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The dynamical mixing of light and pseudoscalar fields

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Abstract. We solve the general problem of mixing of electromagnetic and scalar or pseudoscalar fields coupled by axion-type interactions $\mathcal{L}_{\rm int} = g_\phi \phi\, \epsilon_{\mu\nu\alpha\beta} F^{\mu\nu} F^{\alpha\beta}$. The problem depends on several dimensionful scales, including the magnitude and direction of background magnetic field, the pseudoscalar mass, plasma frequency, propagation frequency, wave number, and finally the pseudoscalar coupling. We apply the results to the first consistent calculations of the mixing of light propagating in a background magnetic field of varying directions, which show a great variety of fascinating resonant and polarization effects.

Keywords. Axions; pseudoscalar–photon mixing; wave propagation in background magnetic field; polarization.

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0. Introduction

For about 20 years the mixing of light and pseudoscalar fields in propagation has been studied with fascination [1–10]. The subject generated renewed attention in the context of cosmological observables that can probe exceedingly small couplings [11–14]. One approach proposes that the dimming of supernova light might be explained by transition of light into unobserved pseudoscalar, or 'axion', modes [15], although this effect might be limited by observations of radio galaxies [16]. It has also been pointed out that pseudoscalar field can generate magnetic fields due to their coupling with photons [17]. Polarization observables are even more sensitive than intensity: for coupling constants many orders of magnitude too small to cause dimming, the cumulative evolution of phase shifts can generate phenomena clearly violating the Maxwell equations in plasmas [18]. Several laboratory experiments have also sought the spontaneous resonant conversion of dark matter axions to photons, and explored the possibilities of conversion in lab-made magnetic fields.

There is a well-established theoretical technology of mixing light with a background magnetic field transverse to propagation. Yet despite long study, we do not know of any complete solution to the mixing problem which depends on a possible variables. And there is no wonder, as there are many dimensionful scales, including the magnitude and direction of background magnetic field, the pseudoscalar mass, plasma frequency, propagation frequency, wave number, and finally the pseudoscalar coupling. By approaching the problem with new methods here, we will be able to survey various limits used in the literature and also present a convincing resolution of the dynamics in a slowly varying background field of arbitrary direction.

The basic Lagrangian assumes a pseudoscalar [18a] field ϕ coupled to the electromagnetic field strength $F_{\mu\nu}$ by the action

$$S = \int \mathrm{d}^4x\sqrt{-g}\Big[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + g_\phi\phi\,\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta} + j_\mu A^\mu + \frac{1}{2}\partial_\mu\phi\partial_\mu\phi - \frac{1}{2}m_\phi^2\phi^2 - V(\phi)\Big]. \quad (1)$$

We include a coupling to a current j_μ for completeness. For the purpose of linear propagation, the potential $V(\phi)$ can be ignored as a small perturbation, and the metric g can be replaced by a given background form. Certain non-local plasma effects, described by the plasma frequency, Faraday rotation, etc., may also need to be incorporated. The limit on the coupling g_ϕ may be obtained by considering the cooling rates of red giants. The current limits imply $g_\phi < 6 \times 10^{-11}$ GeV^{-1} if we assume negligible pseudoscalar mass m_ϕ [19,20]. These limits can be evaded if the pseudoscalars have sufficiently large self-coupling [21]. The applications we have in mind are propagation of electromagnetic radiation over cosmological distances or through the local supercluster. Here the magnetic fields are relatively small, of the order of 10^{-7} to 10^{-9} Gauss with plasma density n_e in the range 10^{-6} to 10^{-8} cm^{-3}. Another interesting application is propagation through the pulsar magnetosphere. Here the magnetic fields are very strong, of the order of 10^{12} Gauss with the plasma density of order 10^{11} cm^{-3}.

By translational symmetry, certain eigenmodes will evolve like e^{ik_iz} in propagation over a distance z, where k_i are wave numbers to be determined in terms of the frequency ω. This is simple and obvious. Yet one might claim the opposite that k should be fixed, while frequency ω remains to be determined, as so common in quantum mechanics and neutrino oscillations. Indeed some literatures solve for ω eigenvalues without discussion. Physics is local, and the time dependence of the waves is fixed by the known time dependence, $\mathrm{e}^{-i\omega t}$, of the source. By Huygen's principle, i.e. the use of causal Green functions, one may then determine the wave numbers k_i for on-shell propagation. There are no sources of fixed k, and so one is required to solve for k_i as a function of ω, just as in careful work on neutrino oscillations [22,23]. We also give extra attention to maintaining gauge invariance, which we have not seen before. The physics turns out to be surprisingly intricate.

We apply the revised propagation equations to the interesting problem of light traveling in a background magnetic field of varying direction. For the parameter values of axion masses, magnetic fields and couplings commonly assumed, the magnitude of new changes is often non-negligible. This fact aside, the results themselves are fascinating, and full of remarkable complexity and structure, somewhat like a generalized version of the resonant propagation of neutrinos. We think this is very interesting: The possible existence of axions can be probed in polarization observables for parameters ranges far smaller than will cause a dimming of light by direct conversion. Although axion-related dimming is given some credence it is usually assumed that there are no exotic polarization effects to be observed. We find that the *absence* of exotic polarization effects would be able to rule out the light-dimming hypothesis. Confrontation with data on polarization, of course, needs a detailed study of many potential backgrounds to any signal, and would go beyond the scope of this paper. Our main task is simply to get the propagation equations resolved once and for all.

1. Gauge invariant methods

1.1 *Equations for E and ϕ*

To eliminate difficulties of gauge invariance we first obtain the non-covariant form of the Maxwell equations with no approximations [24]:

$$\nabla \cdot \vec{E} = g_\phi \nabla\phi \cdot (\vec{\mathcal{B}} + \vec{B}) + \rho, \tag{2}$$

$$\nabla \times \vec{E} + \frac{\partial(\vec{\mathcal{B}} + \vec{B})}{\partial t} = 0, \tag{3}$$

$$\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = g_\phi \left(\vec{E} \times \nabla\phi - (\vec{\mathcal{B}} + \vec{B}) \frac{\partial \phi}{\partial t} \right) + \vec{j}, \tag{4}$$

$$\nabla \cdot (\vec{\mathcal{B}} + \vec{B}) = 0. \tag{5}$$

Here $\mathcal{B}_i + B_i = \frac{1}{2}\epsilon_{ijk}F^{jk}$ and $E_i = F^{0i}$ are the usual magnetic and electric fields. Here $\vec{\mathcal{B}}$ and $\vec{B}$ represent the magnetic field due to the background and due to the electromagnetic wave respectively. The background field is assumed to be independent of time.

