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On a Raychaudhuri equation for hot gravitating fluids

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Abstract. We generalise the Raychaudhuri equation for the evolution of a self gravitating fluid to include an Abelian and non-Abelian hybrid magneto fluid at a finite temperature. The aim is to utilise this equation for investigating the dynamics of astrophysical high temperature Abelian and non-Abelian plasmas.

Keywords. Raychaudhuri equation; plasmas; magneto fluids.

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1. Introduction

It is perhaps one of the finest tributes to the simplicity and elegance of Einstein's theory of gravitation and to the human spirit that a laboratory scientific assistant, who later became a teacher sitting in Kolkata, was able to conceive the equation for the evolution of a gravitating fluid now known as Raychaudhuri's equation [1]. The equation served as a lemma for the Penrose–Hawking singularity theorems and for the study of exact solutions in general relativity [2,3]. It provides a simple validation of our expectation that gravitation should be a universal attractive force between any two particles in general relativity [4]. This equation has stood the test of time and has been generalised in many ways. It has found applications in modern theories of strings and membranes [5,6]. We attempt a more modest generalisation to tackle the statistical properties of a hot astrophysical plasma. This may be an electromagnetic or a chromomagnetic (quark gluon) plasma. We are guided here by a recent formalism that has been used to investigate the dynamics of a hot charged fluid in terms of a hybrid magneto-fluid [7]. The changes brought about to the Raychaudhuri equation by the introduction of statistical attributes associated with finite temperature are many and interesting. We illustrate some of these changes in the context of the evolution of gravitating non-Abelian plasmas [8] in the early Universe.

2. Raychaudhuri's equation for a 'unified' charged gravitating fluid

For a congruence of time-like vectors U^μ, the standard Raychaudhuri equation can be written down in the following form [9]:

$$\dot{\Theta} = \frac{q}{m} F^{\alpha}{}_{\gamma;\alpha} U^{\gamma} + \frac{q}{m} F^{\alpha}{}_{\gamma} \Omega^{g\gamma}{}_{\alpha} - \Sigma^{g}{}_{\alpha\beta} \Sigma^{g\beta\alpha} - \Omega^{g}{}_{\alpha\beta} \Omega^{g\beta\alpha} - \frac{1}{3}\Theta^2 - 8\pi (t_{\mu\nu} - \frac{1}{2} t g_{\mu\nu}) U^\mu U^\nu. \tag{1}$$

With our notation, $(U_\mu U^\mu = -1)$, the limit $F_{\mu\nu} \longrightarrow 0$ is the (geodesic) Raychaudhuri equation. For a geodesic congruence, $\Sigma^g_{\alpha\beta} = \frac{1}{2}(U_{\alpha;\beta} + U_{\beta;\alpha}) - \frac{1}{3} U^{\gamma}{}_{;\gamma}(g_{\alpha\beta} + U_\alpha U_\beta)$ is a symmetric, tracefree tensor and is called the shear tensor, $\Omega^g_{\alpha\beta} = \frac{1}{2}(U_{\alpha;\beta} - U_{\beta;\alpha})$ is called the vorticity tensor and $\Theta = U^{\alpha}{}_{;\alpha}$ is called the expansion scalar. Also, Einstein's equations have been used to replace the Ricci tensor by the energy-momentum tensor.

For the use of the Raychaudhuri equation to describe astrophysical plasmas in which gravitational effects are compatible with the fluid attributes, we have to account for the temperature and pressure of the fluid. For this, we incorporate into a small volume element of the fluid a statistical factor f which represents a temperature-dependent statistical attribute of the fluid, and is related to the enthalpy h, the scalar density in the rest frame n and the mass m of the fluid particles by the relation $h = mnf(T)$. When one does the kinetic theory of high temperature plasmas, $f(T)$ seems to emerge as the most useful variable to represent temperature effects. For relativistic plasmas, $h = mnK_3(m_s/T)/K_2(m_s/T)$ and $f(T)$ is purely a positive function of temperature. The velocity vector of the fluid is obtained as the average velocity of this small volume of the fluid and is written as $V^\mu = fU^\mu$. This drastically alters the character of the terms in the evolution equation. For example, now $V^\mu V_\mu = -f^2$, in contrast with $U^\mu U_\mu = -1$, and, unlike $U^\alpha U_{\alpha;\beta} = \frac{1}{2}(U^\alpha U_\alpha)_{;\beta} = 0$, now $V^\alpha V_{\alpha;\beta} = -f\partial_\beta f$. These terms significantly change the spatial terms of the Raychaudhuri equation and necessitate a generalisation to account for these statistical factors. Indeed, this statistical limit for the fluid velocity [7], unlike the particle limit, provides a natural factor for producing acceleration forces from within the fluid due to pressure and temperature gradients. In the unified magnetofluid picture one can write the equation of motion of a magneto-fluid with entropy σ as

