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Dirac constraint analysis and symplectic structure of anti-self-dual Yang–Mills equations

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Abstract. We present the explicit form of the symplectic structure of anti-self-dual Yang–Mills (ASDYM) equations in Yang's J- and K-gauges in order to establish the bi-Hamiltonian structure of this completely integrable system. Dirac's theory of constraints is applied to the degenerate Lagrangians that yield the ASDYM equations. The constraints are second class as in the case of all completely integrable systems which stands in sharp contrast to the situation in full Yang–Mills theory. We construct the Dirac brackets and the symplectic 2-forms for both J- and K-gauges. The covariant symplectic structure of ASDYM equations is obtained using the Witten–Zuckerman formalism. We show that the appropriate component of the Witten–Zuckerman closed and conserved 2-form vector density reduces to the symplectic 2-form obtained from Dirac's theory. Finally, we present the Bäcklund transformation between the J- and K-gauges in order to apply Magri's theorem to the respective two Hamiltonian structures.

Keywords. Integrable equations in physics; integrable field theories; Dirac constraint analysis; symplectic structure; anti-self-dual Yang–Mills equations.

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1. Introduction

Self-dual gauge fields were originally investigated as instanton solutions of the Yang–Mills field equations which provide dominant contribution to the Euclidean path integral in the quantization of Yang–Mills fields [1]. However, the interest in ASDYM equations has now shifted to a study of its remarkable mathematical properties as a completely integrable system. Most of the completely integrable nonlinear partial differential equations that we know can be obtained as reductions of ASDYM equations and the general expectation is that the ASDYM system itself is *the* framework for integrable equations, which is first conjectured by Ward [2]. We refer to Mason and Woodhouse [3] and Ablowitz and Clarkson [4] for a complete account of these considerations and related topics. There is, however, an exception to this folk-lemma which is provided by Monge–Ampère equations [5].

There is no precise definition of complete integrability but one of the properties we expect from such a system is the bi-Hamiltonian structure. This enables us to obtain an infinite set of conserved quantities which are in involution with respect to Poisson brackets defined by both Hamiltonian structures through the theorem of Magri [6]. Schiff [7] has considered the bi-Hamiltonian structure of ASDYM system. The approach we shall follow here differs from that of Schiff.

In this paper we shall discuss the symplectic structure of ASDYM equations in Yang's two formulations [8]. We shall apply Dirac's theory of constraints [9] to the degenerate Lagrangians that yield the ASDYM equations. We find that the constraints are second class as in the case of all integrable systems [5]. This is in marked contrast to the full Yang–Mills theory where the constraints are first class. Nevertheless, it is expected because in the first place Yang's equations for self-duality are obtained for two particular choices of gauge. The constraint analysis yields the Dirac brackets, or the Hamiltonian operators in the language of integrable systems. The symplectic 2-form is obtained by evaluating the Poisson bracket of Dirac's constraints. It is also the inverse of the Hamiltonian operator.

The usual approach to Hamiltonian structure starts with a choice of time variable and is necessarily non-covariant. For Euclideanized ASDYM equations, the definition of the independent variable with respect to which ASDYM equations can be formulated as a Hamiltonian system becomes a critical matter as physically there cannot be a distinguished independent variable that can be regarded as 'time' coordinate. However, in Dirac's theory the analysis of the constraints can be carried out formally by choosing an arbitrary independent variable to play the role of time and this variable can even be complex. Furthermore, we have the covariant formulation of symplectic structure due to Witten [10] and Zuckerman [11] where the symplectic 2-form is a closed and conserved vector density. We shall construct the Witten–Zuckerman symplectic 2-forms for ASDYM equations. The results we find for both Dirac and Witten–Zuckerman approaches coincide, i.e., the 'time'-component of the Witten–Zuckerman symplectic 2-form is the same as the symplectic 2-form obtained from Dirac's theory.

