

## Quantum manifestation of systems on the macro-scale – the concept of transition state and transition amplitude wave

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**Abstract.** Quantum effects which have usually been associated with micro-scale phenomena can also arise on the macro-scale in situations other than the well-known macro-quantum phenomena of superconductivity and superfluidity. Such situations have been shown here to arise in processes involving inelastic scattering with bound or partially bound systems (not bound in all degrees of freedom), and the macro-quantum behaviour is associated with the state of the total system in transition in the process of scattering. Such a state is designated as a ‘transition-state’. It is pointed out that we have already observed such manifestations for a particular system, the charged particles in a magnetic field where interference effects involving macro-scale matter waves along the magnetic field have been reported [R K Varma *et al*, *Phys. Rev.* **E65**, 026503 (2002)].

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### 1. Introduction

In a series of papers over the last few years the present author has developed the concept of a ‘transition amplitude wave’ in relation to the problem of charged particle dynamics in a magnetic field [1–3]. The concept which is essentially of quantum origin, predicts the existence of one-dimensional matter waves for charged particles moving along magnetic field which are of macroscopic dimensions with an astonishingly large wavelength of  $\sim 5$  cm for typical parameters (electron energy  $\varepsilon \sim 1$  keV and magnetic field  $B \sim 100$  G). Such matter waves with the above macro-scale have indeed been observed [2,4,5], in the form of one-dimensional interference effects. A remark must, of course, be inserted immediately to ward off any misunderstanding, namely, that these matter waves are essentially ‘derived’ ones, and belong to a partially bound system, as distinct from the basic de Broglie waves.

A general formalism for the derivation of the equation of evolution for the transition amplitude of the above problem was given in ref. [1]. However, a more direct

qualitative derivation of the required expression was presented in the Appendix of the experimental paper [2]. It is desirable, however, to give a more precise and fuller formal derivation of the above-mentioned expression. The purpose of this paper is to provide such a derivation. Set in the Feynman path integral formalism, it will be seen to be more general in its applicability. Hence its importance!

## 2. Transition amplitude in the path-integral representation

The transition amplitude plays a central role in the evolution of quantum phenomena. This is expressed by an integral equation of the form

$$\psi(\mathbf{X}_2, t_2) = \int d\mathbf{X}_1 K_V(\mathbf{X}_2, t_2; \mathbf{X}_1, t_1) \psi(\mathbf{X}_1, t_1) \quad (1)$$

with  $K_V$  representing the transition amplitude, and  $\psi(\mathbf{X}_1, t_1)$  and  $\psi(\mathbf{X}_2, t_2)$  denoting the state at  $(\mathbf{X}_1, t_1)$  and  $(\mathbf{X}_2, t_2)$ ,  $t_2 > t_1$ . Specifically,  $K_V(\mathbf{X}_2, t_2; \mathbf{X}_1, t_1)$  denotes the amplitude that a particle emitted at  $\mathbf{X}_1$  at time  $t_1$  transists to the point  $\mathbf{X}_2$  at the time  $t_2$ . It is also called the ‘propagator’.

Following Feynman and Hibbs [6], the ‘transition amplitude’  $K_V$  is represented by a functional integral of the form

$$K_V(\mathbf{X}_2, t_2; \mathbf{X}_1, t_1) = \int_1^2 \exp \left[ \frac{i}{\hbar} \int_{t_1}^{t_2} dt L(\mathbf{X}(t), \dot{\mathbf{X}}(t); t) \right] \mathcal{D}\mathbf{X}(t), \quad (2)$$

where  $L$  denotes the Lagrangian for the dynamical system, and where  $\mathcal{D}\mathbf{X}(t)$  denotes a functional differential element for the paths over which the integration is to be carried out between the points  $\mathbf{X}_1$  and  $\mathbf{X}_2$ . Two of the properties of the propagator  $K_V$  may be noted: First, if we regard  $K_V$  as a function of  $(\mathbf{X}_2, t_2)$ , keeping  $(\mathbf{X}_1, t_1)$  fixed then  $K_V(\mathbf{X}_2, t_2)$  as a function of  $(\mathbf{X}_2, t_2) \equiv (\mathbf{X}, t)$  is a Schrödinger probability amplitude and satisfies the Schrödinger equation.

$$i\hbar \frac{\partial}{\partial t} K_V(\mathbf{X}, t) = \left( -\frac{\hbar^2}{2m} \nabla^2 + V \right) K_V(\mathbf{X}, t). \quad (3)$$

