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Painlevé test for integrability and exact solutions for the field equations for Charap's chiral invariant model of the pion dynamics

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Abstract. It has been shown that the field equations for Charap's chiral invariant model of the pion dynamics pass the Painlevé test for complete integrability in the sense of Weiss *et al.* The truncation procedure of the same analysis leads to auto-Backlund transformation between two pairs of solutions. With the help of this transformation non-trivial exact solutions have been rediscovered.

Keywords. Painlevé analysis; integrability; auto-Backlund transformations; exact solutions; pion dynamics; chiral invariance.

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1. Introduction

In this paper we have observed that the field equations [1,2] for Charap's chiral invariant model of the pion dynamics pass the test for integrability in the sense of Painlevé analysis due to Weiss *et al* [3–5]. The formalism of the truncation of a series solution as advocated by Weiss *et al* [3] leads to auto-Backlund transformation between two pairs of solutions. From the transformation, the nontrivial exact solutions have been rediscovered.

According to Weiss *et al*, the Painlevé test is as follows: If the singularity manifold is determined by

$$u(z_1, z_2, z_3, \dots, z_n) = 0 \tag{1.1}$$

and $\Phi = \Phi(z_1, z_2, z_3, \dots, z_n)$ is a solution of the partial differential equation, then we require that

$$\Phi = u^\alpha \sum_{j=0}^{\infty} \Phi_j u^j, \tag{1.2}$$

where $\Phi_0 \neq 0, \Phi_j = \Phi_j(z_1, \ldots, z_n)$ and $u = u(z_1, z_2, z_3, \ldots, z_n)$ are analytic functions of z_j in the neighbourhood of the manifold (1). The condition that u should be noncharacteristic (for the PDE) ensures that expansion (2) will be well-defined, in the sense of the Chauchy–Kovalevskaya theorem [6]. Substitution of (2) into the PDE determines the value(s) of α, and defines the recursion relations for $\Phi_j, j = 0, 1, 2, \ldots$. When expansion (2) is well-defined and contains the maximum number of arbitrary functions allowed at the 'resonances' [3,7–9], the PDE is said to possess the Painlevé property and is conjectured to be integrable. Informally, the resonances are the values of j for which ϕ_j are not 'fixed' by the recursion relations (i.e. are arbitrary).

The equations under study are given by

$$\phi = \eta^{\mu\nu}(\partial\phi/\partial x^\mu) \cdot (\partial\beta/\partial x^\nu), \tag{1.3a}$$
$$\psi = \eta^{\mu\nu}(\partial\psi/\partial x^\mu) \cdot (\partial\beta/\partial x^\nu), \tag{1.3b}$$
$$\chi = \eta^{\mu\nu}(\partial\chi/\partial x^\mu) \cdot (\partial\beta/\partial x^\nu), \tag{1.3c}$$

where

$$\begin{aligned} \eta^{\mu\nu} &= 0 \quad \text{for } \mu \neq \nu \\ &= 1 \quad \text{for } \mu = \nu \neq 4 \\ &= -1 \quad \text{for } \mu = \nu = 4 \end{aligned} \tag{1.3d}$$
$$\beta = \ln(f_\pi^2 + \phi^2 + \psi^2 + \chi^2) \tag{1.3e}$$
$$f_\pi = \text{constant}. \tag{1.3f}$$

One arrives at eqs (1.3) through tangential parametrization of the field equation for the chiral invariant model of the pion dynamics [1,2].

In order to apply the Painlevé analysis in the sense of Weiss *et al*, eqs (1.3) are rewritten as follows:

$$\begin{aligned} \phi_{11} + \phi_{22} + \phi_{33} - \phi_{44} = {} & 2\phi[\exp(-\beta)](\phi_1^2 + \phi_2^2 + \phi_3^2 - \phi_4^2) \\ & +2\psi[\exp(-\beta)](\phi_1\psi_1 + \phi_2\psi_2 + \phi_3\psi_3 - \phi_4\psi_4) \\ & +2\chi[\exp(-\beta)](\phi_1\chi_1 + \phi_2\chi_2 + \phi_3\chi_3 - \phi_4\chi_4), \end{aligned} \tag{1.4a}$$

$$\begin{aligned} \psi_{11} + \psi_{22} + \psi_{33} - \psi_{44} = {} & 2\psi[\exp(-\beta)](\psi_1^2 + \psi_2^2 + \psi_3^2 - \psi_4^2) \\ & +2\phi[\exp(-\beta)](\phi_1\psi_1 + \phi_2\psi_2 + \phi_3\psi_3 - \phi_4\psi_4) \\ & +2\chi[\exp(-\beta)](\psi_1\chi_1 + \psi_2\chi_2 + \psi_3\chi_3 - \psi_4\chi_4), \end{aligned} \tag{1.4b}$$