$$T\partial^\nu \sigma = gM^{\mu\nu} U_\mu, \tag{2}$$

where

$$M^{\mu\nu} = F^{\mu\nu} + \frac{m}{g} S^{\mu\nu} \tag{3}$$

and

$$S_{\mu\nu} = \partial_\mu (fU_\nu) - \partial_\nu (fU_\mu) \tag{4}$$

are antisymmetric second rank 'flow' tensors defined in [7]. In this sense, $M_{\mu\nu}$ represents an anti-symmetric 'unified' field-flow tensor constructed from the kinematic (U^μ), statistical $(f(T))$ and electromagnetic $(F_{\mu\nu})$ attributes of the magneto-fluid.

While the statistical factors provide us with additional acceleration terms which alter the purely spatial character of the shear and vorticity tensors, there are some relations that remain unaltered. A trivial example is that V^μ remains orthogonal to the hypersurface with metric $h_{\alpha\beta}$ defined as the projector $h_{\alpha\beta} = g_{\alpha\beta} + U_\alpha U_\beta$. We notice that

$$\begin{aligned} V^\mu{}_{;\alpha\beta} &= (fU^\mu{}_{;\alpha} + U^\mu f_{;\alpha})_{;\beta} \\ &= f_{;\beta}U^\mu{}_{;\alpha} + U^\mu{}_{;\beta} f_{;\alpha} + U^\mu f_{;\alpha\beta} + fU^\mu{}_{;\alpha\beta}. \end{aligned} \tag{5}$$

It is easy to see that anti symmetrization in indices α, β allows us to write

$$V^\mu{}_{;\alpha\beta} - V^\mu{}_{;\beta\alpha} = f(U^\mu{}_{;\alpha\beta} + U^\mu{}_{;\beta\alpha}) = fR^\mu{}_{\sigma\alpha\beta}U^\sigma = R^\mu{}_{\sigma\alpha\beta}V^\sigma, \tag{6}$$

and its character is unchanged in the transition to a hot fluid.

The question that arises now is the following. Is it possible to define the generalizations of the standard definitions of the shear and vorticity tensors in a similar manner, and can they be constructed to be purely spatial tensors?

Let us decompose, following the standard procedure, $\tilde{B}_{\alpha\beta} = V_{\alpha;\beta}$ into its irreducible parts:

$$\tilde{B}_{\alpha\beta} = \tilde{\Sigma}_{\alpha\beta} + \tilde{\Omega}_{\alpha\beta} + \frac{1}{3}\tilde{\Theta}h_{\alpha\beta} + \frac{1}{f}V_\alpha \partial_\beta f. \tag{7}$$

The trace of $\tilde{B}_{\alpha\beta}$ defines the expansion scalar $\tilde{\Theta}$ via

$$\tilde{B}^\alpha{}_\alpha = \tilde{\Theta} + \frac{1}{f}V^\alpha \partial_\alpha f = V^\alpha{}_{;\alpha}, \tag{8}$$

where we have assumed that the generalized shear tensor $\tilde{\Sigma}_{\mu\nu}$ is symmetric and traceless. If we now define

$$\tilde{\Sigma}_{\alpha\beta} = \frac{1}{2}(V_{\alpha;\beta} + V_{\beta;\alpha}) - \frac{1}{3}\tilde{\Theta}h_{\alpha\beta} - \frac{1}{2f}(V_\alpha \partial_\beta f + V_\beta \partial_\alpha f), \tag{9}$$

then its trace

$$\tilde{\Sigma}^\mu{}_\mu = V^\mu{}_{;\mu} - \tilde{\Theta} - \frac{1}{f}V^\mu \partial_\mu f \tag{10}$$

goes to zero, because of eq. (8); the trace-free condition also reproduces the required definition of $\tilde{\Theta}$.