Secondly, it can be shown that

$$K_V(\mathbf{X}_2, t_2; \mathbf{X}_1, t_1) \xrightarrow[t_2 \rightarrow t_1 + 0]{} \delta(\mathbf{X}_2 - \mathbf{X}_1) \quad (4)$$

and

$$i\hbar \frac{\partial K_V(2, 1)}{\partial t_2} - H_2 K_V(2, 1) = \delta(\mathbf{X}_2 - \mathbf{X}_1) \delta(t_2 - t_1) \quad (5)$$

with

$$H_2 = -\frac{\hbar^2}{2m} \nabla_2^2 + V(\mathbf{X}_2). \quad (6)$$

By virtue of eq. (3),  $K_V(2,1)$  is a Schrödinger probability amplitude with respect to the point  $(\mathbf{X}_2, t_2)$  of its argument. But taking note also of its argument  $(\mathbf{X}_1, t_1)$ , it may also be interpreted as a ‘conditional probability amplitude’: namely the amplitude of finding the particle at  $(\mathbf{X}_2, t_2)$  if it was at  $\mathbf{X}_1$  at the time  $t_1$ . Taking note of this interpretation and the fact of its being a transition amplitude, allows us to impart it a novel physically observable attribute which we shall discuss in what follows. Note that in eq. (1) an integration over  $\mathbf{X}_1$  with the wave function  $\psi(\mathbf{X}_1, t_1)$  gives the contribution to  $\psi(\mathbf{X}_2, t_2)$  from *all* points  $\{\mathbf{X}_1\}$ . But if the source of particles is only at the point  $\mathbf{X}$ , then we write  $\psi(\mathbf{X}_1, t_1) = \delta(\mathbf{X} - \mathbf{X}_1)$ , so that

$$\psi(\mathbf{X}_2, t_2) = K_V(\mathbf{X}_2, t_2; \mathbf{X}, t_1), \quad (7)$$

the contribution coming only from the point  $\mathbf{X}$ . The wave amplitude  $\psi_2(\mathbf{X}_2)$  at  $\mathbf{X}_2$  (regarding  $\mathbf{X}$  fixed) is then essentially the transition amplitude  $K_V(\mathbf{X}_2, t_2; \mathbf{X}, t_1)$ . The state represented by  $\psi_2(\mathbf{X}_2, t_2; \mathbf{X}, t_1)$  then has its origin attached to it through the argument  $(\mathbf{X}, t_1)$ . In that sense it is a kind of ‘conditional state’. One would interpret it as the amplitude for finding the particle at  $(\mathbf{X}_2, t_2)$  if it originated at  $(\mathbf{X}, t_1)$ . Since  $K_V$  is a Schrödinger (conditional) probability amplitude (by virtue of eqs (3) and (7)) it must satisfy the integral equation (1), so that we have

$$K_V(\mathbf{X}_2, t_2; \mathbf{X}_1, t_1) = \int d\mathbf{X}_3 K_V(\mathbf{X}_2, t_2; \mathbf{X}_3, t_3) K_V(\mathbf{X}_3, t_3; \mathbf{X}_1, t_1) \quad (8)$$

another important property of the propagator. Note that (7) also follows from (8) in making use of (4). Hence the mutual consistency of the above set of equations and definitions.

The conditional Schrödinger amplitude specified by the propagator  $K_V$  which carries the label of the initial state (besides that of the final state through its argument) is, as we shall see, a rather interesting object since it carries the specific information of the transition involved. We shall discuss, in what follow, some of its interesting quantum properties.

### 3. Perturbation expansion of the Feynman kernel

We shall first define an unperturbed one-dimensional system specified by a time-independent potential  $U$  for which the Schrödinger eigenvalue problem can be solved exactly with eigenvalues  $E_n$  and eigenfunctions  $\phi_n$ . The total potential  $V$  is then split into  $V = U + W$ , where  $W$  represents, in general, a time-dependent perturbation.

If we denote by  $K_U$ , the propagator with the potential  $U$ , it is well-known that

$$K_U(X_2, t_2; X_1, t_1) = \sum \phi_n(X_2) e^{-iE_n(t_2-t_1)/\hbar} \phi_n^*(X_1). \quad (9)$$