The formulation of ASDYM equations as an integrable system follows the work of Yang [8] who pointed out that by introducing complex coordinates z, w on Euclideanized space–time the requirement of self-duality reduces to the simple conditions

$$F_{zw} = 0, \quad F_{\bar{z}\bar{w}} = 0, \quad F_{z\bar{z}} + F_{w\bar{w}} = 0 \tag{1}$$

on the components of the field tensor. He further pointed out two choices of gauge, the J- and K-gauges, whereby eq. (1) assumes a particularly simple form. In the K-gauge, Yang proposed the *Ansatz* for the components of the Yang–Mills connection 1-form

$$A_{\bar{z}} = K_w, \quad A_{\bar{w}} = -K_z, \tag{2}$$

where K is a matrix which is an element of the structure group of the Yang–Mills equations and its subscripts denote partial derivatives. Yang's equations (1) reduce to

$$K_{z\bar{z}} + K_{w\bar{w}} + [K_w, K_z] = 0 \tag{3}$$

for this choice of gauge. In the J-gauge, components of the connection 1-form are given by

$$A_{\bar{z}} = J^{-1}\, J_{\bar{z}}, \quad A_{\bar{w}} = J^{-1}\, J_{\bar{w}} \tag{4}$$

and eq. (1) becomes

$$(J^{-1} J_{\bar{z}})_z + (J^{-1} J_{\bar{w}})_w = 0, \tag{5}$$

where J is another matrix which is an element of the structure group. We note that there is an asymmetry between independent variables z, w and their complex conjugates $\bar{z}, \bar{w}$ in both eqs (3) and (5). In the J-gauge this asymmetry is responsible for the deviation of eq. (5) from the standard structure of a harmonic map, or non-linear sigma model. The difference is a Wess–Zumino term. This asymmetry, which is essential in the identification of the ASDYM equations as a multi-dimensional completely integrable system, will appear in everything that follows.

2. *K*-gauge

It is well-known [12] that the Lagrangian

$$\mathcal{L}_{K2} = \tfrac{1}{2} K_z K_{\bar{z}} + \tfrac{1}{2} K_w K_{\bar{w}} + \tfrac{2}{3} K[K_w, K_z] \tag{6}$$

yields the ASDYM equation (3). Here and in the following the trace operation will be understood in all Lagrangians where the variables to be varied independently are matrices. We shall take the independent variable z to act as the 'time' variable and for purposes of Hamiltonian analysis we need to start with a first-order Lagrangian. This is given by

$$\mathcal{L}_K = \tfrac{1}{2} M\tilde{M} - \tfrac{1}{2} K_w K_{\bar{w}} - \tfrac{1}{2}\tilde{M} K_z - \tfrac{1}{2} M K_{\bar{z}} + \tfrac{1}{3} M[K, K_w] \tag{7}$$

because it can be verified that we obtain the Euler–Lagrange equations

$$\begin{aligned} K_z &= M, \\ \tilde{M} &= K_{\bar{z}} + \tfrac{2}{3}[K_w, K] \\ \tilde{M}_z &= -M_{\bar{z}} - 2K_{\bar{w}w} + \tfrac{2}{3}[M_w, K] - \tfrac{4}{3}[K_w, M] \end{aligned} \tag{8}$$

which together result in eq. (3). In this first-order formulation we have introduced $M, \tilde{M}$ as new variables which is double the number required. This is due to the asymmetry, already noted above, between independent variables and their complex conjugates in the K-gauge equation (3) itself. If eq. (3) had possessed such a symmetry then $\tilde{M}$ would simply have been its complex conjugate $\tilde{M}$. The first-order ASDYM field equations can be written as

$$X_z^A = \mathbf{X}_K\,(X^A) \tag{9}$$

with the vector field defining the flow

$$\mathbf{X}_K = \Big(- 2K_{\bar{w}w} - M_{\bar{z}} + \frac{2}{3}[M_w, K] + \frac{4}{3}[M, K_w]\Big)\frac{\delta}{\delta\tilde{M}} + M\frac{\delta}{\delta K} \tag{10}$$

for the K-gauge equations (8). We note that the basis vector field $\delta/\delta M$ is missing above because its coefficient vanishes identically due to the absence of M_z in eq. (8). This is a consequence of the asymmetry and we only need to take

$$X^1 \equiv \bar{M}, \quad X^2 \equiv K$$

in eq. (9). Only the variables $\bar{M}, K$ play a significant role in phase-space and the symplectic structure of ASDYM equations in the K-gauge can be discussed without reference to M.

