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Damage spreading on the 3-12 lattice with competing Glauber and Kawasaki dynamics

Z Z GUO[1,*], XIAO-WEI WU[2] and CHUN-AN WANG[1]
[1]Department of Physics, Inner Mongolia Normal University, Huhhot 010022, China
[2]School of Science, Inner Mongolia Agriculture University, Huhhot 010018, China
[*]Corresponding author. E-mail: zzguo@imnu.edu.cn

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Abstract. The damage spreading of the Ising model on the 3-12 lattice with competing Glauber and Kawasaki dynamics is studied. The difference between the two kinds of nearest-neighboring spin interactions (interaction between two 12-gons, or interaction between a 12-gon and a triangle) are considered in the Hamiltonian. It is shown that the ratio of the interaction strength F between the two kinds of interactions plays an important role in determining the critical temperature $T_{\rm d}$ of phase transition from frozen to chaotic. Two methods are used to introduce the bond dilution on the Ising model on the 3-12 lattice: regular and random. The maximum of the average damage spreading $\langle D\rangle_{\rm max}$ can approach values lower than 0.5 in both cases and the reason can be attributed to the 'survivors' among the spins. We have also, for the first time, presented the phase diagram of the mixed G–K dynamics in the 3-12 lattice which shows what happens when going from pure Glauber to pure Kawasaki.

Keywords. Ising model; damage spreading; Glauber dynamics; Kawasaki dynamics; bond dilution.

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1. Introduction

The random growth processes, such as the formation of a snowflake, the roughness of a crack surface, the corroding process in iron and virus spreading, are so complicated that it is difficult to make direct simulations or descriptions [1]. In order to understand these phenomena in a simple way, we have constructed many mathematical growth models, of which the damage spreading (DS) is a very important one. The DS technique requires one to follow the behavior of two similar samples (slightly different initial conditions) under the same thermal noise. Usually, the evolution is done by means of a growth rule: Glauber dynamics, heat-bath dynamics, Kawasaki dynamics, etc. Recently many researches have started paying attention to the competing Glauber and Kawasaki (G–K) dynamics [2,3], which describes that a system is in thermal equilibrium with a heat reservoir and is subjected to

a continuous flux of energy and this has been proved to be of great importance in describing the time evolution of many dynamical systems.

Although the concept of DS was introduced in the context of biologically motivated systems by Kuffman in 1969 [4], now it has been proved to be a useful technique in the study of dynamical properties of statistical models [1,5], especially in magnetic models, like Ising, Clock, Potts, spin glass, etc. With the DS technique, one can learn how a perturbation spreads throughout a cooperative system composed of interacting subunits. This technique has also been applied in the study of Coulomb glass [6]. The DS technique can be applied in the research of many economic and social phenomena [7].

Many interesting results of DS have been reported since it was applied in statistical physics. In the case of pure Ising ferromagnets, a sharp dynamical phase transition is observed and the dynamical phase transition temperature separates the two phases [1,5,7]: the frozen (order) and chaotic (disorder) phases. In the frozen phase the spin distance is independent of the initial distance and vanishes rapidly while in the chaotic phase the distance remains finite for a long time. In the more complicated systems such as fully frustrated, spin-glass and XY model, a third phase between these two phases was found. It is called the intermediate phase where the distance does not vanish but becomes independent of the initial distance [8]. Many authors had discussed the relation between the DS transition temperature T_{d} and the Curie temperature T_{c} [2]. For the two-dimensional Ising model, it is shown that for the heat-bath dynamics T_{d} coincides with T_{c} whereas for Glauber and Metropolis dynamics T_{d} is close to but smaller than T_{c}.

Many elements which characterize the DS process have been considered in literatures, including the interactions (ferromagnetic, antiferromagnetic, spin-glass, etc.), the Monte Carlo rules (heat bath, Glauber, Metropolis, etc.). As for the lattice geometry, many work has been done on the square, triangular, hexagonal, and other lattices. In recent years, much attention has been paid to the DS study on complex networks [9,10].