Then the perturbation expansion of the propagator  $K_V$  around  $K_U$  can be written in the form [6]:

$$\begin{aligned}
 K_V(2, 1) &= K_U(2, 1) - \frac{i}{\hbar} \int K_U(2, 3)W(X_3, t_3) \\
 &\quad \times \left[ K_U(3, 1) - \frac{i}{\hbar} \int K_U(3, 4)W(X_4, t_4)K_U(4, 1)dX_4dt_4 \right. \\
 &\quad \left. + \dots \right] dX_3dt_3, \tag{10}
 \end{aligned}$$

where the arguments (2, 3), (3, 4) etc. of  $K_U$  denote the totality of variables  $(X_2, t_2)$ ,  $(X_3, t_3)$ ,  $(X_4, t_4)$  etc. Writing the expression (9) for  $K_U$  in (10) we obtain

$$\begin{aligned}
 K_V(2, 1) &= \sum_n \phi_n(X_2)\phi_n^*(X_1)e^{-iE_n(t_2-t_1)/\hbar} \\
 &\quad - \frac{i}{\hbar} \sum_{n,m} \int \phi_m(X_2)\phi_m^*(X_3)W(X_3, t_3)e^{-iE_m(t_2-t_3)} \\
 &\quad \times \phi_n(X_3)\phi_n^*(X_1)e^{-iE_n(t_3-t_1)/\hbar} dX_3 dt_3 + \dots \\
 &= \sum_{n,m} \phi_m(X_2)\lambda_{m,n}(t_2, t_1)\phi_n^*(X_1), \tag{11}
 \end{aligned}$$

where the coefficients  $\lambda_{mn}(t_2, t_1)$  are the ‘transition matrix elements’, and may be written as a perturbation expansion

$$\lambda_{mn} = \delta_{mn}e^{-iE_n(t_2-t_1)/\hbar} + \lambda_{mn}^{(1)} + \lambda_{mn}^{(2)} + \dots \tag{12}$$

with

$$\begin{aligned}
 \lambda_{mn}^{(1)} &= -\frac{i}{\hbar} \int dX_3 \int_{t_1}^{t_2} dt_3 \phi_m^*(X_3)W(X_3, t_3)\phi_n(X_3) \\
 &\quad \times e^{\frac{i}{\hbar}[E_m(t_3-t_2)-E_n(t_3-t_1)]}. \tag{13}
 \end{aligned}$$

With the form (11),  $\lambda_{mn}(t_2, t_1)$  includes all orders of perturbation corresponding to one or more scatterings, with  $\lambda_{mn}^{(1)}$  giving the first-order contribution, while the first term of (12) pertains to no scattering and thus no change of state. Hence the Kronecker  $\delta_{mn}$ .

We now seek a physical interpretation of the terms in the summation in the expression for  $K_V$ . Consider one of the terms such as

$$A_{mn}(X_2, t_2; X_1, t_1) = \phi_m(X_2)\lambda_{mn}(t_2, t_1)\phi_n^*(X_1). \tag{14}$$

Following Feynman and Hibbs [6] (p. 116) it can be interpreted as follows:  $\phi_n^*(X_1)$  is the amplitude that if we are at point  $X_1$  we are in the state  $n$ ,  $\lambda_{mn}$  is the transition amplitude matrix element for going from the state  $n$  at the time  $t_1$  to the state  $m$  at the time  $t_2$ . Thus  $\lambda_{mn}(t_2, t_1)\phi_n^*(X_1)$  may be interpreted as the amplitude for the system being initially (time  $t_1$ ) at the point  $X_1$  and the state  $n$  and transiting over a time  $(t_2 - t_1)$  to the state  $m$ , being still at the point  $X_1$ . Finally interpreting the factor  $\phi_m(X_2)$  as the amplitude for the system to be found at the point  $X_2$ , knowing that we are in the state  $m$ ,  $A_{mn}(X_2, t_2, X_1, t_1)$  is finally interpreted as the amplitude for being initially in the state  $n$  at the time  $t_1$  and to be found in the state  $m$  at the later time  $t_2$  as a result of the time-dependent perturbation  $W(X, t)$ .

#### 4. Equation of evolution for the ‘conditional state’

We introduced in §1 the terminology of the conditional state through eq. (7), as one example. We shall discuss here this concept in more general terms and in more detail. As already discussed in §1, the propagator  $K(X_3, t_3; X_1, t_1)$  can be interpreted as a ‘conditional’ probability amplitude. The equation of evolution (8) of the propagator  $K_V$  is the proper equation of evolution for the conditional state.