The generalised vorticity tensor $\tilde{\Omega}_{\alpha\beta}$ may also be written as

$$\tilde{\Omega}_{\alpha\beta} = \frac{1}{2}(V_{\alpha;\beta} - V_{\beta;\alpha}) - \frac{1}{2f}(V_\alpha \partial_\beta f - V_\beta \partial_\alpha f). \tag{11}$$

We see that the tensor $S_{\mu\nu}$ defined in eq. (4) allows us the following identification:

$$\tilde{\Omega}_{\alpha\beta} = \frac{1}{2}S_{\beta\alpha} - \frac{1}{2f}(V_\alpha \partial_\beta f - V_\beta \partial_\alpha f). \tag{12}$$

From an earlier work, we know that the gradient of f, for a perfect fluid, is related to the pressure gradient and to Lorentz force from the presence of an electromagnetic field [7]. So, unlike the standard particle picture of the shear and vorticity, the unified picture of shear and vorticity automatically includes accelerations coming from the internal forces of the fluids. This in turn ensures that the shear is now not a purely spatial tensor and the vorticity tensor behaves in a similar manner. It is the sum of the shear and vorticity tensors that leads us to the following simplification:

$$V^\alpha(\tilde{\Sigma}_{\alpha\beta} + \tilde{\Omega}_{\alpha\beta}) = 0. \tag{13}$$

This implies that

$$V^\alpha \tilde{B}_{\alpha\beta} = 0 \tag{14}$$

and

$$\tilde{B}_{\alpha\beta}V^\beta = (\tilde{\Sigma}_{\alpha\beta} + \tilde{\Omega}_{\alpha\beta} + \frac{1}{f}V_\alpha\partial_\beta f)V^\beta = V_{\alpha;\beta}V^\beta = \dot{V}_\alpha. \tag{15}$$

The decomposition of $V_{\alpha;\beta}$ into its irreducible components can therefore be written as

$$V_{\alpha;\beta} = \tilde{B}_{\alpha\beta} = \tilde{\Sigma}_{\alpha\beta} + \tilde{\Omega}_{\alpha\beta} + \frac{1}{3}\tilde{\Theta}h_{\alpha\beta} + \frac{1}{f}V_\alpha\partial_\beta f. \tag{16}$$

So, while the generalized expansion scalar, shear and vorticity tensors show evidence of internal fluid forces, the tensor, $\tilde{B}_{\alpha\beta} = V_{\alpha;\beta}$ itself shows similarity with the original tensor, $B_{\alpha\beta} = U_{\alpha;\beta}$.

From the definitions of $\tilde{\Theta}$, $\tilde{\Sigma}_{\mu\nu}$ and $\tilde{\Omega}_{\mu\nu}$, the following equations relate the scalar of expansion, shear and vorticity for a geodesic congruence in the particle view to those in the unified view of gravitating fluids with acceleration forces:

$$\begin{aligned}
&\tilde{\Theta} = f\Theta, \\
&\tilde{\Sigma}_{\mu\nu} = f\Sigma^g{}_{\mu\nu}, \\
&\tilde{\Omega}_{\mu\nu} = f\Omega^g{}_{\mu\nu},
\end{aligned} \tag{17}$$

which then imply

$$\tilde{B}_{\mu\nu} = fB_{\mu\nu} + V_\mu\partial_\nu \ln f. \tag{18}$$

In analogy with the derivation of the standard Raychaudhuri equation [1], if we consider the parallely transported $\tilde{B}_{\alpha\beta;\gamma} = V_{\alpha;\beta\gamma}$, then from eq. (6), we get

$$\begin{aligned}
\tilde{B}_{\alpha\beta;\gamma}V^\gamma &= (V_{\alpha;\gamma\beta}V^\gamma - R_{\alpha\sigma\gamma\beta}V^\sigma V^\gamma) \\
&= ((V_{\alpha;\gamma}V^\gamma)_{;\beta} - V_{\alpha;\gamma}V^\gamma{}_{;\beta} - R_{\alpha\sigma\gamma\beta}V^\sigma V^\gamma) \\
&= (\dot{V}_\alpha)_{;\beta} - \tilde{B}_{\alpha\gamma}\tilde{B}^\gamma{}_\beta - R_{\alpha\sigma\gamma\beta}V^\sigma V^\gamma).
\end{aligned} \tag{19}$$