Compared to conventional Monte Carlo methods, the DS technique is less sensitive to statistical fluctuations. But it is found that DS depends to a great extent on the dynamics chosen and, particularly for the 'heat-bath' dynamics, on the type of initial configurations [11]. These properties are very important because they are contrary to the usual statistical Monte Carlo modeling, where all the dynamics give the same values for the magnetization, susceptibility and specific heat, differing only in the convergence rate. Nobre *et al* [11] investigated the DS in the Ising model on a triangular lattice, for ferro- and antiferromagnetic interactions, using Glauber dynamics. They employed two procedures for updating spins: the sequential and parallel procedures. They found that the DS depends on the procedures for updating spins and only the sequential algorithm leads to reasonable results. Trivalent structures are of great importance due to their topological stability and general occurrence in nature. One of the authors, cooperating with others, investigated the DS dependence on the topology of two kinds of trivalent structures: a set of hierarchical regular lattices generated by star–triangle transformation on hexagonal lattice (the 3-12, the 3-6-24 and the 3-6-12-48 lattices) [7] and random lattices (soap froth and Voronoi [12,13], here Voronoi is the name of a class of two-dimensional random lattices). It is shown that for the hierarchical lattices the smallest polygons

(the smallest polygon in which the number of sides is the smallest) in the lattice play a very important role – the transition temperature T_{d} increases with the increase in the number of the smallest polygons N_{s} when the system is under the pure G-dynamics and T_{d} decreases fast as N_{s} increases when K-dynamics becomes important.

The 3-12 lattice is also referred to as the $(3,12^2)$ Archimedean lattice [14]. It can be generated from the pure hexagonal lattice [7]. On each vertex of the hexagon, replace the star (Y) by a triangle (∇), so that we have a triangle on each vertex of the original hexagon. That is the so-called star–triangle transformation. In doing the star–triangle transformation, the side length of the 12-gon is so chosen that the 12-gon is a regular polygon. The result is a crystal made with 12-gons and triangles. Although this lattice is simple, it may be the simplest model which can describe the Ising spins with different nearest-neighbor exchange interaction coefficients weighted by the related areas of cells and it presents the general features of the set of the hierarchical lattices. In this paper, we give a further discussion on the 3-12 lattice. We will focus our study on two aspects: the effects of the difference of the interaction coefficients between the two kinds of polygons and the bond dilution dependence of the DS on this lattice.

2. Model and theory

We put spins on the center of the cells (the 12-gons or triangles). It is obvious that the interaction strength of spins between two 12-gons and 12-gon and triangle should be different since the triangles and 12-gons have different areas. Taking this factor into consideration, we write the Hamiltonian in the following way:

$$H = -\sum J_{ij} s_i s_j, \tag{1}$$

where $J_{ij} > 0$ is the ferromagnetic exchange interaction coefficient between the nearest-neighbor sites i and j. We assume

$$J_{12:12} = J \tag{2a}$$

and

$$J_{3:12} = J_{12:3} = J * F, \tag{2b}$$

where F is the ratio of interaction strength between the two kinds of interactions (12-gon to 12-gon and 12-gon to triangle). In this calculation we let $F = 1$ to S_3/S_{12}, here $S_3(S_{12})$ is the area of the triangle (12-gon) and $F = 1$ corresponds to the case where the area influence is not considered ($S_3 = S_{12}$) while $F = S_3/S_{12}$ means $J_{ij} \propto (S_i/S_j)$. Note that if we take $F = 0$, the interaction between 12-gons and triangles are broken, resulting in a kind of bond dilution for the Ising spins in the lattice.

We deal with the DS problem on the 3-12 lattice by using Monte Carlo method. We first allow the system A to evolve for a long time to reach equilibrium, then a replica B of the system is made. At $t = 0$, the spin in the center cell of the lattice

B is flipped (damaged) and fixed all the time. The Hamming distance (or damage) in phase-space is calculated by

$$D(T) = \frac{1}{N}\sum_{i=1}^{N}(1 - \delta_{s_i^{\mathrm{A}}(t),s_i^{\mathrm{B}}(t)}), \tag{3}$$

where $\{s_i^{\mathrm{A}}(t)\}$ and $\{s_i^{\mathrm{B}}(t)\}$ are the two spin configurations of the system which are subjected to the same thermal noise and the same set of random numbers, and N is the number of total spins on the lattice studied.

