin of the Lagrangian for dissipative systems in the context of global geometry de Ritis *et al* [9] found a Lagrangian for (1) which is explicitly time-independent. Relatively recently, the corresponding time-independent integral of the motion was noted by two of us [10]. We shall, however, use the Lagrangian in (2) to study the Noether's symmetries and concomitant conservation laws for the damped harmonic oscillator.

In the next section we outline our scheme for symmetry analysis using Noether's theorem. Here we work with the generalized coordinates written as $q_i(t)$. In §3 we specialize ourselves to the Cartesian coordinates as used in (1) and present the main results of this work for the relation between symmetries and conservation laws. Our results also include the generators of the symmetry transformations together with the algebra satisfied by them. Moreover, we present all appropriate results for the constants of the motion. Finally, in §4, we summarize our outlook on the present work.

2. Symmetries and conservation laws

The key element for the Noether-symmetry analysis consists of studying the infinitesimal criterion for the invariance of a variational problem under a group of transformations that map 'points' in configuration space and time into their infinitesimal neighbourhood, i.e. $(\vec{q}, t) \to (\vec{q}\,', t')$. Here $\vec{q} = \{q_i\}$, $i = 1, ..., n$, stands for the set of generalized coordinates representing the dynamical system under consideration and, as usual, t is the time parameter. Formally, such point transformations are represented as

$$t' = t + \delta t, \quad \delta t = \epsilon\xi(\vec{q}, t) \tag{3a}$$

and

$$q_i' = q_i + \delta q_i, \quad \delta q_i = \epsilon\eta_i(\vec{q}, t) \tag{3b}$$

with ϵ, an infinitesimal parameter. The generator of the infinitesimal point transformations in (3) is given by

$$U = \xi(\vec{q}, t)\frac{\partial}{\partial t} + \sum_{i=1}^{n} \eta_i(\vec{q}, t)\frac{\partial}{\partial q_i} \tag{4}$$

and represents a vector field on $(\vec{q}, t)$ since it assigns a tangent vector to each point within $(\vec{q}, t)$. The first prolongation of U written as [11]

$$U^{(1)} = U + \sum_{i=1}^{n} (\dot{\eta}_i(\vec{q}, t) - \dot{\xi}(\vec{q}, t)\dot{q}_i)\frac{\partial}{\partial \dot{q}_i} \tag{5}$$

is such that

$$\delta v = \epsilon U^{(1)} v(\vec{q}, \dot{\vec{q}}, t) \tag{6}$$

represents the variation of an arbitrary well-behaved function $v(\vec{q}, \dot{\vec{q}}, t)$ in the velocity phase-space.

To write the Noether's theorem we consider, among the general set of point transformations defined by (3), only those that leave the action $L\mathrm{d}t$ invariant and we demand that

$$L(\vec{q}_i, \dot{\vec{q}}_i, t) \stackrel{!}{=} L'(\vec{q}\,'_i, \dot{\vec{q}}\,'_i, t'). \tag{7}$$

In order to satisfy the condition in (7), we have allowed the Lagrangian to change its functional form ($L \to L'$). The functional relation between L' and L may be expressed by introducing a gauge function $f(\vec{q}, t)$ [12] such that

$$L'(\vec{q}\,'_i,\ \dot{\vec{q}}\,'_i, t') = L(\vec{q}\,'_i, \dot{\vec{q}}\,'_i, t') - \epsilon\frac{\mathrm{d}f(\vec{q}, t)}{\mathrm{d}t}. \tag{8}$$

From (7) and (8) we have

$$L(\vec{q}\,'_i, \dot{\vec{q}}\,'_i, t')\mathrm{d}t' = L(\vec{q}_i, \dot{\vec{q}}_i, t)\mathrm{d}t + \epsilon\frac{\mathrm{d}f(\vec{q}, t)}{\mathrm{d}t}\mathrm{d}t. \tag{9}$$

On the other hand, using L for v in (6) we have

$$L(\vec{q}\,'_i, \dot{\vec{q}}\,'_i, t') = L(\vec{q}_i, \dot{\vec{q}}_i, t) + \epsilon U^{(1)} L(\vec{q}_i, \dot{\vec{q}}_i, t). \tag{10}$$