Taking the trace over the indices $\alpha\beta$ and using eq. (8) we have

$$(\tilde{\Theta} + \frac{1}{f}V^{\mu}\partial_{\mu}f)_{;\gamma}V^{\gamma} = (\dot{V^{\alpha}}_{;\alpha} - \tilde{B}_{\alpha\gamma}\tilde{B}^{\gamma\alpha} - R_{\sigma\gamma}V^{\sigma}V^{\gamma}). \tag{20}$$

We now use Einstein's equation to get

$$\begin{aligned}&\left(\tilde{\Theta} + \frac{1}{f}V^{\mu}\partial_{\mu}f\right)_{;\gamma}V^{\gamma}\\ &\quad = \left(\dot{V^{\alpha}}_{;\alpha} - \tilde{B}_{\alpha\beta}\tilde{B}^{\beta\alpha} - 8\pi\left(t_{\mu\nu} - \frac{1}{2}tg_{\mu\nu}\right)V^{\mu}V^{\nu}\right).\end{aligned} \tag{21}$$

The acceleration forces are two-fold, one coming from the EM forces within the fluid and the second from the fluid forces themselves (pressures, etc.). To examine what these are, we return to the equation of motion for a fluid with entropy σ in the unified picture eq. (2) and use the fact that

$$S_{\mu\nu} = \partial_{\mu}(f(T)U_{\nu}) - \partial_{\nu}(f(T)U_{\mu}) = V_{\nu;\mu} - V_{\mu;\nu}. \tag{22}$$

For an isentropic fluid,

$$U^{\mu}\partial_{\mu}\sigma = \sigma_{;\mu}V^{\mu} = 0. \tag{23}$$

Therefore, we can write

$$\begin{aligned}T\partial_{\nu}\sigma &= -TU^{\mu}[U_{\mu}\partial_{\nu}\sigma - U_{\nu}\partial_{\mu}\sigma]\\ &= -\frac{T}{f^2}V^{\mu}[V_{\mu}\partial_{\nu}\sigma - V_{\nu}\partial_{\mu}\sigma].\end{aligned} \tag{24}$$

Putting this together with (3), we find

$$-\frac{T}{f^2}V^{\mu}[V_{\mu}\partial_{\nu}\sigma - V_{\nu}\partial_{\mu}\sigma] = qU^{\mu}\left(F_{\nu\mu} + \frac{m}{q}S_{\nu\mu}\right) \tag{25}$$

or

$$mV^{\mu}S_{\mu\nu} = qV^{\mu}F_{\nu\mu} + \frac{T}{f}V^{\mu}[V_{\mu}\partial_{\nu}\sigma - V_{\nu}\partial_{\mu}\sigma]. \tag{26}$$

Therefore,

$$V^{\mu}V_{\nu;\mu} - V^{\mu}V_{\mu;\nu} = \frac{q}{m}\left(V^{\mu}F_{\nu\mu} + \frac{mT}{qf}V^{\mu}[V_{\mu}\partial_{\nu}\sigma - V_{\nu}\partial_{\mu}\sigma]\right) \tag{27}$$

and

$$\dot{V}_{\nu} = -f\partial_{\nu}f + \frac{q}{m}\left(V^{\mu}F_{\nu\mu} + \frac{mT}{qf}V^{\mu}[V_{\mu}\partial_{\nu}\sigma - V_{\nu}\partial_{\mu}\sigma]\right). \tag{28}$$

Defining a pure fluid factor $N_{\mu\nu}$ as

$$N_{\mu\nu} = \frac{T}{f}(\sigma_{;\mu}V_{\nu} - \sigma_{;\nu}V_{\mu}), \tag{29}$$

the acceleration vector simplifies to

$$\dot{V}_\nu = -f\partial_\nu f + \frac{q}{m}V^\mu\left(F_{\nu\mu} + \frac{m}{q}N_{\nu\mu}\right). \tag{30}$$