The Lagrangian (7) is degenerate because its Hessian

$$\det\left|\frac{\partial^2\mathcal{L}}{\partial X_z^A\, \partial X_z^B}\right| = 0 \tag{11}$$

vanishes identically. Hence, it is a system subject to constraints and the passage to its Hamiltonian structure requires the use of Dirac's theory of constraints [9]. We start with the canonical momenta

$$\Pi_A \equiv \frac{\partial\mathcal{L}}{\partial X_z^A} \tag{12}$$

which cannot be inverted due to eq. (11). The definition of the momenta therefore gives rise to the constraints

$$\begin{aligned} \Phi_1 &= \Pi_{\tilde{M}} \\ \Phi_2 &= \Pi_K + \tfrac{1}{2}\tilde{M} \\ \Phi_3 &= \Pi_M \end{aligned} \tag{13}$$

which must vanish weakly. In order to determine the class of these constraints we need to obtain the Poisson bracket of the constraints

$$C_{AB}(w, w') = \{\Phi_A(w), \Phi_B(w')\} \tag{14}$$

using the canonical Poisson brackets

$$\{X^A(w), \Pi_B(w')\} = \delta_B^A \delta(w - w') \tag{15}$$

between the dynamical variables and their conjugate momenta. For the first two constraints we find

$$C_{AB}(w, w') = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\delta(w - w') \tag{16}$$

which shows that they are second class. It is understood that $\bar{w}$ enters into Poisson bracket relations in exactly the same way as w. We shall follow this loose practice in the rest of this paper. The remaining constraint Φ_3 is first class, but only superficially. It is quite unlike the first-class constraints that arise from important principles such as gauge invariance. The origin of Φ_3 as a first-class constraint can be traced back to the pathology that the component of the vector field (10) along M vanishes identically. We shall henceforth ignore M and its conjugate momentum Φ_3 completely as they do not play any significant role in phase-space.

The symplectic 2-form is given by [5]

$$\omega = \delta X^{A} \wedge C_{A\,B}\,\delta X^{B} \tag{17}$$

and from eq. (16) we find that

$$\omega_K = \delta K \wedge \delta\tilde{M}, \tag{18}$$

where we see that $\tilde{M}$ is the momentum map.

In order to obtain the Hamiltonian for the degenerate Lagrangian (7) we first construct the free Hamiltonian obtained by Legendré transformation

$$\begin{aligned} H_0 &= \Pi_A X^A_{\ z} - \mathcal{L} \\ &= -\tfrac{1}{2}M\tilde{M} + \tfrac{1}{2}K_w K_{\bar{w}} + \tfrac{1}{2}MK_{\bar{z}} + \tfrac{1}{3}M[K_w, K] \end{aligned} \tag{19}$$

and the total Hamiltonian density of Dirac is given by

$$H_{\mathrm{T}} = H_0 + \lambda^A \Phi_A, \tag{20}$$

where λ^A are Lagrange multipliers. For second-class constraints, the Lagrange multipliers are determined from the solution of

$$\{H_{\mathrm{T}}, \Phi_A\} = 0 \tag{21}$$

which ensure that the constraints hold for all values of z. The Lagrange multipliers are given by

$$\begin{aligned} &\lambda^1 = -M_{\bar{z}} - 2K_{\bar{w}w} + \tfrac{2}{3}[M_w, K] - \tfrac{4}{3}[K_w, M], \\ &\lambda^2 = M, \end{aligned}$$

which consist of the coefficients of the vector field (10) defining the flow. This is a general property of second-class constraints which are linear in the momenta.