From (9) and (10) it is easy to see that

$$\frac{\mathrm{d}f(\vec{q},t)}{\mathrm{d}t} = \dot{\xi}L + \xi\frac{\partial L}{\partial t} + \sum_{i=1}^{n}\left(\eta_i\frac{\partial L}{\partial q_i} + (\dot{\eta}_i - \dot{\xi}\dot{q}_i)\frac{\partial L}{\partial \dot{q}_i}\right). \tag{11}$$

In writing (11) we have made use of the results in (4) and (5). We, therefore, infer that the action is invariant under those point transformations whose constituents ξ and η_i satisfy (11). The terms of (11) can be rearranged to write

$$\frac{\mathrm{d}I}{\mathrm{d}t} + \sum_{i=1}^{n}(\xi\dot{q}_i - \eta_i)\left(\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \tag{12}$$

with

$$I = \sum_{i=1}^{n}(\xi\dot{q}_i - \eta_i)\frac{\partial L}{\partial \dot{q}_i} - \xi L + f(\vec{q},t). \tag{13}$$

Along the trajectory of the system, the Euler–Lagrange equations hold good such that the second term in (12) is zero. Thus I given in (13) is a conserved quantity or a constant of the motion. The invariant given by (13) and the differential equation for the gauge function in (11) are commonly stated as the Noether's theorem.

In the Hamiltonian formulation of classical mechanics the Noether's invariant can be written as

$$I = \xi(\vec{q},t)H(\vec{q},\vec{p},t) - \sum_{i=1}^{n}\eta_i(\vec{q},t)p_i + f(\vec{q},t). \tag{14}$$

We have obtained (14) from (13) using the relation between H and L as given by the usual Legendre transformation

$$L(\vec{q},\dot{\vec{q}},t) = \sum_{i=1}^{n}p_i\dot{q}_i - H(\vec{q},\vec{p},t), \quad p_i = \frac{\partial L}{\partial \dot{q}_i}. \tag{15}$$

In terms of the Hamiltonian the differential equation (11) for $f(\vec{q},t)$ now reads as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\xi(\vec{q},t)H(\vec{q},\vec{p},t) - \sum_{i=1}^{n}\eta_i(\vec{q},t)p_i + f(\vec{q},t)\right] = 0. \tag{16}$$

Clearly, the expression inside the squared bracket in (16) stands for the conserved quantity given in (14). Equation (16) provides a natural basis to carry out Noether-symmetry analysis for Newtonian systems.

3. The damped harmonic oscillator

The Lagrangian in (2) is explicitly time-dependent. Usual formulation of the Noether's theorem runs into trouble when applied to the symmetry analysis of systems characterized by such Lagrangians [13]. However, we shall presently see that the form of Noether's theorem as given by (14) and (16) is free from this difficulty. The Hamiltonian for the Lagrangian in (2) is given by

$$H = \frac{1}{2}\left(p_x^2 \mathrm{e}^{-\lambda t} + \omega^2 x^2 \mathrm{e}^{\lambda t}\right) \tag{17}$$

with the canonical momentum

$$p_x = \dot{x}\mathrm{e}^{\lambda t}. \tag{18}$$

For H in (17), (16) can be written in the form

$$\begin{aligned} &\frac{\partial f}{\partial t} + p_x \mathrm{e}^{-\lambda t}\frac{\partial f}{\partial x} + \frac{1}{2}\left(\frac{\partial \xi}{\partial t} + p_x \mathrm{e}^{-\lambda t}\frac{\partial \xi}{\partial x}\right)\left(p_x^2 \mathrm{e}^{-\lambda t} + \omega^2 x^2 \mathrm{e}^{\lambda t}\right) \\ &+ \frac{\lambda}{2}\xi(-p_x^2 \mathrm{e}^{-\lambda t} + \omega^2 x^2 \mathrm{e}^{\lambda t}) - \left(\frac{\partial \eta}{\partial t} + p_x \mathrm{e}^{-\lambda t}\frac{\partial \eta}{\partial x}\right) p_x + \omega^2 \eta x \mathrm{e}^{\lambda t} = 0. \end{aligned} \tag{19}$$

In writing (19) we have made use of the canonical equations

$$\dot{x} = \frac{\partial H}{\partial p_x} = p_x \mathrm{e}^{-\lambda t} \quad \text{and} \quad \dot{p}_x = -\frac{\partial H}{\partial x} = -\omega^2 x \mathrm{e}^{\lambda t}. \tag{20}$$