3. Damage spreading with competing G–K dynamics

We use competing G–K dynamics [2,3] to give a weighted transition probability per unit time from state s to s':

$$\omega(s, s') = p\omega_{\mathrm{G}}(s, s') + (1 - p)\omega_{\mathrm{K}}(s, s'). \tag{4}$$

The first term on the right side of eq. (4) corresponds to G-dynamics with p the weighted probability. The single-spin flipping probability is written as

$$\omega_{\mathrm{G}}(s, s') = \sum_{i=1}^{M} \delta_{s'_1 s_1}\delta_{s'_2 s_2}\cdots\delta_{s'_i - s_i}\cdots\delta_{s'_M s_M}\omega_i(s) \tag{5}$$

with $w_i(s) = \min[1, \exp(-(\Delta E_i/k_{\mathrm{B}}T)]$, where $\omega_i(s)$ is the probability of flipping spin i. The contact with the heat reservoir at temperature T follows the Metropolis prescription with ΔE_i being the change in energy in flipping spin i.

The second term on the right side of eq. (4) is based on Kawasaki (K-) dynamics using two-spin exchange with the probability

$$\omega_{\mathrm{K}}(s, s') = \sum_{\langle i,j\rangle}^{M} \delta_{s'_1 s_1}\cdots\delta_{s'_i s_j}\cdots\delta_{s'_j s_i}\cdots\delta_{s'_M s_M}\omega_{ij}(s) \tag{6}$$

with

$$w_{ij}(s) = \begin{cases} 0, & \text{for } \Delta E_{ij} \le 0 \\ 1, & \text{for } \Delta E_{ij} > 0 \end{cases},$$

where $\omega_{ij}(s)$ is the probability of exchange between the nearest-neighbor spins i and j.

In the following calculation we choose $M = 15$, where M is the size of the lattice (the number of the 12-gons along one direction, see figure 1). The results are averaged over 100 configurations.

Figure 2 shows the typical phase transition curves of the DS of Ising model on the 3-12 lattice for the case $J_{12:12} = J$ and $J_{3:12} = J_{12:3} = J^* S_3/S_{12}$ when there is competing G-K dynamics. There exists obvious phase transition from frozen to chaotic. The maximum values of $\langle D\rangle$ in the long time limit are all 0.5. The

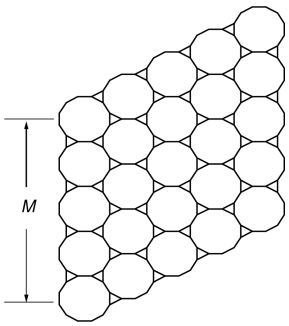

Figure 1. The 3-12 lattice.

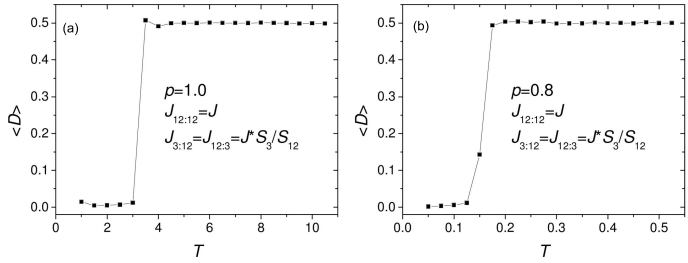

Figure 2. Averaged damage spreading of the 3-12 lattice as a function of temperature for the cases $F = S_3/S_{12}$ and $p = 1.0$ (**a**) and 0.8 (**b**). T is in units of J/k_{B} and $M = 15$.

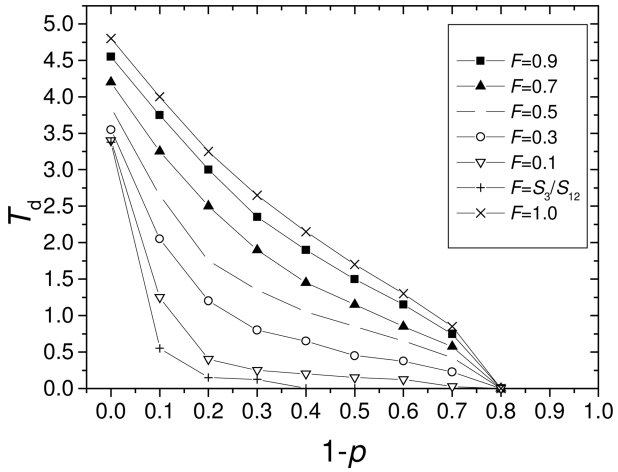

Figure 3. The critical temperatures as a function of $1 - p$ for different interaction strengths F.

temperature at which $\langle D \rangle$ rises to half of its maximum value in the long time limit is chosen as the critical temperature. In this way, we obtain the critical temperatures as a function of $1 - p$ for various F as shown in figure 3.