The Dirac bracket is a modification of the Poisson bracket designed to vanish on the surface defined by the constraints. For two smooth functionals F, G of the canonical variables, the Dirac bracket is given by

$$[F, G]_K = [F, G] - [F, \Phi_A] Z_K^{AB} [\Phi_B, G], \tag{22}$$

where Z_K is obtained by inverting the matrix of the Poisson bracket of the constraints

$$\int C_{AB}(w, w'') Z_K^{BC}(w'', w')\,\mathrm{d}w'' = \delta_A^C\,\delta(w - w') \tag{23}$$

and we note [5] that the inverse of the Poisson bracket of the constraints is known as the Hamiltonian operator in the literature of integrable systems. From eq. (16) we have

$$Z_K^{A\,B}(w,w') = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \delta(w-w') \tag{24}$$

and eq. (8) can be written in the Hamiltonian form

$$X_z^A = Z_K^{A\,B} \frac{\delta H_0}{\delta X^B}, \tag{25}$$

where integration over primed variables is implied. Finally, Hamilton's equations can be written in the form

$$i_{\mathbf{X}} \omega_K = -\delta H_0, \tag{26}$$

where $i_{\mathbf{X}}$ denotes contraction with respect to the vector field (10) of the symplectic 2-form (18). The Hamiltonian function H_0 given by eq. (19) is conserved. On shell

$$H_0 = -K_z K_{\bar{w}} \tag{27}$$

and

$$-(K_w K_{\bar{w}})_z + (K_z K_z)_{\bar{z}} + (K_z K_{\bar{w}})_w + (K_w K_z)_{\bar{w}} = 0$$

is the associated continuity equation.

The discussion of the symplectic structure of ASDYM equations in the K-gauge contains some unfamiliar features owing to the choice of a complex variable as 'time'-parameter and the asymmetry between the independent variables and their complex conjugates in eq. (3). We shall now turn to the covariant Witten–Zuckerman formulation of symplectic 2-form vector density ω^μ which is closed

$$\delta\omega^\mu = 0 \tag{28}$$

and conserved

$$\partial_\mu \omega^\mu = 0 \tag{29}$$

and the fact that it is covariant relieves us of the encumbrance involved in justifying the choice of 'time' variable. Starting with the Lagrangian (7) we find that the Witten–Zuckerman symplectic 2-form is given by

$$\begin{aligned}
\omega^z &= -\tfrac{1}{2}\delta\tilde{M} \wedge \delta K, \\
\omega^{\bar{z}} &= -\tfrac{1}{2}\delta M \wedge \delta K, \\
\omega^w &= -\tfrac{1}{2}\delta K_{\bar{w}} \wedge \delta K - \tfrac{1}{3}\delta[K,M] \wedge \delta K, \\
\omega^{\bar{w}} &= \tfrac{1}{2}\delta K_w \wedge \delta K,
\end{aligned} \tag{30}$$

where we note that

$$\omega^z = \omega_K \tag{31}$$

is the expression for the symplectic 2-form (17) obtained from Dirac's theory of constraints.

3. *J*-gauge

We have remarked on the asymmetry between the complex coordinates and their complex conjugates in ASDYM equations which is due to a Wess–Zumino term. This is an essential feature of their complete integrability but leads to problems in writing a Lagrangian for Yang's ASDYM equations in the J-gauge. The nonlinear σ-model Lagrangian does not yield eq. (5). The trace operation implied in this Lagrangian will always result in equations of motion symmetric in derivatives of J with respect to $z, \bar{z}$ and $w, \bar{w}$. Nair and Schiff [13] have written down a Lagrangian in five dimensions with the Wess–Zumino term that will take care of this essential asymmetry but the explicit result which will come from its restriction to a four-dimensional boundary has not been carried out. The explicit expression for the Lagrangian in the J-gauge was given by Pohlmeyer [14] for gauge group $SU(2)$. It depends on each entry of J. For $SU(2)$ Yang parametrized J in terms of Poincaré coordinates for the forward mass hyperboloid

$$J = \frac{1}{\phi}\begin{pmatrix} 1 & \bar{\rho} \\ \rho & \phi^2 + \rho\bar{\rho} \end{pmatrix}, \tag{32}$$

where ϕ is real and ρ is complex. Then eq. (5) reduces to

$$\begin{aligned}
&\phi\phi_{w\bar{w}} + \phi\phi_{z\bar{z}} - \phi_w\phi_{\bar{w}} - \phi_z\phi_{\bar{z}} + \bar{\rho}_w\rho_{\bar{w}} + \bar{\rho}_z\rho_{\bar{z}} = 0,\\
&\phi\bar{\rho}_{w\bar{w}} + \phi\bar{\rho}_{z\bar{z}} - 2\phi_{\bar{w}}\bar{\rho}_w - 2\phi_{\bar{z}}\bar{\rho}_z = 0,\\
&\phi\rho_{w\bar{w}} + \phi\rho_{z\bar{z}} - 2\phi_w\rho_{\bar{w}} - 2\phi_z\rho_{\bar{z}} = 0
\end{aligned} \tag{33}$$