If in Eq. (8), we let [from (4)]

$$K_V(X_3, t_3; X_1, t_1) \Rightarrow \lim_{t_3 \rightarrow t_1} K_V = \delta(X_3 - X_1) \quad (15)$$

then we essentially recover eq. (7) except for the suffixes on  $X$  and  $t$ .

$$K_V(X_2, t_2; X_1, t_1) = \int dX_3 K_V(X_2, t_2; X_3, t_3) \delta(X_3 - X_1) \quad (16)$$

$$\psi(X_2, t_2; X_1, t_1) = K_V(X_2, t_2; X_1, t_1). \quad (17)$$

Note that the expression (7) for  $\psi(X_2, t_2; X_1, t_1)$  or expression (17) for  $\psi(X_2, t_2; X_1, t_1)$  in terms of  $K_V$  as the conditional state, was obtained with  $K_V(X_3, t_3; X_1, t_1)|_{t_3 \rightarrow t_1} = \delta(X_3 - X_1)$ . The interpretation for the relation (15) is that it gives the probability amplitude to be at  $X_1$  at the time  $t_1$ , and to be at  $X_3$  at the time  $t_3 \rightarrow t_1$ . The  $\delta$ -function describes the fact that there is no evolution.

So far the conditional state was defined with the initial state being localized at a point  $X$ , defined by the  $\delta$ -function  $\delta(X_1 - X)$ . However, a more interesting situation arises, when the conditional state is defined with the initial state being one of the eigenstates of the unperturbed system or sharply centered around it. Therefore, in what follows, we shall gradually pass on from the initial state  $\delta(X_1 - X)$  to the one corresponding to one of the eigenstates and obtaining the final state corresponding to it.

To this end, we begin by expressing the  $\delta$ -function in terms of the eigenfunction  $\phi_n(X)$  as a completeness property

$$\delta(X_1 - X) = \sum_n \phi_n(X_1) \phi_n^*(X). \quad (18)$$

Then a given term of the sum,  $\phi_\alpha(X_1) \phi_\alpha^*(X)$ , is to be interpreted as:  $\phi_\alpha^*(X)$ , the amplitude to be at  $X$ , but in the state  $\alpha$ , while  $\phi_\alpha(X_1)$  for the particle to be in state  $\alpha$ , but at the position  $X_1$ . The  $\delta$ -function is ensured only if all the states  $n$  are allowed to operate in the sum with equal weights.

If, on the other hand, the sum is centred around a particular eigenstate  $N$ , we have the function

$$\Delta(X_3 - X_1; N) = \sum_\nu \phi_{N+\nu}(X_3) \phi_{N+\nu}^*(X_1) p(\nu), \quad (19)$$

where  $p(\nu)$  denotes the weight function for  $\nu$ . If  $p(\nu) = \exp[-(\nu/\nu_0)^2]$ , then the dominant term in the sum is  $\phi_N(X_1) \phi_N^*(X)$ , if  $\nu_0 \ll N$ .

If we now use the function (19) instead of  $\delta(X_3 - X_1)$  in eq. (16), we have, employing the expression (11) for  $K_V(2, 3)$

$$K_V(X_2, t_2; X_1, t_1) = \int dX_3 \sum_{m,n} \phi_m(X_2) \lambda_{mn}(t_2, t_1) \phi_n^*(X_3) \times \sum_{\nu} \Phi_{N+\nu}(X_3) \phi_{N+\nu}^*(X_1) p(\nu). \quad (20)$$

Carrying out integration over  $X_3$  on the right-hand side we get

$$K_V(X_2, t_2; X_1, t_1; N) = \sum_{m,\nu} \phi_m(X_2) \lambda_{m,N+\nu}(t_2, t_1) p(\nu) \phi_{N+\nu}^*(X_1), \quad (21)$$

where we have added a label  $N$  to the kernel  $K_V(X_3, t_3; X_1, t_1; N)$  on the left, indicating the fact that the distribution of eigenstates defining the initial state is peaked at  $n = N$ . Considered as a Schrödinger wave amplitude with respect to the argument  $(X_2, t_2)$ , the  $K_V(X_2, t_2; X_1, t_1; N)$  is then a probability amplitude of finding the particle at  $(X_2, t_2)$ , if initially it had the amplitude at  $X_1$  with a sum of eigenstates centred around  $N$  as given by (19). The initial distribution (19) in  $X_1$  is a broad one as against the  $\delta$ -function,  $\delta(X_2 - X_1)$ . In the limit  $\nu_0 \rightarrow 0$ , so that  $p(\nu) = 0$ , for  $\nu \neq 0$ , there is only one term,  $\nu = 0$  in the summation (21), and the situation is completely complementary to the  $\delta$ -function case. The corresponding  $K_V$  is then a ‘conditional’ probability amplitude with the initial eigenstate label  $N$ , rather than the initial position label  $X_1$  as in eq. (7) or (17).

$$K_V(X_3, t_3, X_1, t_1; N) = \sum_m \phi_m(X_3) \lambda_{mN} \phi_N^*(X_1). \quad (22)$$

This state is now comprised of the eigenstates  $\phi_m$  obtained through transitions  $\lambda_{mN}$  from the well-defined initial state  $\phi_N$ . We may call this a ‘transition state’ (for want of a more suitable name) as it is related to a state in transition. This is also essentially a conditional state but with the initial state being an eigenstate. This nomenclature emphasizes this specification. It is found to have some rather interesting properties as we shall see in what follows.