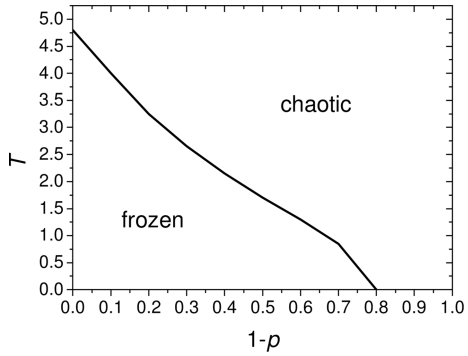

Figure 4. The damage spreading phase diagram for the case $F = 1$.

Figure 3 tells us that F plays an important role in determining the critical temperatures. The critical temperature decreases with the decrease in F and it decreases sharply as $1 - p$ increases (p decreases). When Kawasaki dynamics is dominant (p is small), $T_{\rm d}$ approaches zero and no phase transition occurs. From figure 3 we also know that the difference of $T_{\rm d}$ between the two limited cases $F = 1$ and $F = S_3/S_{12}$ becomes bigger with the decrease of p.

As was pointed out before, the damage transition temperature $T_{\rm d}$ separates two phases: the frozen and chaotic phases. In fact, figure 3 is just the damage spreading phase diagram. In order to show this, we replot the temperature curve as a function of $1 - p$ for the case $F = 1$ (see figure 4), which tells us what happens when going from pure Glauber to pure Kawasaki ($p = 0$ to $p = 1$).

It is known that the damage measures the fractions of spins that are different in configurations A and B and at higher temperature ($T > T_{\rm d}$) and in long time limit, $\langle D(t \to \infty)\rangle = 0.5$ holds for G-dynamics whatever the lattice geometry is [7]. This is confirmed in our calculation. Our calculation shows that $\langle D(t \to \infty)\rangle = 0.5$ also holds when there is competition of G–K dynamics. The only exception is found in the case of very small F for $p = 1$. Figures 5a and 5b show the maximum of $\langle D\rangle$ as a function of F. Only when $p = 1$, $\langle D\rangle_{\max}$ can approach values lower than 0.5 for the cases with very small F. This result is interesting and unexpected for the G-dynamics ($p = 1$). We will show that this phenomenon originates from the bond dilution of spins, which will be discussed later. When p is a little smaller, this phenomenon disappears and only one value (0.5) of $\langle D\rangle_{\max}$ is found (see figure 5b).

In the above calculations, we have fixed the 12-gons in the center area of the lattice damaged. The 12-gons are the main polygons in the 3-12 lattice with the contiguity number $f = 12$. The damage on a 12-gon would spread easier and faster due to its bigger contiguity number. But no obvious difference occurs if we fix the triangle (the contiguity number $f = 3$) in the center area of the lattices in the long time limit. A reasonable explanation may be that the memory of the original damaged cell is lost after a long time evolution. But if we focus our attention on the beginning of the damage spreading, the time dependence of damage will tell the dependence of damage on topology. Figure 6 shows the time dependence of damage at a given temperature for the cases where we fix different cells in the center area of the 3-12 lattice. It is easy to understand that, the more linkage a cell has, when it

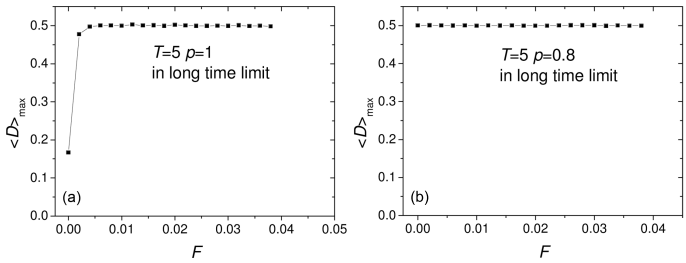

Figure 5. Averaged damage spreading of the 3-12 lattice as a function of F for the cases $p = 1.0$ (**a**) and 0.8 (**b**). T is in units of J/k_{B} and $M = 15$.