$$\begin{aligned} \chi_{11} + \chi_{22} + \chi_{33} - \chi_{44} = {} & 2\chi[\exp(-\beta)](\chi_1^2 + \chi_2^2 + \chi_3^2 - \chi_4^2) \\ & +2\phi[\exp(-\beta)](\phi_1\chi_1 + \phi_2\chi_2 + \phi_3\chi_3 - \phi_4\chi_4) \\ & +2\psi[\exp(-\beta)](\psi_1\chi_1 + \psi_2\chi_2 + \psi_3\chi_3 - \psi_4\chi_4), \end{aligned} \tag{1.4c}$$

where

$$\beta = \ln(f_\pi^2 + \phi^2 + \psi^2 + \chi^2).$$

Equations (1.4) with a physical origin stated have been found to have interesting solutions and mathematical characteristics. Charap [2] obtained solutions for (1.4) under the assumption that ϕ, ψ and χ are all functions of $(k_1x^1+k_2x^2+k_3x^3+k_4x^4)$ where k_i is any four vector. Ray [10] presented two types of solutions for (1.4). For obtaining the first type of solution, Ray [10] (rediscovered by Chakraborty and Chanda [11]) used the ansatz

$$\phi = \phi(v), \quad \psi = \psi(v), \quad \chi = \chi(v), \tag{1.5}$$

where v is an unspecified function of x^1, x^2, x^3, x^4.

This type of solution is a generalization of the solutions obtained by Charap mentioned above and includes a soliton solution as a special case. For obtaining the second type of solutions, Ray [10] used the ansatz

$$\phi = \phi(x^1, x^2, x^3 - x^4), \tag{1.6a}$$

$$\psi = \psi(x^1, x^2, x^3 - x^4), \tag{1.6b}$$

$$\chi = \chi(x^1, x^2, x^3 - x^4). \tag{1.6c}$$

Chanda *et al* [12] further generalized them considerably. The reduced equations for this ansatz are conformally invariant, i.e. the form of the equations remains invariant under the transformation $(x^1, x^2) \to (g, h)$ where g and h are two mutually conjugate solutions of Laplace's equations in x^1 and x^2. Hence from any solution of the reduced equations one can immediately generate infinitely many other solutions of the same equations simply by replacing (x^1, x^2) by (g, h) where g and h are two mutually conjugate solutions of Laplace's equations.

Chakraborty and Chanda [13] presented two other types of exact solutions where the dependence on x^3 and x^4 is more generalized than that stated in (1.6). They obtained two types of solutions. In the first case the ansatz was

$$\phi = \phi(\tau, \sigma), \quad \psi = \psi(\tau, \sigma), \quad \chi = \chi(\tau, \sigma), \tag{1.7a}$$

where

$$\tau = \tau(x^1, x^2), \quad \sigma = \sigma(x^3, x^4). \tag{1.7b}$$

In the second case they sought a class of solutions by changing variables to functions of space–time coordinates, which were restricted in the following way:

$$(x^1, x^2, x^3, x^4) \to (X, Y, Z, W) \tag{1.8a}$$

such that

$$X_1 = Y_2, \quad X_2 = -Y_1 \tag{1.8b}$$

and

$$Z_3 = W_4, \quad Z_4 = W_3, \tag{1.8c}$$

where

$$X = X(x^1, x^2), \quad Y = Y(x^1, x^2),$$

$$Z = Z(x^3, x^4), \quad W = W(x^3, x^4).$$

Here also the reduced equations admitted infinite number of solutions.

In a recent publication [14], Chakraborty and Chanda have shown graphically that the solutions corresponding to (1.5) of this paper obtained by Ray [10] and rediscovered by Chakraborty and Chanda [11] represent solitary wave with oscillatory profile.

Chakraborty and Chanda [11] have found eqs (1.4) to be interesting from another angle of view. They observed that the equations reported by Charap [1,2] for the chiral invariant model of pion dynamics under tangential parametrization and the equations reported by Yang [15] while discussing the condition of self-duality of $SU(2)$ gauge fields on Euclidean four-dimensional space have some common characteristics which are mathematically interesting. This has been elaborated in their publication [11].

With this motivation they have combined the two sets of equations and have obtained a new set of equations wherefrom the previous two sets can be obtained as particular cases. It has been found that [11] the solutions of the combined equations, still having interesting physical character, deviate much from Charap's equations. In this connection it may be mentioned that another class of rather generalized solutions for the nonlinear sigma model of chiral theories has been found by Enikova *et al* [16] and more recently been rediscovered by Anslem [17].

The results showing that eqs (1.4) pass the Painlevé test for integrability (in the sense of Weiss *et al*) and admit truncation of series leading to auto-Backlund transformation between two pairs of exact solutions wherefrom nontrivial exact solutions can be rediscovered add to the importance of Charap's equations (1.4).