### 5. ‘Transition state’ in two dimensions

It will be necessary to consider a generalization of the above formalism to two degrees of freedom to uncover these properties. The two degrees of freedom may refer to the two dimensions of a given particle. Let  $x$  and  $y$  refer to the two coordinates of the particle. Then the two-dimensional counterpart of eq. (11) is given by

$$K_V(2, 1) = \sum_{\substack{n,m \\ \mu\nu}} \phi_m(x_2) \chi_{\mu}(y_2) \lambda_{mn;\mu\nu} \phi_n^*(x_1) \chi_{\nu}^*(y_1), \quad (23)$$

where we assume that the perturbation potential  $W(x, y)$  is time-independent, so that  $\lambda_{mn;\mu\nu}$  is also time-independent.

We recall that  $K_V(2, 1)$  is a Schrödinger amplitude with respect to the coordinates '2'  $\equiv (x_2, y_2, t_2)$  of the particle, subject to the initial state being given as fixed. If the initial state is weighted as in (19), centred around a  $(N, K)$  with the weight functions  $p(\sigma, \tau) \simeq \exp[-(\sigma/\sigma_0)^2 - (\tau/\tau_0)^2]$  so that (23) leads to

$$K_V(2, 1; N, K) = \sum_{\substack{m, \sigma \\ \mu, \tau}} \phi_m(x_2) \chi_\mu(y_2) \lambda_{m, N+\sigma; \mu, K+\tau} \\ \times \phi_{N+\sigma}^*(x_1) \chi_{K+\tau}^*(y_1) p(\sigma, \tau), \quad (24)$$

where we have written  $n = N + \sigma$ ,  $\nu = K + \tau$ , and included the labels  $(N, K)$  on  $K_V$ , signifying the fact that the initial state is 'biased' to be centred around the eigenstate  $(N, K)$ . The final state denoted by the left-hand side is then a conditional state. Since it is essentially defined by the transition  $\lambda_{m, N; \mu, K}$  we have called it a 'transition state' in the limit  $\sigma_0, \tau_0 \ll 1$ .

### 5.1 Scattering problem with one 'bound' and one 'free' degree of freedom

We shall now apply the above formalism to a scattering problem where the particle is bound in its one degree of freedom and free in the other: the potential  $U$  in the  $x$ -coordinate, and a constant potential along the  $y$ -coordinate. Since the perturbation  $W$  which induces the transition  $\lambda_{m, N; \mu, K}$  is assumed to be time-independent, the total energy is conserved across the transition so that

$$E_m + E_\mu = E_N + E_K, \quad (25)$$

where  $E_N$  and  $E_K$  are respectively the energy of the initial 'bound' and 'free' degrees of freedom. Correspondingly  $\phi_N, \phi_m$  represent bound state eigenfunctions while  $\chi_K, \chi_\mu$  represent plane waves along  $y$ .

$$\chi_K = e^{iKy}, \quad \chi_\mu = e^{i\mu y} \quad (26)$$

with  $(K, \mu)$  being continuous wave number variables, so that

$$E_K = \frac{(\hbar K)^2}{2m}, \quad E_\mu = \frac{(\hbar \mu)^2}{2m}. \quad (27)$$

Now from (23) the transition state (with the initial labels  $N, K$ ) is given by (with  $\sigma_0 = \tau_0 = 0$ )

$$K_V(2, 1; N, K) = \sum_{m, \mu} \phi_m(x_2) \phi_N^*(x_1) \chi_\mu(y_2) \chi_K^*(y_1) \lambda_{m, N; \mu, K}. \quad (28)$$

Note that,

$$\chi_\mu(y_2) \chi_K^*(y_1) = e^{i\mu y_2 - iKy_1} = e^{i(\mu - K)y_2} e^{iK(y_2 - y_1)}. \quad (29)$$

Using (25) and (27) one can calculate the quantity  $(\mu - K)$  in (29). Assuming that the transition  $N \rightarrow m = N + \alpha$ , is such that  $\alpha \ll N$ , then from the energy conservation (24) we have

$$E_\mu - E_K = E_N - E_m = -\alpha \frac{\partial E_N}{\partial N}. \quad (30)$$