In anticipation we note that the revised 'Gauss' law' (eq. (2)) couples the longitudinal electric field to $\vec{\nabla}\phi$. This creates a qualitative change compared to light in free space, where the longitudinal mode does not normally propagate. If ϕ propagates we now have a propagating longitudinal light field. If there is a plasma, then the ordinary Gauss' law becomes $\vec{\nabla} \cdot \epsilon E = \rho_{\text{free}}$, where ϵ is the dielectric constant (or 'permittivity'). Since $\epsilon = \epsilon(\omega)$ is not local in the time domain, we will incorporate it below in the Fourier-transformed equations.

The pseudoscalar field's equation of motion is

$$\frac{\partial^2 \phi}{\partial t^2} - \nabla^2\phi + m_\phi^2 \phi = -g_\phi \vec{E} \cdot (\vec{\mathcal{B}} + \vec{B}). \tag{6}$$

Gauge invariance is explicit, and one can check current conservation directly,

$$\vec{\nabla} \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0.$$

Assume that $\vec{\mathcal{B}}$ solves the zeroeth-order Maxwell equations with no ϕ background. The linearized equations for $\vec{E}/c \ll \vec{\mathcal{B}}$, $\vec{B} \ll \vec{\mathcal{B}}$ are

$$\nabla \cdot \vec{E} = g_\phi \nabla\phi \cdot \vec{\mathcal{B}} + \rho, \tag{7}$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0, \tag{8}$$

$$\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = -g_\phi \vec{\mathcal{B}} \frac{\partial \phi}{\partial t} + \vec{j}, \tag{9}$$

$$\nabla \cdot \vec{B} = 0. \tag{10}$$

Proceed to get a wave equation for $\vec{E}$ by taking the curl of Faraday's law,

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\vec{\nabla}^2 \vec{E} + \vec{\nabla}\vec{\nabla} \cdot \vec{E} = -\frac{\partial}{\partial t} \vec{\nabla} \times \vec{B},$$

and substituting eqs (2) and (4), we obtain

$$-\vec{\nabla}^2 \vec{E} + \frac{\partial^2 \vec{E}}{\partial t^2} = g_\phi \vec{\mathcal{B}} \frac{\partial^2 \phi}{\partial t^2} - g_\phi \vec{\nabla}(\vec{\nabla}\phi \cdot \vec{\mathcal{B}}). \tag{11}$$

In this equation the longitudinal part of $\vec{E}$ mixes with $\vec{\nabla}\phi$. Take the transverse (sub-T) and longitudinal parts (sub-L) of the electric wave equation, for wave number $\vec{k}$, with $E_{\mathrm{L}} = \hat{k} \cdot \vec{E}$:

$$(k^2 - \omega^2) E_{\mathrm{T}} = -g_\phi \omega^2 \mathcal{B}_{\mathrm{T}} \phi, \tag{12}$$

$$(k^2 - \omega^2) E_{\mathrm{L}} = g_\phi (k^2 - \omega^2) \mathcal{B}_{\mathrm{L}} \phi. \tag{13}$$

There clearly exists no gauge in which the longitudinal electric field decouples from the problem. If we limit the study to $\vec{\nabla}\phi \cdot \vec{\mathcal{B}} = 0$, then Gauss' law makes $\vec{E}$ transverse. Everything in the literature is perfectly consistent.

1.2 *Equations for D and ϕ*

Another method is needed when $\vec{k} \cdot \vec{\mathcal{B}} \neq 0$. Many linearized electromagnetic theories can be encompassed by the equations:

$$\vec{\nabla} \cdot \vec{D} = 0, \tag{14}$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0, \tag{15}$$

$$\nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = 0, \tag{16}$$

$$\vec{\nabla} \cdot \vec{B} = 0. \tag{17}$$

The purpose of the 'archaic' representation via $\vec{D}$ is to have a field which is perfectly transverse. With $\vec{D}$ the transverse wave operator is greatly simplified:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{D}) \to -\vec{\nabla}^2 \vec{D}.$$

This effectively reduces the freedom of the propagating gauge fields from 3 to 2: one would have to use 4-state mixing of three $\vec{E}$ components and one ϕ if this were not arranged.

Can we make $\vec{D}$ and $\vec{H}$ serve in eqs (7)–(10), and also include plasma effects? We find eqs (14)–(17) consistent with the definitions:

$$\vec{D} = \epsilon \vec{E} - g_\phi \phi \vec{\mathcal{B}}, \tag{18}$$

$$\vec{H} = \vec{B}. \tag{19}$$

The asymmetry here comes from having a magnetic background. In our work we will assume the contribution to ϵ due to the plasma frequency $\omega_{\rm p}$, via

$$\epsilon = \left(1 - \frac{\omega_{\rm p}^2}{\omega^2}\right).$$

The plasma frequency is given by

$$\omega_{\rm p}^2 = \frac{4\pi\alpha n_e}{m_e} = \frac{n_e}{10^{-8}\ {\rm cm}^{-3}} \left(3.7 \times 10^{-15}\ {\rm eV}\right)^2, \tag{20}$$

where n_e is the plasma density, m_e the electron mass and α the fine structure constant. In table 1 we list typical values of the various parameters relevant for intergalactic propagation. Here we also correct a numerical error in ref. [18] in the conversion of magnetic field to Mpc.

1.3 *Decoupling*

From Faraday's law and the $\vec{D}$ equation we have

$$\frac{1}{1 - (\omega_{\rm p}^2/\omega^2)} \vec{\nabla} \times (\vec{D} + g_\phi \vec{\mathcal{B}} \phi) = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial \vec{H}}{\partial t},$$

Table 1. Typical values of dimensionful scales relevant for intergalactic propagation. If not otherwise specified we use $\hbar = c = 1$.