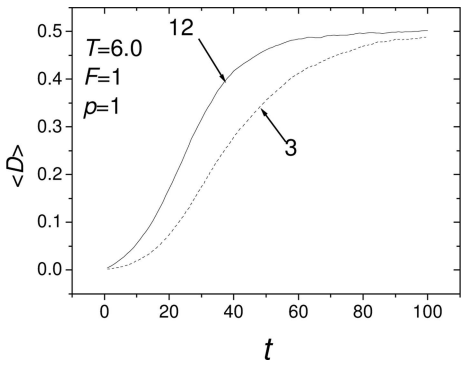

Figure 6. Time dependence of the averaged damage spreading of the 3-12 lattice at $T = 6$ for $p = 1$ and $F = 1$. '3' and '12' indicate that a triangle or a 12-gon in the center area of the lattice is damaged and fixed. $M = 15$.

is damaged, the more cells will be affected more quickly. So when we fix the 12-gon in the center area of the lattice, the system needs lesser time to reach the damage equilibrium. But the triangle has only three sides ($f = 3$) and when it is damaged, less cells will be affected, and so the DS is difficult when compared with the 12-gon ($f = 12$). In other words, it needs much time to reach the equilibrium state (the so-called chaotic).

4. Damage spreading on the bond-diluted 3-12 Ising lattice

Note that we have observed the phenomenon $\langle D\rangle_{\max} < 0.5$ in figure 5 and we have claimed that it is related with bond dilution. In fact, $F = 0$ results in the bond breaking between the 12-gons and triangles. So it is a kind of bond dilution of Ising spins (when $F = 0$, the dilute content is $x = 1/6$).

Quenched site and bond-diluted Ising models are often used to describe the magnetic properties of different materials. They are very important for understanding the behavior of disordered ferromagnets [15,16]. But till now, few reports have been presented for the DS on diluted Ising models. In the above $F = 0$ case, the dilution is introduced regularly: all the bonds between 12-gons and triangles are broken. Next, we will study the DS for the bond-diluted Ising ferromagnets on

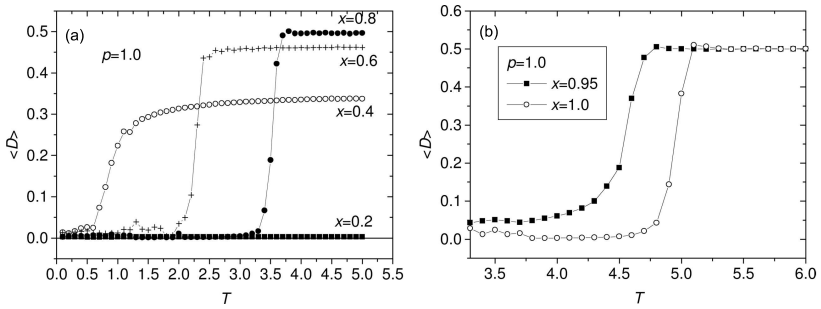

Figure 7. Averaged damage spreading of the diluted 3-12 lattice as a function of T for the cases $p = 1.0$ and various dilute contents.

the 3-12 lattice further. Here the bond dilution is chosen randomly. To determine J_{ij}, we generate a random number $0 < \xi < 1$. If $\xi > x$, $J_{ij} > 0$, else $J_{ij} = 0$. That means that the interaction breaks randomly, not only between a 12-gon and a triangle but also between two 12-gons. Figures 7a and 7b show the temperature dependence of the averaged damage spreading for the case $p = 1$ (the pure Glauber case). For a fixed dilute content x, two states exist, the state A (with smaller $\langle D \rangle$) and the state B (with bigger $\langle D \rangle$). Compared with the $x = 1$ case (no dilution), the maximum value of $\langle D \rangle$ is lower in state B while the minimum value of $\langle D \rangle$ is higher in state A (for pure frozen state, $\langle D \rangle = 0$, but here when $x < 1$, $\langle D \rangle > 0$. For example, $\langle D \rangle = 0.25$ for $x = 0.2$, so no frozen state exists). As x is decreased, the minimum value of $\langle D \rangle$ in state A approaches zero ($x = 0$) gradually, while as x is increased, the maximum value of $\langle D \rangle$ in state B approaches 0.5 ($x = 1$) gradually. But the difference between the states A and B disappears gradually as the Glauber probability p decreases. When the dynamics is pure K-dynamics, no damage spreads out and only frozen state exists.