2. Painlevé test for integrability of eqs (1.4) in the sense of Weiss *et al*

For eqs (1.4) we define the singularity manifold given by

$$u = u(x^1, x^2, x^3, x^4) = 0 \tag{2.1}$$

and set

$$\phi = u^\alpha \sum \phi_j u^j, \quad \psi = u^\beta \sum \psi_j u^j, \quad \chi = u^\gamma \sum \chi_j u^j, \tag{2.2}$$

where $\phi(x^1, x^2, x^3, x^4), \psi(x^1, x^2, x^3, x^4), \chi(x^1, x^2, x^3, x^4)$ are a set of solutions of (1.4); ϕ_j, ψ_j, χ_j are all analytic functions of (x^1, x^2, x^3, x^4) in the neighborhood of the manifold (2.1); $\phi_0 \neq 0, \psi_0 \neq 0, \chi_0 \neq 0$.

Now, the test may be divided into three main steps after the substitution of (2) in the differential equations concerned:

(I) Make the leading order analysis (where one gets all possible $\alpha, \beta, \gamma, \phi_0, \psi_0, \chi_0$ in (2.2)).

(II) Define the recursion relations for u_j for the leading orders obtained in step I and determine the resonance positions (those values of j for which the relations are not defined).

(III) Check whether the expansions allow requisite number of arbitrary functions at the resonance positions.

2.1 *Leading order analysis*

We assume

$$\phi \sim \phi_0 u^\alpha, \quad \psi \sim \psi_0 u^\beta, \quad \chi \sim \chi_0 u^\gamma. \tag{2.3}$$

We substitute (2.3) in (1.4) and equate the coefficients of the negative powers of u (considering that all α, β and γ are negative). This leads to $\alpha = -1, \beta = -1, \gamma = -1$ so that

$$\phi = u^\alpha \sum_{j=0}^{\infty} \phi_j u^j, \quad \psi = u^\beta \sum_{j=0}^{\infty} \psi_j u^j, \quad \chi = u^\gamma \sum_{j=0}^{\infty} \chi_j u^j, \tag{2.4}$$

$\phi_0 =$ arbitrary, $\psi_0 =$ arbitrary, $\chi_0 =$ arbitrary.

2.2 *Resonance positions*

We directly substitute (2.4) in (1.4). We have not written explicitly the recursion relation because of their involved structure. In order to have an idea one can consult the recursion relations of the work of Chanda and Roy Chowdhuri [5]. Here the resonance positions are $R = -1$, 0, 0, 0, 1, 1.

(i) $R = -1$ indicates that the singularity manifold defined in (2.1) is required to be arbitrary.
(ii) $R = 0, 0, 0$ indicate that all of the coefficients ϕ_0, ψ_0 and χ_0 are required to be arbitrary.
(iii) $R = 1, 1$ indicate that any two of the coefficients ϕ_1, ψ_1 and χ_1 are required to be arbitrary.

2.3. *To check whether the expansions allow requisite number of arbitrary functions at the resonance positions*

(i) The singularity manifold, by definition, is arbitrary.
(ii) The terms involving ϕ_0, ψ_0 and χ_0 cancel each other. Hence all ϕ_0, ψ_0 and χ_0 are arbitrary.
(iii) We get only one equation involving $\phi_1, \psi_1, \sigma_1, \phi_0, \psi_0, \chi_0$. Hence two of the coefficients of ϕ_1, ψ_1, χ_1 can be kept arbitrary when the third is determined in terms of those arbitrary functions and ϕ_0, ψ_0, χ_0.

With the above observations one can conclude that eqs (1.4) pass the Painlevé test for integrability in the sense of Weiss *et al.*

3. Truncation of the series (2.2), auto-Backlund transformation and exact solutions

Here we forcefully make the coefficients ϕ_j, ψ_j, χ_j of u^{j-1} in the expansions (2.4) zero for $j > 1$.

The coefficients ϕ_1, ψ_1 and χ_1 in (2.4) are rewritten as p, q and r respectively in order to differentiate them from $(\partial\phi/\partial x^1), (\partial\psi/\partial x^1), (\partial\chi/\partial x^1)$. Then from (2.4) one gets

$$\phi = \phi_0 u^{-1} + p, \quad \psi = \psi_0 u^{-1} + q, \quad \chi = \chi_0 u^{-1} + r \tag{3.1}$$

subject to the condition that the three equations in (3.2) are satisfied:

$$Pu^{-5} + Qu^{-4} + Cu^{-3} + Eu^{-2} + Fu^{-1} + G = 0, \tag{3.2a}$$
$$P'u^{-5} + Q'u^{-4} + C'u^{-3} + E'u^{-2} + F'u^{-1} + G' = 0, \tag{3.2b}$$
$$P''u^{-5} + Q''u^{-4} + C''u^{-3} + E''u^{-2} + F''u^{-1} + G'' = 0, \tag{3.2c}$$