and Pohlmeyer showed that the second-order Lagrangian

$$\mathcal{L}_{J2} = (2\phi^2)^{-1}\left[\phi_z\phi_{\bar{z}} + \phi_w\phi_{\bar{w}} + \bar{\rho}_z\rho_{\bar{z}} + \bar{\rho}_w\rho_{\bar{w}}\right] \tag{34}$$

yields eq. (33). We need to cast this Lagrangian into first-order form. It can be verified that

$$\begin{aligned}
\mathcal{L}_J = &-\tfrac{1}{2}P\bar{P} + (2\phi)^{-1}P\phi_{\bar{z}} + (2\phi)^{-1}\bar{P}\phi_z + (2\phi^2)^{-1}\phi_w\phi_{\bar{w}}\\
&-(2\phi^2)^{-1}(Q\bar{Q} - Q\bar{\rho}_z - \bar{Q}\rho_{\bar{z}}) + (2\phi^2)^{-1}\rho_{\bar{w}}\bar{\rho}_w
\end{aligned} \tag{35}$$

gives rise to the Euler–Lagrange equations

$$\begin{aligned}
&P = \phi^{-1}\phi_z, \quad \bar{P} = \phi^{-1}\phi_{\bar{z}}, \quad Q = \rho_{\bar{z}}, \quad \bar{Q} = \bar{\rho}_z,\\
&P_{\bar{z}} + \bar{P}_z + 2\phi^{-1}\phi_{w\bar{w}} - 2\phi^{-2}\phi_w\phi_{\bar{w}} + 2\phi^{-2}\rho_{\bar{w}}\bar{\rho}_w + 2\phi^{-2}Q\bar{\rho}_z = 0,\\
&(Q\phi^{-2})_z + (\phi^{-2}\rho_{\bar{w}})_w = 0,\\
&(\bar{Q}\phi^{-2})_{\bar{z}} + (\phi^{-2}\bar{\rho}_w)_{\bar{w}} = 0,
\end{aligned} \tag{36}$$

which together result in eq. (5). It is yet another consequence of the asymmetry between independent variables and their complex conjugates in eq. (5) that derivatives of ρ, P and $\bar{Q}$ with respect to z do not appear in eq. (36). Phase-space for $SU(2)$-ASDYM equations in the J-gauge is spanned by the variables $\bar{\rho}, \bar{P}, Q, \phi$ only. The meaningful variables $X^1 = \bar{\rho}, X^2 = \bar{P}, X^3 = Q, X^4 = \phi$ satisfy first-order field equations

$$X^A_z = \mathbf{X}_J(X^A), \tag{37}$$

with the vector field defining the flow

$$\mathbf{X}_J = (2\phi^{-2}\phi_w\phi_{\bar{w}} - 2\phi^{-1}\phi_{w\bar{w}} - 2\phi^{-2}\rho_{\bar{w}}\bar{\rho}_w - P_{\bar{z}} - 2\phi^{-2}Q\bar{\rho}_z)\frac{\delta}{\delta\bar{P}} \\ +P\phi\frac{\delta}{\delta\phi} + \bar{Q}\frac{\delta}{\delta\bar{\rho}} + [2PQ - \phi^2(\phi^{-2}\rho_{\bar{w}})_w]\frac{\delta}{\delta Q} \tag{38}$$

for eq. (36). The symplectic structure of the $SU(2)$ J-gauge equations can be discussed without reference to ρ, P and $\bar{Q}$.