Equation (19) can be globally satisfied for any particular choice of the momenta provided the sum of momentum-independent terms, the coefficients of linear, quadratic and cubic terms in p_x vanish separately. Following this viewpoint we write

$$p_x^0: \quad \frac{\partial f}{\partial t} + \frac{\omega^2}{2}x^2 \mathrm{e}^{\lambda t}\frac{\partial \xi}{\partial t} + \frac{\lambda \omega^2}{2}x^2 \mathrm{e}^{\lambda t}\xi + \omega^2 \eta x \mathrm{e}^{\lambda t} = 0, \tag{21a}$$

$$p_x^1: \quad \mathrm{e}^{-\lambda t}\frac{\partial f}{\partial x} + \frac{\omega^2}{2}x^2 \mathrm{e}^{-2\lambda t}\frac{\partial \xi}{\partial x} - \frac{\partial \eta}{\partial t} = 0, \tag{21b}$$

$$p_x^2: \quad \frac{1}{2}\mathrm{e}^{-\lambda t}\frac{\partial \xi}{\partial t} - \frac{\lambda}{2}\mathrm{e}^{-\lambda t}\xi - \mathrm{e}^{-\lambda t}\frac{\partial \eta}{\partial x} = 0 \tag{21c}$$

and

$$p_x^3: \quad \frac{1}{2}\mathrm{e}^{-2\lambda t}\frac{\partial \xi}{\partial x} = 0. \tag{21d}$$

Equation (21a) signifies that we have equated the sum of p-independent terms to zero while (21b)–(21d) have been obtained by equating the sum of the coefficients of p_x^1, p_x^2 and p_x^3 to zero. From (21d) we see that ξ is not a function of x. Thus

$$\xi(x,t) \equiv \xi(t) = \beta(t) \quad \text{(say)}. \tag{22}$$

In view of (22), we can write (21a), (21b) and (21c) as

$$\frac{\partial f}{\partial t} + \frac{\omega^2}{2} x^2 \mathrm{e}^{\lambda t} \dot{\beta} + \frac{\lambda \omega^2}{2} x^2 \mathrm{e}^{\lambda t} \beta + \omega^2 \eta x \mathrm{e}^{\lambda t} = 0, \tag{23a}$$

$$\mathrm{e}^{-\lambda t} \frac{\partial f}{\partial x} - \frac{\partial \eta}{\partial t} = 0, \tag{23b}$$

and

$$\frac{1}{2} \mathrm{e}^{-\lambda t} \dot{\beta} - \frac{\lambda}{2} \mathrm{e}^{-\lambda t} \beta - \mathrm{e}^{-\lambda t} \frac{\partial \eta}{\partial x} = 0. \tag{23c}$$

We can solve (23c) for η to write

$$\eta = \frac{1}{2} x \dot{\beta} - \frac{\lambda}{2} x \beta + \psi(t), \tag{24}$$

with $\psi(t)$, a constant of integration. From (23b) and (24) we have

$$f = \left(\frac{1}{4} x^2 \ddot{\beta} - \frac{\lambda}{4} x^2 \dot{\beta} + \dot{\psi} x \right) \mathrm{e}^{\lambda t}. \tag{25}$$

Using the expressions for η and f from (24) and (25) in (14) we obtain the invariant I in the form

$$I = I_\beta + I_\psi, \tag{26}$$

where

$$I_\beta = \frac{1}{4} (x^2 \ddot{\beta} - \lambda x^2 \dot{\beta}) \mathrm{e}^{\lambda t} - \frac{1}{2} x p_x \dot{\beta} + \frac{1}{2} \left(\lambda x p_x + p_x^2 \mathrm{e}^{-\lambda t} + \omega^2 x^2 \mathrm{e}^{\lambda t} \right) \beta \tag{27a}$$

and

$$I_\psi = x \dot{\psi} \mathrm{e}^{\lambda t} - \psi p_x. \tag{27b}$$

In writing (27) we also used (17) and (22). Each of the I's in (27) is expected to form a separate constant. This can be seen as follows.