Quantity	Typical values	Alternate units
$\mathcal{B}$	$0.01\ \mu$G $(1.95 \times 10^{-28}$ GeV$^2)$	4.78×10^{48} Mpc^{-2}
g_ϕ	10^{-11} GeV^{-1}	6.4×10^{-50} Mpc
$\omega_{\rm p}$	$3.7 \times 10^{-24}\ \text{GeV}\sqrt{\frac{n_e}{10^{-8}\,\text{cm}^{-3}}}$	$5.7\ \times 10^{14}\sqrt{\frac{n_e}{10^{-8}\,\text{cm}^{-3}}}\ \text{Mpc}^{-1}$
ω	10^{-5}–1 eV	$1.6\ \times 10^{24}$–1.6×10^{29} Mpc^{-1}

$$\vec{\nabla} \times \vec{\nabla} \times (\,\vec{D} + g_\phi \vec{\mathcal{B}} \phi\,) = \left(1 - \frac{\omega_{\rm p}^2}{\omega^2}\right) \frac{\partial^2 \vec{D}}{\partial t^2}. \tag{21}$$

Together with the ϕ propagation from eq. (6), the equations have been simplified as much as possible without loss of generality: the coupled system of ϕ, D_x, D_y, D_z have one locally decoupled mode, no longitudinal mode, and are equivalent to two coupled partial differential equations with no approximations other than linearization.

We now drop terms of order $\vec{\nabla}\mathcal{B}/B$ as negligible compared to other length scales, including the splitting of modes, setting up the usual adiabatic limit. We seek local plane-wave solutions with $\vec{\nabla} \to i\vec{k}$. The component of $\vec{D}$ perpendicular to $\vec{\mathcal{B}}$ decouples:

$$(k^2 + \omega_{\rm p}^2 - \omega^2)\vec{D} \times \hat{\mathcal{B}} = 0. \tag{22}$$

The other transverse projection of the $\vec{D}$ wave equation becomes

$$(k^2 + \omega_{\rm p}^2 - \omega^2)\vec{D} \cdot \hat{\mathcal{B}}_{\rm T} + k^2 g_\phi \mathcal{B}_{\rm T} \phi = 0. \tag{23}$$

Notice that in using $\vec{D}$ the equation of motion, involving the curl of $\vec{B}$, is not used: in fact it is satisfied as an identity. Conversely, when Faraday's law is substituted into the $\vec{E}$ wave equation, then Faraday's law is satisfied as an identity, and the equation of motion is solved (eq. (11)). By subtracting eq. (23) from the (in principle) independent wave eq. (11) for $\vec{E}$ at the compatible point, we obtain a nice consistency check.

We turn to the coupled system:

$$(k^2 + \omega_{\rm p}^2 - \omega^2)\vec{D} \cdot \hat{\mathcal{B}}_{\rm T} + k^2 g_\phi \mathcal{B}_{\rm T} \phi = 0, \tag{24}$$

$$\frac{g_\phi \mathcal{B}_{\rm T}}{1 - \omega_{\rm p}^2/\omega^2}\, \vec{D} \cdot \hat{\mathcal{B}}_{\rm T} + \left(k^2 + m_\phi^2 - \omega^2 + \frac{g_\phi^2 \mathcal{B}^2}{1 - \omega_{\rm p}^2/\omega^2}\right) \phi = 0. \tag{25}$$

The system can be solved directly for the dispersion relation $k^2 = k^2(\omega)$ by setting to zero the determinant of the corresponding matrix M, defined by

$$M \begin{pmatrix} \vec{D} \cdot \hat{\mathcal{B}}_{\rm T} \\ \phi \end{pmatrix} = 0.$$

However, the eigenvalues k^2 needed are not on the diagonal. Moreover, M is not symmetric, and non-symmetric matrices have eigenvectors which are not orthogonal.

Much the same occurs in optics [25], where the corresponding equations for propagation with a tensor dielectric constant ϵ_{ij} are

$$\begin{aligned}&(k^2\delta^T(k)\epsilon^{-1} - \omega^2)D = 0,\\ &\delta^T_{ij}(k) = \delta_{ij} - \hat{k}_i\hat{k}_j.\end{aligned} \tag{26}$$

One seldom finds $\delta^T(k)\epsilon^{-1}$ to be symmetric. Yet since

$$\delta^T(k)D = D,$$

multiplication on the left by $\delta^T(k)$ yields a symmetric eigenvalue equation:

$$(k^2\delta^T(k)\epsilon^{-1}\delta^T(k) - \omega^2)D = 0. \tag{27}$$

The propagation eigenstates are obtained from the 2×2 matrix $\delta^T(k)\epsilon^{-1}\delta^T(k)$ in the sector transverse to $\vec{k}$. This is considerably more subtle than (say) diagonalizing ϵ_{ij} first, and simply taking a transverse part.

This indicates that further transformations are needed for a useful solution.

1.4 *Orthogonal modes*

First, $\hat{D} \times \hat{\mathcal{B}}$ decouples from ϕ and propagates like ordinary light (including plasma frequency) with wave number $k_0 = \sqrt{\omega^2 - \omega_{\mathrm{p}}^2}$.

We made the rest of the transformation by inspection. Define

$$\begin{aligned}&\bar{\phi} = k_0\phi,\\ &\bar{D} = \frac{D\cdot\hat{\mathcal{B}} + g\mathcal{B}_{\mathrm{T}}\phi}{\sqrt{1 - (\omega_{\mathrm{p}}^2/\omega^2)}}.\end{aligned} \tag{28}$$

Now the propagation matrix is symmetric and eigenvalue k^2 lies on the diagonal:

$$\begin{pmatrix} k^2 + \omega_{\mathrm{p}}^2 - \omega^2 & g_\phi\mathcal{B}_{\mathrm{T}}\omega \\ g_\phi\mathcal{B}_{\mathrm{T}}\omega & k^2 + \tilde{m}_\phi^2 - \omega^2 \end{pmatrix} \begin{pmatrix} \bar{D} \\ \bar{\phi} \end{pmatrix} \tag{29}$$

with

$$\tilde{m}_\phi^2 = m_\phi^2 + \frac{g_\phi^2\mathcal{B}_{\mathrm{L}}^2}{1 - (\omega_{\mathrm{p}}^2/\omega^2)}\ . \tag{30}$$