Using the expression (27) for  $E_K$  and  $E_\mu$ , we get

$$(\mu - K) = - \left( \frac{\hbar K}{m} \right)^{-1} \frac{\alpha \partial E_N}{\hbar \partial N} = -\frac{\alpha \varpi}{v}, \quad (31)$$

where we have approximated  $(\mu + K) \simeq 2K$ , in view of  $\alpha \ll N$ ,  $\alpha$  being the level interval across which transition has taken place, and where  $v = \hbar K/m$  is the velocity along  $y$ , while  $\varpi = (\partial E_N / \hbar \partial N)$  is a frequency corresponding to the bound motion in the  $x$ -coordinate in the correspondence limit.

### 5.2 Harmonic oscillator in the bound degree of freedom

We now specialize to the case where the bound degree of freedom is a harmonic oscillator with the potential  $U(x) = \frac{1}{2}m\Omega^2 x^2$ ; with the frequency  $\Omega$ , so that  $E_N = (N + \frac{1}{2})\hbar\Omega$ . The expression for  $(\mu - K)$  is then obtained from (30), as  $(\mu - K) = -\alpha\Omega/v$ .

Consider now the complete expression for  $K_V(2, 1; N, K)$  of eq. (28) with  $m = N + \alpha$ . Note that because of the energy conservation (25), the summation (or integral) over  $\mu$  is restricted to values given by (31), which is in turn related to  $\alpha = m - N$ . We thus have from (28) for the case under consideration.

$$\begin{aligned} K_V(2, 1; N, K) &= \sum_{\alpha} \phi_{N+\alpha}(x_2) \phi_N^*(x_1) \exp[i(\mu - K)y_2 + iK(y_2 - y_1)] \\ &\quad \times \left\{ \delta_{N+\alpha, N} \delta_{\mu K} e^{-i(E_N + E_K)(t_2 - t_1)/\hbar} \right. \\ &\quad \left. - i\pi e^{-i(E_N + E_K)(t_2 - t_1)/\hbar} \right. \\ &\quad \left. \times \delta(E_N + E_K - E_m - E_\mu) W_{N+\alpha, N; \mu K}, \right. \end{aligned} \quad (32)$$

where  $W_{N+\alpha, N; \mu K}$  is the matrix element of the perturbation  $W(x, y)$  between the states  $(\phi_{N+\alpha}\chi_\mu)$  and  $(\phi_N\chi_K)$ , and the  $\delta$ -function in energy arises from the time integration in the expression for  $\lambda_{mn; \mu K}^{(1)}$ , when the perturbation  $W$  is time-independent. Note that the  $\delta$ -function term  $\sim \delta_{N+\alpha, N} \delta_{\mu K}$  in (32) corresponds to the unperturbed part of  $K_V(2, 1; N, K)$ , which would be non-zero only for  $N + \alpha = N$ ; that is  $\alpha = 0$ , and for  $\mu = K$ .

On the other hand, the factor  $W_{N+\alpha, N; \mu K}$  in the expression for  $\lambda_{N+\alpha, N; \mu K}^{(1)}$ , which is the matrix element of the perturbation  $W(x, y)$ , is also defined for  $\alpha = 0$ . This pertains to the elastic part of the transition amplitude. Separating it from the inelastic part we have

$$\begin{aligned} K_V(2, 1; N, K) &= \phi_N(x_2) \phi_N(x_1) \exp[iK(y_2 - y_1)] \\ &\quad \times \exp[-i(E_N + E_K)(t_2 - t_1)/\hbar] \\ &\quad \times \left\{ 1 - i\pi \delta(\bar{E}) \left[ W_{NN; KK} + (\phi_N(x_2) \phi_N(x_1))^{-1} \right. \right. \end{aligned}$$

$$\times \sum_{\alpha \neq 0} \left. \left. \phi_{N+\alpha}(x_2) \phi_N(x_1) W_{N+\alpha, N; \mu K} e^{i(\mu-K)y_2} \right] \right\}, \quad (33)$$

where we have denoted for brevity  $\bar{E} = (E_N + E_K - E_{N+\alpha} - E_\mu)$ . If the perturbation potential  $W(x, y)$  in the expression for  $\lambda_{mN, \mu K}^{(1)}$  is centred around  $y = y_0$ , then the matrix element  $W_{mN; \mu K}$  is

$$\begin{aligned} W_{mN; \mu K} &= \int dx dy e^{-i\mu y} \phi_m(x) W(x, y - y_0) e^{iKy} \phi_N(x) \\ &= e^{-i(\mu-K)y_0} \tilde{W}_{mN; \mu K}. \end{aligned} \quad (34)$$

Now define  $K_V^{(1)}(2, 1; N, K)$  as the kernel describing the transition  $(N, K \Rightarrow N + \alpha, \mu)$  for all integers  $\alpha$ . The matrix element  $W_{N+\alpha, N; \mu K}$ , of the perturbation  $W$  would be non-zero for the set of values  $(\alpha, \mu)$  which are consistent with the energy conservation given by (30) to the lowest order.