Defining

$$G_{\mu\nu} = F_{\mu\nu} + \frac{m}{q}N_{\mu\nu}, \tag{31}$$

substitution into (21) yields

$$\begin{aligned}(\tilde{\Theta} + \frac{1}{f}V^\mu\partial_\mu f)_{;\gamma}V^\gamma = \left(-f\partial^\alpha f + \frac{q}{m}V_\beta G^{\alpha\beta}\right)_{;\alpha} - \tilde{B}_{\alpha\beta}\tilde{B}^{\beta\alpha} \\ - 8\pi t_{\mu\nu}V^\mu V^\nu - 4\pi f^2 t,\end{aligned} \tag{32}$$

which simplifies to

$$\begin{aligned}\dot{\tilde{\Theta}} + \dot{\zeta} = (-f\partial^\alpha f)_{;\alpha} + \frac{q}{m}V_{\beta;\alpha}G^{\alpha\beta} + \frac{q}{m}V_\beta G^{\alpha\beta}{}_{;\alpha} \\ -\tilde{B}_{\alpha\beta}\tilde{B}^{\beta\alpha} - 8\pi t_{\mu\nu}V^\mu V^\nu - 4\pi f^2 t,\end{aligned} \tag{33}$$

where we have defined $V^\mu\partial_\mu \ln f = \zeta$. Using the relationship between $B_{\mu\nu}$ and $\tilde{B}_{\mu\nu}$, we find

$$\begin{aligned}\dot{\tilde{\Theta}} + \dot{\zeta} &= (-f\partial^\alpha f)_{;\alpha} + \frac{q}{m}V_{\beta;\alpha}G^{\alpha\beta} + \frac{q}{m}V_\beta G^{\alpha\beta}{}_{;\alpha} \\ &\quad -fB_{\alpha\beta}fB^{\beta\alpha} - 2\tilde{B}^{\alpha\gamma}V_\gamma\partial_\alpha \ln f + \zeta^2 - 8\pi t_{\mu\nu}V^\mu V^\nu - 4\pi f^2 t \\ &= \frac{q}{m}V_{\beta;\alpha}G^{\alpha\beta} + \frac{q}{m}V_\beta G^{\alpha\beta}{}_{;\alpha} - fB_{\alpha\beta}fB^{\beta\alpha} \\ &\quad +\zeta^2 - f^2(\partial^2 \ln f) - 8\pi t_{\mu\nu}V^\mu V^\nu - 4\pi f^2 t.\end{aligned} \tag{34}$$

Simplifying the left-hand side, we have

$$\begin{aligned}\dot{\tilde{\Theta}} = \frac{q}{m}V_{\beta;\alpha}G^{\alpha\beta} + \frac{q}{m}V_\beta G^{\alpha\beta}{}_{;\alpha} - fB_{\alpha\beta}fB^{\beta\alpha} + \zeta^2 \\ -\dot{\zeta} - f^2(\partial^2 \ln f) - 8\pi t_{\mu\nu}V^\mu V^\nu - 4\pi f^2 t.\end{aligned} \tag{35}$$

On further simplification of the $B_{\mu\nu}B^{\nu\mu}$ terms, we have

$$\begin{aligned}\dot{\tilde{\Theta}} = \frac{q}{2m}S_{\alpha\beta}G^{\alpha\beta} + \frac{q}{m}V_\beta G^{\alpha\beta}{}_{;\alpha} - \Bigg(\tilde{\Sigma}_{\alpha\beta}\tilde{\Sigma}^{\beta\alpha} \\ +\tilde{\Omega}_{\alpha\beta}\tilde{\Omega}^{\beta\alpha} + \frac{1}{3}\tilde{\Theta}^2\Bigg) + \zeta^2 - \dot{\zeta} - f^2(\partial^2 \ln f) \\ -8\pi t_{\mu\nu}V^\mu V^\nu - 4\pi f^2 t.\end{aligned} \tag{36}$$

This final form of the generalised Raychaudhuri equation, incorporating the statistical attributes of the gravitating hot plasma, reduces to the standard equation in the limit $f(T) \longrightarrow 1$.