As a consequence propagation generates unitary rotations of $(\bar{D}, \bar{\phi})$. Go to a new basis

$$|\eta\rangle = O|\psi_\Lambda\rangle; \quad O = \begin{pmatrix} \cos\bar{\theta} & -\sin\bar{\theta} \\ \sin\bar{\theta} & \cos\bar{\theta} \end{pmatrix}. \tag{31}$$

The mixing angle diagonalizing propagation is

$$\tan 2\bar{\theta} = \frac{g_\phi \omega \mathcal{B}_{\rm T}}{\tilde{m}_\phi^2 - \omega_{\rm p}^2}. \tag{32}$$

The dispersion relations are

$$k_1^2 = \omega^2 - \frac{1}{2}(\,\tilde{m}_\phi^2 + \omega_{\rm p}^2) - \frac{1}{2}\sqrt{\Omega^4}, \tag{33}$$

$$k_2^2 = \omega^2 - \frac{1}{2}(\tilde{m}_\phi^2 + \omega_{\rm p}^2) + \frac{1}{2}\sqrt{\Omega^4}, \tag{34}$$

where

$$\Omega^4 = 4g_\phi^2 \mathcal{B}_{\rm T}^2 \omega^2 + (\tilde{m}_\phi^2 - \omega_{\rm p}^2)^2. \tag{35}$$

By inspection of these results, the eigenvalues and mixing are just the same as solving the $\mathcal{B}_{\rm L} = 0$ limit and making the replacement $m_\phi^2 \to \tilde{m}_\phi^2 = m_\phi^2 + g_\phi^2 \mathcal{B}_{\rm L}^2/[1 - (\omega_{\rm p}^2/\omega^2)]$.

1.4.1 *Plane-wave simplification*

There are circumstances where neglecting $\vec{\nabla}\mathcal{B}/\mathcal{B}$ may not be possible. Then eqs (21) and (6) cannot be simplified further. However, if the propagation can be reduced to plane-wave modes with constant parameters, there is a simple way to understand the modes.

First solve the longitudinal mode using Gauss' law:

$$\begin{aligned} &\vec{k} \cdot \epsilon \vec{E} = g_\phi(\,\vec{k}\phi) \cdot \vec{\mathcal{B}}, \\ &\epsilon \vec{E}_{\rm L} = \hat{k}\, g_\phi \hat{k}\phi \cdot \vec{\mathcal{B}}. \end{aligned} \tag{36}$$

Here $\hat{k} = \vec{k}/k$ is a non-local operator. Insert the solution where it appears in the propagation of ϕ (eq. (6)):

$$\begin{aligned} (\omega^2 + k^2 + m_\phi^2)\phi &= -g_\phi \vec{E} \cdot \vec{\mathcal{B}}, \\ &\to -g_\phi E_{\rm T} \mathcal{B}_{\rm T} - \frac{g^2 \mathcal{B}_{\rm L}^2 \phi}{\epsilon}. \end{aligned} \tag{37}$$

Observe that the effects on equations for ϕ are the same as replacing $m_\phi^2 \to \tilde{m}_\phi^2 = m_\phi^2 + g_\phi^2 \mathcal{B}_{\rm L}^2/\epsilon$. Meanwhile the *transverse* projection of the electric equation (eq. (11)), also involves only $E_{\rm T}$ and ϕ. Since this subsystem has decoupled, they must have modes which are linear combinations of ϕ and $E_{\rm T}$: finally we recover the transformation to reveal that $\bar{D} = E_{\rm T}$ in this limit.

1.4.2 *Resonant mixing*

In certain situations the mixing angle $\bar{\theta}$ may cross $\pi/2$. This phenomenon of resonant mixing was studied in detail in ref. [18]. Study was restricted to the case where $\mathcal{B}_{\mathrm{L}} \sim 0$, in part to compare with previous literature which violated gauge invariance by assuming early that modes were transverse. Here we find that the phenomenon of resonant mixing undergoes a qualitative change if the value of $g_\phi \mathcal{B}_{\mathrm{L}}$ is sufficiently large. In the limit $\mathcal{B}_{\mathrm{L}} = 0$ the resonant mixing occurs if the plasma frequency depends on position z and at some point along the path becomes equal to the pseudoscalar mass m_ϕ. In this region $m_\phi^2 - \omega_{\mathrm{p}}^2(z) \approx 0$ and $\tan 2\bar{\theta} \approx \infty$. If $\mathcal{B}_{\mathrm{L}} \neq 0$, the resonant mixing may happen at some frequency irrespective of the values of m_ϕ and ω_{p}.

The resonant mixing occurs when

$$m_\phi^2 - \omega_{\mathrm{p}}^2 = -\frac{g_\phi^2 \mathcal{B}_{\mathrm{L}}^2}{1 - \omega_{\mathrm{p}}^2/\omega^2}. \tag{38}$$

If ω_{p} changes along the path then it may be possible to find some position at which the resonant condition is satisfied for a wide range of values of the frequency ω. Alternatively if ω_{p} is roughly uniform, then it may be possible to find some frequency at which the condition for resonance applies.

In the case of cosmological propagation the relevant limit is $\omega/\omega_p \gg 1$ and $g_\phi \mathcal{B}_{\mathrm{L}} \ll \omega_{\mathrm{p}}$. This applies even at radiofrequencies for typical values of plasma frequency in intergalactic or galactic space. Let us assume that the plasma frequency is varying along the path. Then for typical values of galactic or intergalactic magnetic fields and coupling g_ϕ we find that the resonant condition is satisfied at the position $\omega_{\mathrm{p}} \approx m_\phi$. The precise value of ω_{p} of course depends on the value of the longitudinal component of the magnetic field and the frequency ω. Hence in this case the results do not change qualitatively with respect to the case discussed in ref. [18] where the longitudinal component of the magnetic field is neglected.