Substituting the values of η and f in (23a) we get

$$J_\beta + J_\psi = 0, \tag{28}$$

where

$$J_\beta = \frac{1}{4} x^2 \mathrm{e}^{\lambda t} \left(\dddot{\beta} + \left(\omega^2 - \frac{\lambda^2}{4} \right) \dot{\beta} \right) \tag{29a}$$

and

$$J_\psi = x \mathrm{e}^{\lambda t} \left(\ddot{\psi} + \lambda \dot{\psi} + \omega^2 \psi \right). \tag{29b}$$

Using the appropriate Hamilton's equations it is easy to verify that

$$\int J_\beta \mathrm{d}t = I_\beta \tag{30a}$$

and

$$\int J_\psi \mathrm{d}t = I_\psi . \tag{30b}$$

Equations (30a) and (30b) verify our conjecture.

The generator of the infinitesimal transformations leading to the conserved quantities in (27a) and (27b) are obtained by using the values of $\xi(t)$ and η from (22) and (24) in (4). Thus we have

$$U = U_\beta + U_\psi , \tag{31}$$

where

$$U_\beta = \beta \frac{\partial}{\partial t} - \frac{\lambda}{2} x \beta \frac{\partial}{\partial x} + \frac{1}{2} x \dot{\beta} \frac{\partial}{\partial x} \tag{32a}$$

and

$$U_\psi = \psi \frac{\partial}{\partial x} . \tag{32b}$$

To find the symmetries and corresponding conservation laws we first need to calculate the special values of $\beta(t)$ and $\psi(t)$ from

$$J_\beta = 0 \tag{33a}$$

and

$$J_\psi = 0. \tag{33b}$$

Equations (33a) and (33b) give

$$\beta = 1 \quad \text{and} \quad \beta^\pm = \mathrm{e}^{\pm 2i\bar{\omega}t} \tag{34a}$$

and

$$\psi^\pm = \mathrm{e}^{-\frac{\lambda}{2} \pm i\bar{\omega}t} , \tag{34b}$$

where $\bar{\omega} = \sqrt{\omega^2 - \frac{\lambda^2}{4}}$. Equations in (34) clearly show that we are interested in the symmetries of the underdamped oscillator. From (27a) and (32a) we obtain, for $\beta = 1$, the conserved quantity and the associated generator as

$$I_{\beta=1} = \frac{1}{2} \left(\dot{x}^2 + x^2 + \lambda x \dot{x} \right) \mathrm{e}^{\lambda t} \tag{35}$$

and

$$U_{\beta=1} = \frac{\partial}{\partial t} - \frac{\lambda}{2} x \frac{\partial}{\partial x}. \tag{36}$$

For $\lambda = 0$, $I_{\beta=1}$ represents the total energy of the harmonic oscillator with $U_{\beta=1}$ as the time translation operator. For finite values of λ, however, $I_{\beta=1}$ stands for the energy function or Jacobi's integral [2] of the system. Results similar to those in (35), (36) for $\beta^{\pm}$, $\psi^{\pm}$ are given below.

For $\beta^+ = \mathrm{e}^{+2i\bar{\omega}t}$, the invariant I_β gives rise to two real invariants

$$\begin{aligned} I_{\beta^1} = \mathrm{Re}\, I_{\beta^+=\mathrm{e}^{+2i\bar{\omega}t}} = &\left(\frac{1}{2}p_x^2 \mathrm{e}^{-\lambda t} - \frac{1}{2}\omega^2 x^2 \mathrm{e}^{\lambda t} + \frac{\lambda^2}{4} x^2 \mathrm{e}^{\lambda t} + \frac{\lambda}{2} x p_x\right) \cos 2\bar{\omega} t \\ &+ \bar{\omega}\left(\frac{\lambda}{2} x^2 \mathrm{e}^{\lambda t} + x p_x\right) \sin 2\bar{\omega} t \end{aligned} \tag{37}$$

and

$$\begin{aligned} I_{\beta^2} = \mathrm{Im}\, I_{\beta^+=\mathrm{e}^{+2i\bar{\omega}t}} = &\left(\frac{1}{2}p_x^2 \mathrm{e}^{-\lambda t} - \frac{1}{2}\omega^2 x^2 \mathrm{e}^{\lambda t} + \frac{\lambda^2}{4} x^2 \mathrm{e}^{\lambda t} + \frac{\lambda}{2} x p_x\right) \sin 2\bar{\omega} t \\ &- \bar{\omega}\left(\frac{\lambda}{2} x^2 \mathrm{e}^{\lambda t} + x p_x\right) \cos 2\bar{\omega} t. \end{aligned} \tag{38}$$