3. A Raychaudhuri equation for the unified non-Abelian magneto fluid

In a recent paper [8], we had generalised the proposed unification of the Abelian magneto fluid [7] to a non-Abelian magneto fluid. The equations of motion for the magneto fluid clearly indicated the possibility of solitonic solutions which are normally absent in the Abelian case. The inherent non-linearities present in the Yang–Mills magneto fluid allow such possibilities and a natural question arises as to what will be the dynamics of a self-gravitating non-Abelian magneto fluid? This is particularly relevant in view of the compelling experimental evidence from the relativistic heavy-ion collider at Brookhaven National Laboratory (BNL) that the universe in its first few moments may have existed as a quark-gluon fluid. Since, large gravitational fields are also present in this epoch, it provides us with a motivation to give a generalisation of the Raychaudhuri equation for the non-Abelian magneto fluid.

The suggestion in [8] was that each worldline would now carry an internal index i labelling the non-Abelian species. Generalising the Lorentz force law to the non-Abelian case, we had derived the equation of motion for the magneto fluid in terms of a unified antisymmetric tensor, ${M^i}_{\mu\nu} = {F^i}_{\mu\nu} + \frac{m}{g}{S^i}_{\mu\nu}$, where ${F^i}_{\mu\nu}$ is the standard Yang–Mills field strength tensor while ${S^i}_{\mu\nu}$ is given by

$$
\begin{aligned}
{S^i}_{\mu\nu} = \partial_\mu(f{U^i}_\nu) - \partial_\nu(f{U^i}_\mu) - igf[A_\mu, U_\nu]^i + igf[A_\nu, U_\mu]^i \\
-imf^2[U_\mu, U_\nu]^i,
\end{aligned} \tag{37}
$$

with the gauge covariant derivative being defined by $\mathcal{D}_\mu = \partial_\mu - ig[A_\mu,]$. The non-Abelian fluid equations of motion corresponding to eq. (3) for the 'Yang–Mills magneto-fluid', with entropy σ are given by

$$
T\partial^\nu\sigma = gM_a^{\mu\nu}U_{a\mu}. \tag{38}
$$

For a non-Abelian magneto fluid velocity vector ${U^i}_\mu$, we can write

$$
{U^{i\mu}}_{;\nu\sigma} - {U^{i\mu}}_{;\sigma\nu} = {R^\mu}_{\rho\nu\sigma}U^{i\rho}. \tag{39}
$$

Examining ${B^i}_{\mu\nu} = {U^i}_{\mu;\nu}$,

$$
{B^i}_{\mu\nu;\alpha}{U_i}^\alpha = ({U^i}_{\mu;\alpha}{U_i}^\alpha)_{;\nu} - R_{\mu\rho\nu\alpha}U^{i\mu}{U_i}^\rho - {U^i}_{\mu;\alpha}{U_i}^\alpha{}_{;\nu}. \tag{40}
$$

From our earlier work [8], we can write

$$
\mathrm{tr}\dot{U}_\mu = {U^i}_{\mu;\nu}{U_i}^\nu = \frac{g}{m}{F^i}_{\mu\nu}{U_i}^\nu \tag{41}
$$

for the generalisation of the Lorentz force law in the particle picture of the non-Abelian magneto fluid; the trace is over the internal, gauge group indices ($i = 1\ldots N = \dim(\text{gauge group})$). Clearly, since now, ${U^i}_\mu{U_i}^\mu = -N$, we can henceforth, assume that the ${U^i}_\mu$ are normalised so that ${U^i}_\mu{U_i}^\mu = -1$, i.e we assume that ${U^i}_\mu \longrightarrow {U^i}_\mu/\sqrt{N}$. With this proviso, we can define an orthogonal projection tensor as before

$$h_{\mu\nu} = g_{\mu\nu} + U^i{}_\mu U_{i\nu}. \tag{42}$$

It follows that

$$U^{j\mu} h_{\mu\nu} = U^{j\mu} g_{\mu\nu} + U^{j\mu} U^i{}_\mu U_{i\nu} = U^j{}_\nu - \delta^{ij} U_{i\nu} = 0 \tag{43}$$

and

$$h_{\mu\nu} U^{j\nu} = g_{\mu\nu} U^{j\nu} + U^i{}_\mu U_{i\nu} U^{j\nu} = U^j{}_\mu - \delta^{ij} U_{i\mu} = 0. \tag{44}$$