The results change qualitatively when $\omega \approx \omega_{\mathrm{p}}$ or when $g_\phi^2 \mathcal{B}_{\mathrm{L}}^2$ is comparable to the typical value of $|m_\phi^2 - \omega_{\mathrm{p}}^2|$ in the medium. In this case the resonance condition may be satisfied for some frequency even if m_ϕ is not close to ω_{p} anywhere along the path. This is very different from the resonance phenomenon discussed in ref. [18]. It may be observable in some special systems such as pulsars or magnetars. The typical value of magnetic field in pulsars is 10^{12} Gauss. In current models the plasma density is given by $n_e = 7 \times 10^{10} B_{12}/P$ cm^{-3} [26], where B_{12} is the magnetic field in units of 10^{12} Gauss and P is the period of rotation of the pulsar in seconds. Using these values, with $P = 1$ and $B_{12} = 1$, we find that $g_\phi^2 B^2 \approx 10^{-16}$ eV^2 for $g_\phi = 10^{-10}$ GeV^{-2} and $\omega_{\mathrm{p}}^2 \approx 10^{-10}$ eV^2. Hence in this case we expect resonance only for $\omega \approx \omega_{\mathrm{p}}$ if $|\omega_{\mathrm{p}}^2 - m_\phi^2| \gg g_\phi^2 \mathcal{B}_{\mathrm{L}}^2$. If the astrophysical limits are not applicable [21], then we may have much larger value of g_ϕ and can expect resonance to occur at larger values of ω. Physically we expect a large change in the intensity and polarization of the wave as the frequency approaches the resonant frequency.

The resonant effect is very interesting conceptually. Qualitatively, at resonance it appears that the longitudinal mode of a plasma oscillation becomes very strongly mixed with the pseudoscalar field, depending on the difference of masses. As we mentioned earlier, ϕ mixes with $\vec{E}_\mathrm{L}$: indeed due to the constraint of Gauss' law, it is the *same dynamical phenomenon* as the longitudinal field. Let us estimate some magnitudes: when fully mixed, $\bar{\phi} \sim \bar{D}$, or

$$\phi \sim \frac{E_\mathrm{T}}{\omega_\mathrm{p}}.$$

As mentioned earlier $E_\mathrm{L} \sim (g_\phi \phi \mathcal{B}_\mathrm{L}/\epsilon)$. Together the relations predict

$$\frac{E_\mathrm{L}}{E_\mathrm{T}} \sim \frac{g_\phi \mathcal{B}_\mathrm{L}}{\epsilon \omega_\mathrm{p}} \sim \frac{g_\phi \mathcal{B}_\mathrm{L} \omega_\mathrm{p}}{\omega_\mathrm{p}^2 - \omega^2}.$$

Thus there is always a frequency for which we may observe the formerly non-interacting pseudoscalar electromagnetically, and as a form of *longitudinally polarized light*: The E_L being observable and affecting instruments just as much as a longitudinal field in a plasma oscillation. Given sufficiently fine measurements the 'invisible axion' could in principle be 'visible'.

We hope to explore more deeply the potential laboratory and astrophysical repercussions of these phenomena in another paper. Given that most current interest centers on cosmological propagation, we turn to studying the effects of a varying $\vec{\mathcal{B}}$ field in the next section.

2. Three-mode mixing: Varying $\vec{\mathcal{B}}$

We next consider the adiabatic propagation of light through a background magnetic field which varies slowly in direction. This problem has not been solved before. The results are far from trivial, and give substance to many cosmological applications assuming some 'fluctuating' magnetic fields with typical coherence lengths. As we will show, the variety of physical phenomena one can observe is very great. In some limits, writing a transition probability and taking a statistical average may suffice, but in other limits the polarization effects are quite spectacular. The dynamical possibilities for the mixing of light actually exceed those for neutrino-mass mixing, which has been studied for nearly 50 years and still appears inexhaustible.

The physically observable density matrix ρ is given by

$$\rho = \begin{pmatrix} \langle E_{||} E_{||}^* \rangle & \langle E_{||} E_{\perp}^* \rangle \\ \langle E_{\perp} E_{||}^* \rangle & \langle E_{\perp} E_{\perp}^* \rangle \end{pmatrix}, \tag{39}$$

where $\langle\,\rangle$ denotes the statistical averages occurring in propagation [26a].

Orient the z-axis along the direction of the wave. Let angle ξ measure the direction of the background field relative to the x-axis:

$$\vec{\mathcal{B}}_\mathrm{T} = B \cos\xi(z) \hat{i} + B \sin\xi(z) \hat{j}. \tag{40}$$

For direct numerical integration we assume that $\xi(z)$ varies linearly with z. We fix the magnitude of the background magnetic field to identify effects arising due to varying magnetic field direction. A changing background magnetic field magnitude is easily included in the formalism. For the same reason we ignore the variation in plasma density along the path. The effects of varying plasma density for fixed field direction has been studied in detail elsewhere [18].

The wave equation can now be written as

$$(\omega^2 + \partial_z^2) \begin{pmatrix} A_x \\ A_y \\ \phi \end{pmatrix} - M \begin{pmatrix} A_x \\ A_y \\ \phi \end{pmatrix} = 0, \tag{41}$$

where $\vec{A} = \vec{E}/\omega$ and M is the mass or mixing matrix,

$$M = \begin{pmatrix} \omega_{\rm p}^2 & 0 & -gB\omega\cos\xi \\ 0 & \omega_{\rm p}^2 & -gB\omega\sin\xi \\ -gB\omega\cos\xi & -gB\omega\sin\xi & m_\phi^2 \end{pmatrix}. \tag{42}$$

We dropped $g^2\mathcal{B}_{\rm L}^2$ terms as negligible for intergalactic propagation with typical parameters. With a slowly varying background and working in the adiabatic limit, we define transformed fields A'_x, A'_y and ϕ' such that

$$\begin{pmatrix} A_x \\ A_y \\ \phi \end{pmatrix} = \begin{pmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A'_x \\ A'_y \\ \phi' \end{pmatrix}. \tag{43}$$

The wave equation reduces to

$$(\omega^2 + \partial_z^2) \begin{pmatrix} A'_x \\ A'_y \\ \phi' \end{pmatrix} - \begin{pmatrix} \omega_{\rm p}^2 & 0 & 0 \\ 0 & \omega_{\rm p}^2 & -gB\omega \\ 0 & -gB\omega & m_\phi^2 \end{pmatrix} \begin{pmatrix} A'_x \\ A'_y \\ \phi' \end{pmatrix} = 0. \tag{44}$$