The Lagrangian (34) is degenerate and applying Dirac's theory of constraints we find that the definition of momenta give rise to the constraints

$$\Phi_1 = \Pi_{\bar{\rho}} - (2\phi^2)^{-1}Q, \quad \Phi_2 = \Pi_{\bar{P}}, \quad \Phi_3 = \Pi_Q, \\ \Phi_4 = \Pi_\phi - (2\phi)^{-1}\bar{P}, \quad \Phi_5 = \Pi_P, \quad \Phi_6 = \Pi_{\bar{Q}}, \quad \Phi_7 = \Pi_\rho \tag{39}$$

which must vanish weakly. Evaluating the Poisson brackets of these constraints we find that Φ_5, Φ_6 and Φ_7 are spurious first-class constraints which will be ignored. For the remaining constraints the Poisson brackets yield

$$C_{AB} = \begin{pmatrix} 0 & 0 & -(2\phi^2)^{-1} & \phi^{-3}Q \\ 0 & 0 & 0 & (2\phi)^{-1} \\ (2\phi^2)^{-1} & 0 & 0 & 0 \\ -\phi^{-3}Q & -(2\phi)^{-1} & 0 & 0 \end{pmatrix} \delta(w - w') \tag{40}$$

which are again second class. From the definition (17) it follows that the symplectic 2-form is given by

$$w_J = -(2\phi)^{-1}\delta\phi \wedge \delta\bar{P} - (2\phi^2)^{-1}\delta\bar{\rho} \wedge \delta Q - Q\phi^{-3}\delta\bar{\rho} \wedge \delta\phi \tag{41}$$

which is closed modulo divergence. The total Hamiltonian of Dirac is given by

$$H_{\mathrm{T}} = H_0 + \lambda^A\Phi_A \\ H_0 = \tfrac{1}{2}P\bar{P} - (2\phi)^{-1}P\phi_{\bar{z}} - (2\phi^2)^{-1}\phi_w\phi_{\bar{w}} - (2\phi^2)^{-1}\rho_{\bar{w}}\bar{\rho}_w \\ +(2\phi^2)^{-1}(Q\bar{Q} - \bar{Q}\rho_{\bar{z}}),$$

where λ^A are Lagrange multipliers which will be determined from eq. (21) which are strong equations that determine the Lagrange multipliers completely. Since the constraints are linear in the momenta we know that $\lambda^A = i_{\mathbf{X}}\delta X^A$, or explicitly

$$\lambda^1 = \bar{\rho}_z, \\ \lambda^2 = -P_{\bar{z}} - \frac{1}{2\phi}\phi_{\bar{w}w} + \frac{2}{\phi^2}\phi_{\bar{w}}\phi_w - \frac{2}{\phi^2}\bar{\rho}_w\rho_{\bar{w}} - \frac{2}{\phi^2}Q\bar{\rho}_z, \\ \lambda^3 = 2QP - \phi^2\left(\frac{\rho_{\bar{w}}}{\phi^2}\right)_w \\ \lambda^4 = P\phi,$$

and λ^i, $i = 5, 6, 7$ are arbitrary.

Using eq. (23) we get the Hamiltonian operator for the $SU(2)$ J-gauge

$$Z_J^{A\,B} = \begin{pmatrix} 0 & 0 & 2\phi^2 & 0 \\ 0 & 0 & -4Q & -2\phi \\ -2\phi^2 & 4Q & 0 & 0 \\ 0 & 2\phi & 0 & 0 \end{pmatrix} \delta(w - w') \tag{42}$$

and eq. (36) can be written in the Hamiltonian form (25) with the subscript J replacing K. Similarly, eq. (26) holds with the vector field (38), the symplectic 2-form (41) and Hamiltonian function H_0 given by eq. (43). On shell

$$H_0 = (2\phi^2)^{-1}\phi_z\phi_{\bar{w}} + (2\phi^2)^{-1}\bar{\rho}_z\rho_{\bar{w}} \tag{43}$$

and

$$\begin{aligned} &\left[\frac{1}{2\phi^2}(\phi_z\phi_w + \rho_w\bar{\rho}_z)\right]_{\bar{z}} + \left[\frac{1}{2\phi^2}(\phi_w\phi_{\bar{z}} + \rho_{\bar{z}}\bar{\rho}_w)\right]_z \\ &-\left[\frac{1}{2\phi^2}(\phi_z\phi_{\bar{z}} + \rho_{\bar{z}}\bar{\rho}_z)\right]_w + \left[\frac{1}{2\phi^2}(\phi_w^2 + \rho_w\bar{\rho}_w)\right]_{\bar{w}} = 0 \end{aligned}$$

is the associated continuity equation.