Equation (40) can be reduced to

$$B^i{}_{\mu\nu;\alpha} U_i{}^\alpha = (\frac{g}{m} F^i{}_{\mu\alpha} U_i{}^\alpha)_{;\nu} - R_{\mu\rho\nu\alpha} U^{i\mu} U_i{}^\rho - U^i{}_{\mu;\alpha} U_i{}^\alpha{}_{;\nu}. \tag{45}$$

Tracing over indices μ and ν, and defining $\Theta^i = B^i{}_\mu{}^\mu$, we find

$$\dot{\Theta}^i = (\frac{g}{m} F^{i\mu}{}_\alpha U_i{}^\alpha)_{;\mu} - R_{\rho\alpha} U^{i\alpha} U_i{}^\rho - B^i{}_{\mu\alpha} B_i{}^{\alpha\mu}. \tag{46}$$

Once again, we can decompose the tensor $B^i{}_{\mu\nu}$ into its irreducible parts and write

$$B^i{}_{\mu\nu} = U^i{}_{\mu;\nu} = \Sigma^i{}_{\mu\nu} + \Omega^i{}_{\mu\nu} + \frac{1}{3}\Theta^i h_{\mu\nu}. \tag{47}$$

It is to be noted here that we are considering a special sector of the magneto fluid dynamics, that of a non-interacting sector of the full non-Abelian fluid as is seen from our definition of the acceleration vector $a_\mu = U^i{}_{\mu;\nu} U_i{}^\nu$. From a fluid point of view, we are dealing with a multi-species model of the non-Abelian fluid. The full non-Abelian interactions of the fluid will be explored in a future study. Returning to the decomposition of $B^i{}_{\mu\nu}$, we write

$$\begin{aligned}
\Sigma^i{}_{\mu\nu} &= \frac{1}{2}(U^i{}_{\mu;\nu} + U^i{}_{\nu;\mu}) - \frac{1}{3}\Theta^i h_{\mu\nu}, \\
\Omega^i{}_{\mu\nu} &= \frac{1}{2}(U^i{}_{\mu;\nu} - U^i{}_{\nu;\mu}).
\end{aligned} \tag{48}$$

Clearly, these definitions give a decomposition of $B^i{}_{\mu\nu}$ into its irreducible components. As expected, the resulting Raychaudhuri equation, upon substitution of $B^i{}_{\mu\nu} B_i{}^{\nu\mu}$ into eq. (46), is a multi-species generalisation of the standard Raychaudhuri equation. This non-Abelian generalisation of Raychaudhuri's equation gives us a means of studying the shear and vorticity of quark-gluon astrophysical plasmas.

4. Conclusion

High temperature plasmas have an important role to play in the early universe. In what is known as the 'classical' or the 'radiation' epoch of the universe, both gravitational and high temperature effects are of equal importance. Thus, for the evolutionary dynamics of such a hot gravitating plasma, a generalisation of the

Raychaudhuri equation to include finite temperature fluid forces as well as electromagnetic effects was called for. Using a unified magneto-fluid approach to construct a generalised Raychaudhuri equation, we have attempted to respond to this call. Non-linear plasmas also find an application in cosmology through the (now compelling) evidence that a non-Abelian, non-linear quark gluon fluid existed in the early epochs of the universe. To deal with the evolution of this fluid when the gravitational effects are strong, we have laid the foundation of a generalised Raychaudhuri equation for the evolution of non-Abelian gravitating plasmas. Most interesting plasmas involve collective effects which are non-linear even in the special relativistic situation and the inclusion of gravity can only lead to more intriguing highly non linear phenomena. The investigation of further implications of the ideas, germinated in this paper, is a promising avenue for future research. The amalgamation and unification of ideas that cut across barriers always enriches physics. We feel that any work of this kind is a fitting tribute to a man whose physics has crossed international boundaries.

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