Here $\beta = \xi - \pi/2$. The equation reduces to the case of two-component mixing which can be solved along the lines discussed in ref. [18]. Once we have obtained all the correlators between A'_x and A'_y, we can express the required correlators as

$$\begin{aligned}
\langle A_x^*(z)A_x(z)\rangle &= \sin^2\xi(z)\langle A'^*_x(z)A'_x(z)\rangle + \cos^2\xi(z)\langle A'^*_y(z)A'_y(z)\rangle \\
&\quad + \cos\xi(z)\sin\xi(z)\left(\langle A'^*_x(z)A'_y(z)\rangle + \langle A'^*_y(z)A'_x(z)\rangle\right) \\
\langle A_y^*(z)A_y(z)\rangle &= \cos^2\xi(z)\langle A'^*_x(z)A'_x(z)\rangle + \sin^2\xi(z)\langle A'^*_y(z)A'_y(z)\rangle \\
&\quad - \cos\xi(z)\sin\xi(z)\left(\langle A'^*_x(z)A'_y(z)\rangle + \langle A'^*_y(z)A'_x(z)\rangle\right) \\
\langle A_x^*(z)A_y(z)\rangle &= -\cos\xi(z)\sin\xi(z)\left(\langle A'^*_x(z)A'_x(z)\rangle - \langle A'^*_y(z)A'_y(z)\rangle\right) \\
&\quad + \sin^2\xi(z)\langle A'^*_x(z)A'_y(z)\rangle \\
&\quad - \cos^2\xi(z)\langle A'^*_y(z)A'_x(z)\rangle.
\end{aligned} \tag{45}$$

The correlators appearing on the right-hand side of these equations can be calculated using the results in ref. [18].

2.1 *Transition probabilities*

Analytic calculations in the adiabatic limit fail for small frequencies since in this case the transition probabilities between instantaneous eigenstates are large. Even

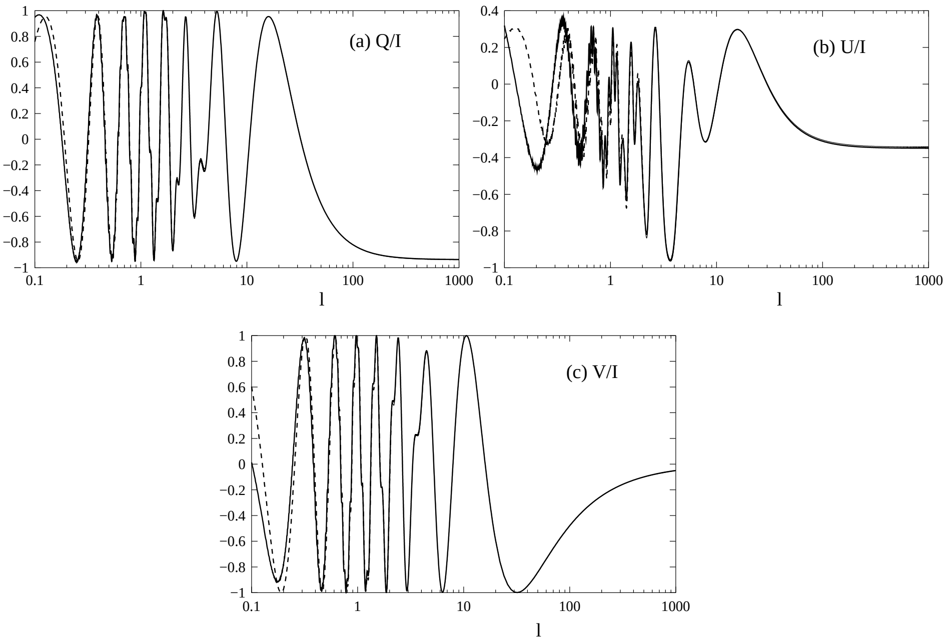

Figure 1. Normalized Stokes parameters (**a**) Q/I, (**b**) U/I and (**c**) V/I as a function of the length parameter l for varying directions of background magnetic field; the magnitude $|\vec{\mathcal{B}}|$ and ω_{p} are constant. Curves are generated by direct numerical integration (solid line) and adiabatic analytic calculation (dashed line). Parameters $gB = 1.0$, $L = 100$, $m_\phi^2/\omega_{\mathrm{p}}^2 = 0.1$, angles $\xi(0) = \pi/2$ and $\xi(L) = \pi/2 - 0.3\pi$; initial polarization ($Q/I = 0,\ U/I = 1.0,\ V/I = 0.0$).

in the large frequency regime the adiabatic limit fails unless the product $gBL \gg 1$. This can be verified explicitly by computing the transition probabilities using the procedure discussed in ref. [18]. Using the Dirac notation we can rewrite the basic wave equation (41) as,

$$(\omega^2 + \partial_z^2)|\psi\rangle - M|\psi\rangle = 0. \tag{46}$$

We obtain the instantaneous eigenstates $|n\rangle$ and eigenvalues μ_n^2 by solving the equation

$$M|n\rangle = \mu_n^2|n\rangle. \tag{47}$$

The solution $|\psi\rangle$ can be expressed as

$$|\psi\rangle = \sum_n a_n(z)\mathrm{e}^{i\int_0^z \mathrm{d}z'\omega_n}|n\rangle, \tag{48}$$

where the frequency ω_n can be determined by substituting this in eq. (46). We find $\omega_n = \omega - \mu_n^2/(2\omega)$. The evolution of the coefficients $a_n(z)$ with z gives an estimate of the transition among different eigenmodes. These coefficients are obtained by solving the equation

$$\partial_z b_m \approx \sum_{n, n\neq m} b_n \frac{\langle m|(\partial_z M)|n\rangle}{\mu_m^2 - \mu_n^2} \mathrm{e}^{i\int_0^z \mathrm{d}z'(\omega_n - \omega_m)} \ , \tag{49}$$

where we have approximated $\omega_n \approx \omega$ and b_m is defined by the equation

$$a_m(z) = \mathrm{e}^{-\frac{1}{2}\int_0^z \mathrm{d}z'(\partial_{z'}\omega_m)/\omega_m} b_m(z) \approx b_m(z). \tag{50}$$