If  $\alpha \ll N$ , as assumed, we may write  $\Phi_{N+\alpha} = \phi_N + \alpha \partial \phi_N / \partial N$ , then the perturbation transition kernel is given by (from (33))

$$\begin{aligned} K_V^{(1)}(2, 1; NK) &= -i\pi \exp[-i(E_N + E_K)(t_2 - t_1)/\hbar] \\ &\quad \times \phi_N(X_2) \phi_N(X_1) \delta(\bar{E}) e^{iK(y_2 - y_1)} \\ &\quad \times \left\{ \tilde{W}_{NN; KK} + \sum_{\alpha} \alpha \frac{\partial \ln \phi_N(X_2)}{\partial N} \right. \\ &\quad \left. \times \tilde{W}_{N+\alpha, N; \mu K} e^{-i\alpha\Omega(y_2 - y_0)/v} + \dots \right\}, \end{aligned} \quad (35)$$

where eq. (33) has been used and  $(\mu - K)$  has been substituted for from (31) identifying  $\varpi$  with  $\omega$ .

Expression (35) has a rather interesting structure which needs to be taken note of. It has under the summation over  $\alpha$ , an exponential  $\exp[-i\alpha\Omega(y_2 - y_0)/v]$  which derives from the term  $\exp[-i(\mu - K)(y_2 - y_0)]$  and which represents a (one-dimensional) plane wave with a difference wave number  $(\mu - K) = -\alpha\Omega/v$  originating at  $y = y_0$ . Since  $\alpha$  takes values  $1, 2, 3, \dots$ , the sum is over all the harmonic waves with the wave number  $(\mu - K) = -\alpha\Omega/v$  originating at the centre  $y_0$  of the potential, while the original (large) de Broglie wave number  $K$  appears only in the overall common factor  $\exp[iK(y_2 - y_1)]$  representing a direct propagation from the initial point  $y_1$  to the final point  $y_2$ .

The consequence of such a structure is that if there are more than one scattering centres denoted by  $(y_0^{(1)}, y_0^{(2)}, \dots)$ , then  $K_V^{(1)}$  would have the form

$$\begin{aligned} K_V^{(1)}(2, 1; NK) &\sim e^{iK(y_2 - y_1)} \left\{ \tilde{W}_{N, K} + \sum_{\alpha} \alpha \frac{\partial \ln \phi(x_2)}{\partial N} \right. \\ &\quad \left. \times \left[ \tilde{W}_{N, \alpha}^{(1)} e^{-i\alpha\Omega(y_2 - y_0^{(1)})/v} + \tilde{W}_{N, \alpha}^{(2)} e^{-i\alpha\Omega(y_2 - y_0^{(2)})/v} + \dots \right] \right\}. \end{aligned} \quad (36)$$

If we then calculate the modulus squared  $|K_V^{(1)}(2, 1; NK)|^2$  to obtain the transition probability, one would get interference terms involving the terms  $\sim e^{-i\alpha\Omega(y_2 - y_0^{(1)})/v}$ ,

and  $e^{i\alpha\Omega(y_2-y_0^{(2)})}$  while the overall factor  $e^{iK(y_2-y_1)}$  (involving the original wave number  $K$ ) is annihilated. The initial de Broglie wave number is thus not involved in the interference pattern. These terms represent a set of plane waves along the  $y$ -direction with the wavelength  $\lambda_\alpha = 2\pi v/\alpha\Omega$ . These waves, which are associated with the transition amplitude, may be called ‘transition amplitude waves’, with  $\alpha$  being the harmonic number. From the nature of their derivation, it is clear that these are matter waves, but are interestingly independent of the Planck quantum  $\hbar$  to the lowest order in the present case.