The generators of I_{β^1} and I_{β^2} as found from (32a) are given by

$$U_{\beta^1} = \mathrm{Re}\, U_{\beta^+=\mathrm{e}^{+2i\bar{\omega}t}} = \cos 2\bar{\omega} t \left(\frac{\partial}{\partial t} - \frac{\lambda}{2} x \frac{\partial}{\partial x}\right) - x\bar{\omega} \sin 2\bar{\omega} t \,\frac{\partial}{\partial x} \tag{39}$$

and

$$U_{\beta^2} = \mathrm{Im}\, U_{\beta^+=\mathrm{e}^{+2i\bar{\omega}t}} = \sin 2\bar{\omega} t \left(\frac{\partial}{\partial t} - \frac{\lambda}{2} x \frac{\partial}{\partial x}\right) + x\bar{\omega} \cos 2\bar{\omega} t \,\frac{\partial}{\partial x}. \tag{40}$$

For $\beta^- = \mathrm{e}^{-2i\bar{\omega}t}$, we have

$$I_{\beta^3} = \mathrm{Re}\, I_{\beta^-=\mathrm{e}^{-2i\bar{\omega}t}} = I_{\beta^1}, \tag{41}$$

$$I_{\beta^4} = \mathrm{Im}\, I_{\beta^-=\mathrm{e}^{-2i\bar{\omega}t}} = -I_{\beta^2} \tag{42}$$

and

$$U_{\beta^3} = \mathrm{Re}\, U_{\beta^-=\mathrm{e}^{-2i\bar{\omega}t}} = U_{\beta^1}, \tag{43}$$

$$U_{\beta^4} = \mathrm{Im}\, U_{\beta^-=\mathrm{e}^{-2i\bar{\omega}t}} = -U_{\beta^2}. \tag{44}$$

The results for the invariants I_Ψ and generators U_Ψ for values of ψ given in (34b) are obtained as

$$I_{\Psi^1} = \mathrm{Re}\, I_{\psi^+ = \mathrm{e}^{(-\frac{\lambda}{2}+i\bar{\omega})t}} = -\left(\frac{\lambda}{2} x \mathrm{e}^{\frac{\lambda}{2}t} + p_x \mathrm{e}^{-\frac{\lambda}{2}t}\right)\cos\bar{\omega}t \\ -\bar{\omega} x \mathrm{e}^{\frac{\lambda}{2}t}\sin\bar{\omega}t, \tag{45}$$

$$I_{\Psi^2} = \mathrm{Im}\, I_{\psi^+ = \mathrm{e}^{(-\frac{\lambda}{2}+i\bar{\omega})t}} = -\left(\frac{\lambda}{2} x \mathrm{e}^{\frac{\lambda}{2}t} + p_x \mathrm{e}^{-\frac{\lambda}{2}t}\right)\sin\bar{\omega}t \\ +\bar{\omega} x \mathrm{e}^{\frac{\lambda}{2}t}\cos\bar{\omega}t, \tag{46}$$

$$U_{\Psi^1} = \mathrm{Re}\, U_{\psi^+ = \mathrm{e}^{(-\frac{\lambda}{2}+i\bar{\omega})t}} = \mathrm{e}^{-\frac{\lambda}{2}t}\cos\bar{\omega}t\frac{\partial}{\partial x}, \tag{47}$$

$$U_{\Psi^2} = \mathrm{Im}\, U_{\psi^+ = \mathrm{e}^{(-\frac{\lambda}{2}+i\bar{\omega})t}} = \mathrm{e}^{-\frac{\lambda}{2}t}\sin\bar{\omega}t\frac{\partial}{\partial x}, \tag{48}$$

$$I_{\Psi^3} = \mathrm{Re}\, I_{\psi^- = \mathrm{e}^{(-\frac{\lambda}{2}-i\bar{\omega})t}} = I_{\Psi^1}, \tag{49}$$

$$I_{\Psi^4} = \mathrm{Im}\, I_{\psi^- = \mathrm{e}^{(-\frac{\lambda}{2}-i\bar{\omega})t}} = -I_{\Psi^2}, \tag{50}$$

$$U_{\Psi^3} = \mathrm{Re}\, U_{\psi^- = \mathrm{e}^{(-\frac{\lambda}{2}-i\bar{\omega})t}} = U_{\Psi^1} \tag{51}$$

and

$$U_{\Psi^4} = \mathrm{Im}\, U_{\psi^- = \mathrm{e}^{(-\frac{\lambda}{2}-i\bar{\omega})t}} = -U_{\Psi^2}. \tag{52}$$