The matrix M given in eq. (42) can be easily diagonalized. We find the eigenvalues, ω_{p}^2 and

$$\lambda_{\pm} = \frac{\omega_{\mathrm{p}}^2 + m_\phi^2}{2} \pm \frac{1}{2}\sqrt{(\omega_{\mathrm{p}}^2 - m_\phi^2)^2 + 4(gB\omega)^2}$$

with the corresponding eigenvectors

$$\begin{pmatrix} \sin\xi \\ -\cos\xi \\ 0 \end{pmatrix}, \frac{1}{\sqrt{(gB\omega)^2 + (\omega_{\mathrm{p}}^2 - \lambda_\pm)^2}} \begin{pmatrix} gB\omega\cos\xi \\ gB\omega\sin\xi \\ \omega_{\mathrm{p}}^2 - \lambda_\pm \end{pmatrix}$$

respectively. The transition probability can now be computed using eq. (49). Let us assume that the magnetic field changes appreciably over a distance scale L. To be specific this means that the change in angle $\xi(z)$ in eq. (40) is of order unity over a distance scale L. Another useful length scale is the oscillation length $l = 2\omega/|\omega_{\mathrm{p}}^2 - m_\phi^2|$. In the limit of small frequencies, $gBl \ll 1$, we find that the transition probabilities are negligible as long as $(gBl)(gBL) \gg 1$. In all other cases we find that the transitions are small as long as $gBL \gg 1$.

2.2 *Results*

In figure 1 we show a sample of results obtained in the case of varying direction of background magnetic field from analytic calculation in the adiabatic limit as well as direct numerical integration. Here angle $\xi(0) = \pi/2$ and $\xi(L) = \pi/2 - 0.3\pi$, i.e. the transverse component of the background magnetic field is aligned along the y-axis initially and evolves to angle 0.3π after a distance L. The parameters used in this figure are $gB = 1.0$, $L = 100$, $m_\phi^2/\omega_{\mathrm{p}}^2 = 0.1$. The initial state of polarization has been chosen such that $Q/I = 0$, $U/I = 1$, and $V/I = 0$. The analytic results in this case are in good agreement with the numerical results, except in the limit of small frequencies. In the large frequency limit the exponent in eq. (49) is approximately equal to $igBL/2$. For the parameters chosen this phase factor is large and hence suppresses the transition probability between different eigenstates.

In figure 2 we show a sample of results obtained in the case of varying direction of background magnetic field for a smaller value of the product gBL. Here we choose $gB = 0.1$, $L = 100$, and $m_\phi^2/\omega_{\mathrm{p}}^2 = 0.1$. In this case we use direct numerical integration since the analytic results are not reliable. The orientation of the background magnetic field is chosen to be same as in figure 1, i.e. $\xi(0) = \pi/2$ and $\xi(L) = \pi/2 - 0.3\pi$. The initial state of polarization has been chosen such that $Q/I = 0$, $U/I = 1$, and $V/I = 0$. The results obtained using this parameter choice

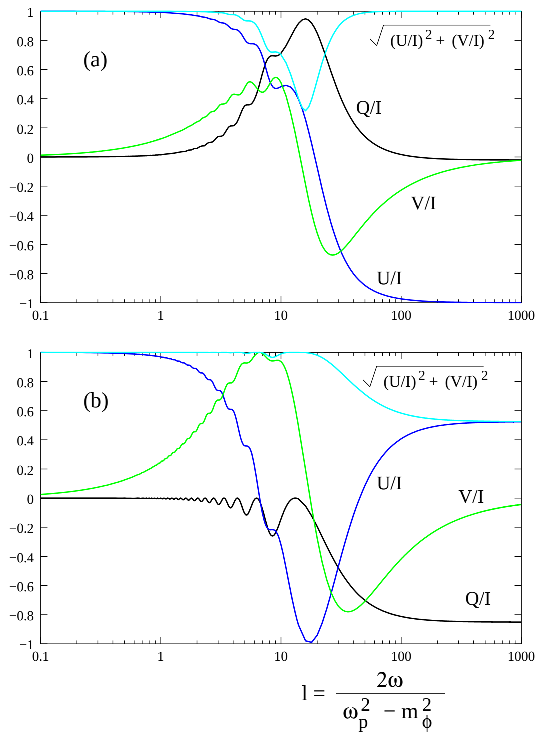

Figure 2. (**a**) Normalized Stokes parameters $Q/I, U/I$ and V/I as a function of the length parameter l for varying background magnetic field directions; the magnitude $|\vec{\mathcal{B}}|$ and $\omega_{\rm p}$ are constant. Parameters $gB = 0.1$, $L = 100$, $m_\phi^2/\omega_{\rm p}^2 = 0.1$; angles $\xi(0) = \pi/2$, $\xi(L) = \pi/2 - 0.3\pi$; initial state of the polarization ($Q = 0$, $U = 1.0$, $V = 0.0$). Results for uniform background magnetic field (**b**) are shown for comparison.

and with uniform magnetic field direction are also shown for comparison. We find that the results obtained for the case of varying background magnetic field direction are considerably different in comparison to what is obtained in the case of uniform direction. As expected, the results agree in the limit of small ω. In figure 3 we show the results for the same parameter choice used in figure 2 but with the wave assumed to be unpolarized initially.

The degree of polarization and the normalized Stokes parameters as a function of distance are shown in figure 4. Here the parameters are taken to be same as for figure 2 with the length parameter $l = 2\omega/(\omega_{\rm p}^2 - m_\phi^2) = 10$ and the wave is assumed to be unpolarized at source. We see that all the parameters $p, Q/I, U/I, V/I$ oscillate with propagation distance.

In figure 5 we show the relationship between Q/I and V/I for several different choice of parameters for the case of varying background magnetic field. The dependence of Q/I and V/I follows approximately an elliptical behaviour. This is in contrast to the case of uniform magnetic field direction, which shows such a relationship between U/I and V/I [18]. As in the case of uniform background, a simple

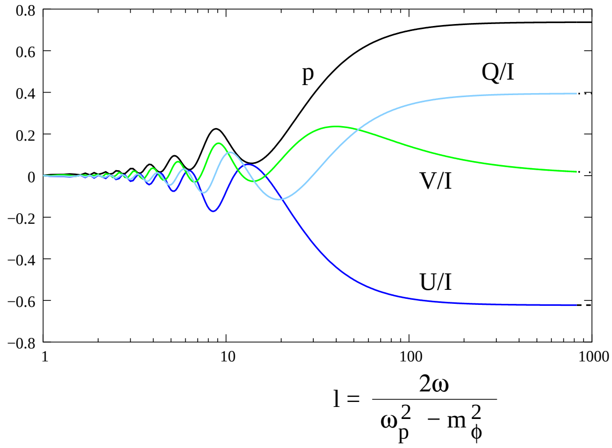

Figure 3. The degree of polarization p and the normalized Stokes parameters $Q/I, U/I$ and V/I as a function of the length parameter l for varying directions of background magnetic fields; the magnitude $|\vec{\mathcal{B}}|$ and $\omega_{\rm p}$ are constant. Parameters $gB = 0.1$, $L = 100$, $m_\phi^2/\omega_{\rm p}^2 = 0.1$; angles $\xi(0) = \pi/2$, $\xi(L) = \pi/2 - 0.3\pi$. The wave is assumed to be unpolarized $(Q = 0,\ U = V = 0)$ at source.