The phenomenon in which $\langle D \rangle_{\max} < 0.5$ with pure G-dynamics ($p = 1$) implies that there exist 'survivors' among the spins. The term 'survivor' was used first by Levitan *et al* [17] to describe the bubbles which do not disappear in the evolving cellular structures. Here we define the survivors as those sites which are always the same in A and B. The 'survivors' phenomenon can also be found in the damage spreading on complex networks with power-law degree distributions (we will report this result later). The reason for the appearance of 'survivors' is due to the existence of the isolated spins in the lattice. The isolated spins increase with decreasing x, resulting in the decrease of $\langle D \rangle_{\max}$, as shown in figure 7.

So, we have introduced the bond dilution in two ways: by letting $F = 0$, the bond dilution is introduced regularly; by using the random numbers and the dilute content x, the bond dilution is introduced randomly. $\langle D \rangle_{\max} < 0.5$ is the general feature of the bond-diluted Ising models.

5. Summary and discussions

In summary, we have studied the damage spreading of the Ising model on the 3-12 lattice with competing G–K dynamics. We include the difference of the two kinds of interactions between spins of two 12-gons and spins of a 12-gon and a triangle in

the Hamiltonian, by considering the different areas of the triangles and the 12-gons. We find that the ratio of the interaction strength between the two kinds of interactions (F) plays an important role in determining the critical temperatures. We introduced the bond dilution in two ways: regularly (by letting $F = 0$) or randomly. In both cases, $\langle D \rangle_{\max}$ can approach values lower than 0.5. This result is unusual for the G-dynamics since we have nearly accepted the opinion that $\langle D(t \to \infty) \rangle = 0.5$ holds for G-dynamics at higher temperatures whatever the lattice geometry is. This phenomenon can be explained in terms of the 'survivors', which are the isolated spins in the lattice due to the bond dilution.

We have also, for the first time, presented the phase diagram of the mixed G–K dynamics in the 3-12 lattice which shows what happens when going from pure Glauber to pure Kawasaki ($p = 0$ to $p = 1$) and how the strength change between the two kinds of interactions affects the phase diagram.

This method can also be extended to study social problems, for example, introducing a model for elections [18]. But for that case, we have to use other dynamics instead of Glauber or Kawasaki dynamics.

Acknowledgments

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References

[1] K Binder (ed.), *The Monte Carlo method in condensed matter physics* (Springer-Verlag, 1992) pp. 93–120
[2] Z Z Guo and K Y Szeto, *Phys. Rev.* **E71**, 016605 (2005)
[3] B C S Grandi and W Figueiredo, *Phys. Rev.* **E53**, 5484 (1996)
[4] S A Kauffman, *J. Theor. Biol.* **22**, 437 (1969)
[5] H Eugene Stanley, Dietrich Stauffer, Janos Kertesz, and Hans J Herrmann, *Phys. Rev. Lett.* **59**, 2326 (1987)
[6] T Wappler and T Vojta, *Phys. Rev.* **B55**, 6272 (1997)
[7] Z Z Guo, K Y Szeto and Xiujun Fu, *Phys. Rev.* **E70**, 016105 (2004)
[8] Ce-Jun Liu, H-B Schutter and Jia-Zhen Hu, *Commun. Theor. Phys.* **35**, 480 (2001)
[9] R Albert and A-L Barabasi, *Rev. Mod. Phys.* **74**, 47 (2002) and references therein
[10] Erik Volz, *Phys. Rev.* **E70**, 056115 (2004)
[11] F D Nobre, A M Mariz and E S Sousa, *Phys. Rev. Lett.* **69**, 13 (1992)
[12] K Y Szeto, Xiujun Fu and W Y Tam, *Phys. Rev. Lett.* **88**, 138302 (2002)
[13] K Y Szeto and W Y Tam, *Phys. Rev.* **E53**, 4213 (1996)
[14] Paul N Suding and Robert M Ziff, *Phys. Rev.* **E60**, 275 (1999)
[15] Z Neda, *J. Phys. I France* **4**, 175 (1994)
[16] U L Fulco, F D Nobre, L R da Silva and L S Lucena, *Physica* **A297**, 131 (2001)
[17] B Levitan, E Slepyan, O Krichevsky, J Stavans and E Domany, *Phys. Rev. Lett.* **73**, 756 (1994)
[18] A T Bernardes, U M S Costa, A D Araujo and D Stauffer, *Int. J. Mod. Phys.* **C12**, 159 (2001)