In the above the odd and even superscripts on β and ψ refer to real and imaginary parts of the invariants and generators as the case may be. Looking closely at eqs (37)–(52) we find that there are only five linearly independent group generators given by

$$G_1 = U_{\beta^1}, \quad G_2 = U_{\beta^2}, \quad G_3 = U_{\Psi^1}, \quad G_4 = U_{\Psi^2} \quad \text{and} \quad G_5 = U_{\beta=1}. \tag{53}$$

We have already seen that G_5 for $\lambda = 0$ represents the time translation operator and the corresponding conserved quantity is the total energy of the undamped oscillator. Similarly, in the limit of no damping all the group generators in (53) coincide with those given by Lutzky [14]. The algebra of our five-parameter Lie group is given in table 1.

To each of the one-parameter subgroups in table 1 there corresponds a constant of the motion (C_i). More explicitly, we write

$$C_1 = I_{\beta^1}, \quad C_2 = I_{\beta^2}, \quad C_3 = I_{\Psi^1}, \quad C_4 = I_{\Psi^2} \quad \text{and} \quad C_5 = I_{\beta=1}. \tag{54}$$

In (54) the conserved quantities that can be treated as independent are C_3 and C_4 because it is easy to show that

Table 1. Commutation relations for the generators in (53), each element in the table being represented by $G_{ij} = [G_i\,, G_j]$.

	G_1	G_2	G_3	G_4	G_5
G_1	0	$2\bar{\omega}G_5$	$\bar{\omega}G_4$	$\bar{\omega}G_3$	$2\bar{\omega}G_2$
G_2	$-2\bar{\omega}G_5$	0	$-\bar{\omega}G_3$	$\bar{\omega}G_4$	$-2\bar{\omega}G_1$
G_3	$-\bar{\omega}G_4$	$\bar{\omega}G_3$	0	0	$\bar{\omega}G_4$
G_4	$-\bar{\omega}G_3$	$-\bar{\omega}G_4$	0	0	$-\bar{\omega}G_3$
G_5	$-2\bar{\omega}G_2$	$2\bar{\omega}G_1$	$-\bar{\omega}G_4$	$\bar{\omega}G_3$	0

$$C_1 = \frac{1}{2}\left(C_3^2 - C_4^2\right), \tag{55a}$$

$$C_2 = \frac{1}{2}C_3C_4 \tag{55b}$$

and

$$C_5 = \frac{1}{2}\left(C_3^2 + C_4^2\right). \tag{55c}$$

Elimination of p_x from C_3 and C_4 yields

$$x = \frac{\mathrm{e}^{-\frac{\lambda}{2}t}}{\bar{\omega}}(C_4\cos\bar{\omega}t - C_3\sin\bar{\omega}t). \tag{56}$$

Since x represents the general solution of the damped harmonic oscillator in (1), the system is completely specified by the two-parameter Abelian symmetry group generated by G_3 and G_4.

4. Concluding remarks

Noether's theorem provides a one-to-one correspondence between the symmetry properties and conserved quantities of a dynamical system. We have chosen to work with a theoretical framework which attributes the reason for this to the properties of some auxiliary equations which can always be written in the form of a total time derivative.

As with the case of uncoupled oscillator [14] we found that a five-parameter group of transformations leaves the action integral of the damped harmonic oscillator invariant. This results in five conserved quantities. Only two of these quantities determine the solution and correspond to a two-parameter Abelian subgroup.

The conserved quantity in (35) was noticed earlier by Lemos [15] while deriving a Hamilton–Jacobi method for the damped harmonic oscillator. The same result for the energy function or Jacobi integral was found by Tapia [13] by adopting the Noether's theorem to parametrized systems in which time is treated as a configuration-space variable. Here we have shown that the direct approach of Noether's theorem yields Jacobi integral in a rather straightforward manner.

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