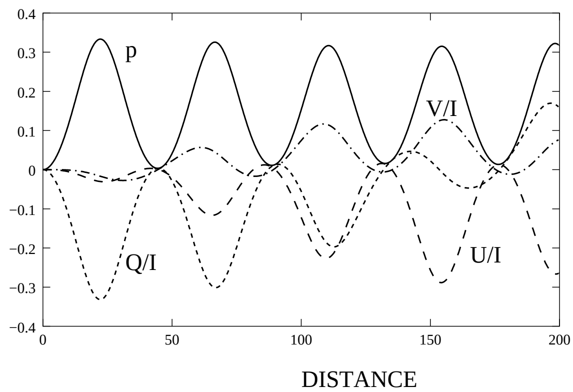

Figure 4. The degree of polarization p and the normalized Stokes parameters $Q/I, U/I$ and V/I as a function of the distance of propagation for varying directions of background magnetic fields; the magnitude $|\vec{\mathcal{B}}|$ and $\omega_{\rm p}$ are constant. The parameters $gB = 0.1$, $m_\phi^2/\omega_{\rm p}^2 = 0.1$, $l = 2\omega/(\omega_{\rm p}^2 - m_\phi^2) = 10$; angles $\xi(0) = \pi/2$, $\xi(L) = \pi/2 - 0.3\pi$. The wave is assumed to be unpolarized $(Q = 0,\ U = V = 0)$ at source.

correlation is seen only for frequencies larger than a minimum frequency. At low frequencies the relationship becomes very complicated.

One may be able to test relationships among different Stokes parameters in future observations. To rule out other possible mechanisms affecting data, certain tests require observations over a sufficiently large frequency interval.

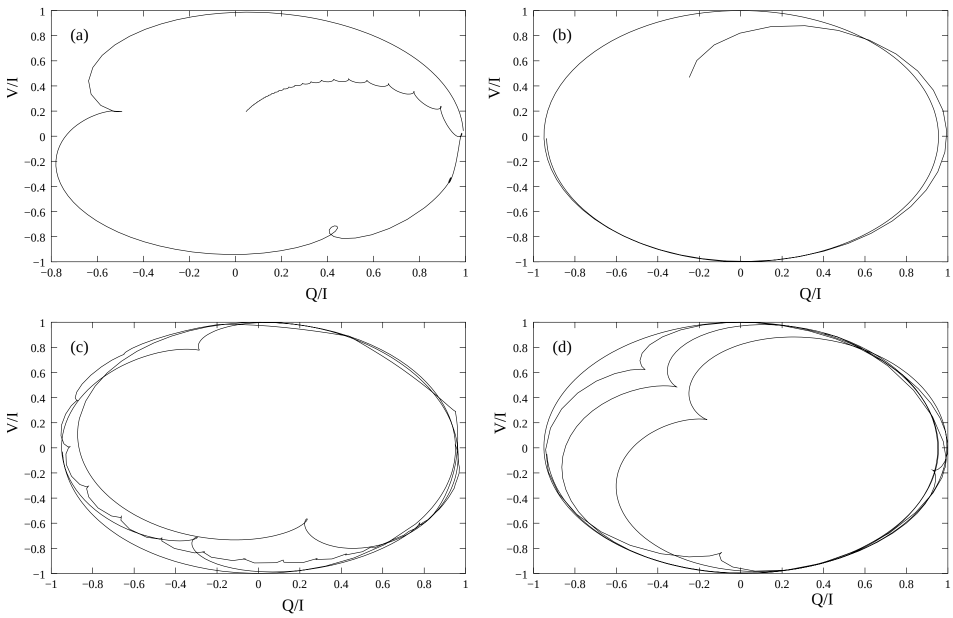

Figure 5. A sample of results showing the correlation between the normalized Stokes parameters Q/I and V/I for some randomly chosen parameters and initial state of polarization. The results are shown for varying background magnetic field directions with the plasma frequency and the magnitude of the magnetic field uniform. Parameters (in arbitrary units) are (**a**) $gB = 2, L = 10, 0.04 < l < 20$, (**b**) $gB = 10, L = 10, 0.4 < l < 800$, (**c**) $gB = 1, L = 50, 0.2 < l < 800$ and (**d**) $gB = 10, L = 10, 0.04 < l < 100$. The ratio $m_\phi^2/\omega_{\rm p}^2 = 0.1$; angles $\xi(0) = \pi/2$ and $\xi(L) = \pi/2 - 0.3\pi$ for all the plots.

3. Summary and conclusion

The general treatment of mixing of electromagnetic waves with pseudoscalars in the presence of background magnetic field is a surprisingly intricate topic. The pseudoscalar mixes with (and indeed becomes) the longitudinal mode of light, a situation potentially generating cumulative deviation compared to treatments assuming the fields stay transverse. Cumulative errors do occur in principle, but for parameters of current interest they are fortunately controlled. The contribution due to the longitudinal component can be accommodated by redefining the pseudoscalar mass parameter $m_\phi^2 \to m_\phi^2 + g_\phi^2 \mathcal{B}_{\rm L}/\epsilon$. This simplification led to exploring the problem of propagation in a magnetic field whose direction may vary along the path. The condition of adiabaticity is found to be rather stringent: For a wide range of parameter space the evolution cannot be assumed to be adiabatic.

Thus the general problem of mixing of light with pseudoscalars has more twists and turns than could have been anticipated early. Stokes parameters show interesting correlations with one another which are distinctively different from those observed for fixed background field direction [18]. Such polarization effects may be observable with current technology, and may eventually serve either to identify new physics, or to put new limits on the pseudoscalar–photon coupling parameters.

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