The covariant Witten–Zuckerman symplectic 2-form vector density w^μ which is closed and conserved follows directly from Lagrangian (35). We find

$$\begin{aligned} w^z &= (2\phi)^{-1}\delta\bar{P} \wedge \delta\phi + (2\phi^2)^{-1}\delta Q \wedge \delta\bar{\rho} - \phi^{-3}Q\delta\phi \wedge \delta\bar{\rho} \\ w^w &= (2\phi^2)^{-1}\delta\phi_{\bar{w}} \wedge \delta\phi + (2\phi^2)^{-1}\delta\bar{\rho}_w \wedge \delta\bar{\rho} - \phi^{-3}\rho_{\bar{w}}\delta\phi \wedge \delta\bar{\rho} \end{aligned} \tag{44}$$

together with their complex conjugates. We note that

$$w^z = w_J \tag{45}$$

is the expression for the symplectic 2-form (41) obtained from Dirac's theory of constraints.

4. Bi-Hamiltonian structure

We have obtained the symplectic structure of ASDYM equations both in the K-gauge and the J-gauge. These results are not sufficient to conclude that the two Poisson brackets form a pencil to which we can apply the theorem of Magri and conclude that ASDYM system admits bi-Hamiltonian structure and is therefore a completely integrable system. In order to be able to arrive at such a result we must express both symplectic structures in the *same* variables. So we turn back to the definition of the Yang–Mills potential 1-forms $A = A_\mu \mathrm{d}x^\mu$ in eqs (2) and (4) that give rise to eqs (3) and (5) respectively. We consider a gauge transformation between these two choices of gauge

$$SA^J = A^K S + \mathrm{d}S, \tag{46}$$

where superscripts on the potential refer to its expression in the gauge indicated. The solution of this equation for S is the definition of the Bäcklund transformation. We find

$$S = J^{-1} K \tag{47}$$

which enables us to relate eqs (3) and (5). Hence we get

$$\begin{aligned} K_w &= J^{-1} J_{\bar{z}} \\ K_z &= -J^{-1} J_{\bar{w}} \end{aligned} \tag{48}$$

which is also the Lax pair. This is the required transformation for bringing our results on the K- and J-gauges together. The Hamiltonian operators in the K- and J-gauges are related by

$$Z_K = \mathcal{S} Z_J \mathcal{S} \tag{49}$$

and from eq. (48) it follows that

$$\mathcal{S} = \left(\frac{\mathrm{d}}{\mathrm{d}w}\right)^{-1} \left(\frac{\mathrm{d}}{\mathrm{d}\bar{z}} J^{-1} + J^{-1} J_{\bar{z}} J^{-1}\right), \tag{50}$$

where d^{-1} denotes the principal value integral. We can now write down the Lenard–Magri recursion relation

$$Z_K \frac{\delta H_{n+1}}{\delta K} = \mathcal{S} Z_J \mathcal{S} \frac{\delta H_n}{\delta K}, \tag{51}$$

where $n = 0, 1, \ldots, \infty$ and H_0 is given by eq. (19). The recursion relation then determines an infinite hierarchy of conserved quantities. The fact that the transformation law (50) is an integro-differential equation presents a considerable obstacle to carry out this task in practice and also prevents a quick check of compatibility of the Hamiltonian operators (24) and (49).

5. Conclusion

We have presented the symplectic structure of ASDYM equations in explicit form. Using Dirac's theory of constraints and the covariant Witten–Zuckerman approach we have obtained the Hamiltonian operators in Yang's J- and K-gauges. The results for the symplectic 2-form coincide in both of these theories. We have also obtained the transformation law, or Bäcklund transformation, for the Hamiltonian operators between these two gauges which establishes that ASDYM system admits two Hamiltonian structures. The complicated nature of the Bäcklund transformation between these two gauges makes it difficult to check the compatibility of the Hamiltonian operators as well as higher conserved Hamiltonians according to the Lenard scheme.

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