Of course, these are not the basic de Broglie waves, but are derived, and have astonishingly long wavelengths. Assuming for example, the frequency of the harmonic oscillator to be  $\Omega \sim 10^9 \text{ rad s}^{-1}$  and a velocity  $v \sim 10^9 \text{ cm s}^{-1}$ , yields a value for  $\lambda_1 = 2\pi v/\Omega \sim 2\pi \text{ cm}$ . This is a rather large value for a matter wavelength, which we have indeed observed in our experiments [1] in the form of one-dimensional matter wave interference effects in a system of charged particles (electrons) moving along a magnetic field. Here the bound  $x$ -motion correspond to the Landau bound states in the normal direction (with the frequency  $\Omega$  being the gyrofrequency  $\Omega = eB/mc$ ) and the free motion along  $y$ , the magnetic field direction.

## 6. Summary and concluding remarks

As mentioned in the Introduction, the concept of transition amplitude wave was developed by the author in relation to the problem of charged particle dynamics in a magnetic field. This problem which involves effectively a harmonic oscillator in the normal direction to the magnetic field (of frequency  $\Omega = eB/mc$ , the particle gyrofrequency), and a free motion along the magnetic field, is completely analogous to the problem discussed in this paper. The magnetic field value which would correspond to an electron gyrofrequency  $\Omega = 10^9 \text{ rad s}^{-1}$ , used in the above estimate is  $B = 100 \text{ G}$ .

The derivation given in this paper culminating in the expression (36) for the transition kernel thus provides a formal justification for the transition amplitude concept introduced in a qualitative manner in the Appendix of ref. [2], with the object  $e^{-i\alpha\Omega(y-y_0)/v}$  in (34) being the wave function describing the macro-matter wave being the same as obtained in ref. [2]. Though the derivation given here is for a simple case (a harmonic oscillator in one of the dimensions and free motion in the other) the concept is applicable to more general cases. The author has, for example, applied this concept to atoms and molecules [7] where the centre of mass motion takes up the role of free motion. Interestingly, the corresponding expression for the macro-matter wavelength has the same form as obtained above, namely  $\lambda_M = 2\pi v/\omega$ , where  $v$  is now the velocity of the centre of mass and  $\omega$  is the orbital frequency of the internal motion of the atom or molecule in the correspondence limit.

What is interesting and remarkable about the expression (36) is that it consists of a sum of the transition amplitude waves  $e^{-i\alpha\Omega(y_2-y_0^{(1)})/v}$ ,  $e^{-i\alpha\Omega(y_2-y_0^{(2)})/v}$  etc. with their points of origin  $y_0^{(1)}, y_0^{(2)}, \dots$  specified therein. This indicates that they arise only as a consequence of and subsequent to the scattering process. An image profile of the interference of these waves would thus carry information about the

relative location of the scattering centres, such as in a crystal or atoms in a long molecule, for example.

In an extension of the experiment already reported in [1], we have demonstrated in a simple manner, the possibility of extracting such an information from the interference patterns. The experiment of ref. [1] was repeated with varying grid positions between the electron gun and the collector plate. As was emphasized, the grid had served as a scatterer of electrons moving along the magnetic field, and hence as the generator of the transition amplitude wave at its position. As the position of the grid is varied, the interference current profile at the collector plate (and the grid) would vary accordingly. This is indeed found to be the case in the performed experiment [5]. Inverting the argument, one can say that by analysing the interference pattern one can deduce information about the relative position of the grid *vis à vis*, the plate. Moreover, the frequency of the internal motion, the gyrofrequency in this case (corresponding to the Landau levels), is also deducible from the experimental data, as also the Landau level structure through the harmonics observed.

Finally, we wish to emphasize the importance of the ‘transition amplitude state’ which results from the process of inelastic scattering of electrons (or any other particle) off a bound system. This is distinct from both the initial state and what we have been calling as the ‘final state’ in the scattering process. It is a kind of a projection of the final state on to the initial state, indicating the ‘extent’ by which the initial state has changed in the process of scattering. The interesting point is that this ‘extent’ of change is directly observable through the interference phenomena in the cases discussed above. We give this state a distinct identity calling it a ‘transition amplitude state’ or for short a ‘transition state’ with the understanding that this name will convey the required meaning. This may also be called an ‘overlap state’ if one so prefers.

Besides the experimental results of ref. [1], a more dramatic consequence of such a state has already been reported [8,9]. This is the observation on the detection of a curl-free vector potential on the macro-scale and in one dimension through the dynamics of the transition amplitude generated as a consequence of scattering of electrons moving in a magnetic field. This is the Aharanov–Bohm-like effect, but occurs on the macro-scale and in one dimension. This effect was predicted by the author in ref. [1], and was subsequently observed [8].

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