where P, Q, C, E, F, G etc. are given below with the notations:

$$u = u_{11} + u_{22} + u_{33} - u_{44}, \quad \mathbf{p}\cdot\mathbf{q} = p_1q_1 + p_2q_2 + p_3q_3 - p_4q_4 \tag{3.3a}$$
$$A = \phi_0 p + \psi_0 q + \chi_0 r, \quad B = \phi_0^2 + \psi_0^2 + \chi_0^2 \tag{3.3b}$$
$$P = 0, \quad P' = 0, \quad P'' = 0 \tag{3.3c}$$

$$\begin{aligned} Q = {} & 2A(\mathbf{u}\cdot\mathbf{u}) - B(\mathbf{u}\cdot\mathbf{u}) + 2\phi_0(\boldsymbol{\phi}_0\cdot\mathbf{u}) + 2\psi_0(\boldsymbol{\psi}_0\cdot\mathbf{u}) \\ & + 2\chi_0(\boldsymbol{\chi}_0\cdot\mathbf{u}) = 0 \end{aligned} \tag{3.4a}$$
$$Q' = Q, \quad Q'' = Q \tag{3.4b}$$

$$\begin{aligned} C = {} & 2\phi_0 e^{\beta}(\mathbf{u}\cdot\mathbf{u}) + \phi_0[2p(\boldsymbol{\phi}_0\cdot\mathbf{u}) + 2q(\boldsymbol{\psi}_0\cdot\mathbf{u}) + 2r(\boldsymbol{\chi}_0\cdot\mathbf{u})] \\ & -2A(\boldsymbol{\phi}_0\cdot\mathbf{u}) - 2\phi_0(\boldsymbol{\phi}_0\cdot\boldsymbol{\phi}_0) - 2\psi_0(\boldsymbol{\phi}_0\cdot\boldsymbol{\psi}_0) - 2\chi_0(\boldsymbol{\phi}_0\cdot\boldsymbol{\chi}_0) \\ & +2\phi_0[\phi_0(\mathbf{u}\cdot\mathbf{p}) + \psi_0(\mathbf{u}\cdot\mathbf{q}) + \chi_0(\mathbf{u}\cdot\mathbf{r})] \\ & +2B(\mathbf{u}\cdot\mathbf{p}) - 2\phi_0 Au + B\phi_0, \end{aligned} \tag{3.5a}$$

$$\begin{aligned} C' = {} & 2\psi_0 e^{\beta'}(\mathbf{u}\cdot\mathbf{u}) + \psi_0[2p(\boldsymbol{\phi}_0\cdot\mathbf{u}) + 2q(\boldsymbol{\psi}_0\cdot\mathbf{u}) + 2r(\boldsymbol{\chi}_0\cdot\mathbf{u})] \\ & -2A(\boldsymbol{\psi}_0\cdot\mathbf{u}) - 2\psi_0(\boldsymbol{\psi}_0\cdot\boldsymbol{\psi}_0) - 2\phi_0(\boldsymbol{\phi}_0\cdot\boldsymbol{\psi}_0) - 2\chi_0(\boldsymbol{\psi}_0\cdot\boldsymbol{\chi}_0) \\ & +2\psi_0[\phi_0(\mathbf{u}\cdot\mathbf{p}) + \psi_0(\mathbf{u}\cdot\mathbf{q}) + \chi_0(\mathbf{u}\cdot\mathbf{r})] \\ & +2B(\mathbf{u}\cdot\mathbf{q}) - 2\psi_0 Au + B\psi_0, \end{aligned} \tag{3.5b}$$

$$\begin{aligned} C'' = {} & 2\chi_0 e^{\beta'}(\mathbf{u}\cdot\mathbf{u}) + \chi_0[2p(\boldsymbol{\phi}_0\cdot\mathbf{u}) + 2q(\boldsymbol{\psi}_0\cdot\mathbf{u}) + 2r(\boldsymbol{\chi}_0\cdot\mathbf{u})] \\ & -2A(\boldsymbol{\chi}_0\cdot\mathbf{u}) - 2\chi_0(\boldsymbol{\chi}_0\cdot\boldsymbol{\chi}_0) - 2\phi_0(\boldsymbol{\phi}_0\cdot\boldsymbol{\chi}_0) - 2\psi_0(\boldsymbol{\psi}_0\cdot\boldsymbol{\chi}_0) \\ & +2\chi_0[\phi_0(\mathbf{u}\cdot\mathbf{p}) + \psi_0(\mathbf{u}\cdot\mathbf{q}) + \chi_0(\mathbf{u}\cdot\mathbf{r})] \\ & +2B(\mathbf{u}\cdot\mathbf{r}) - 2\chi_0 Au + B\chi_0, \end{aligned} \tag{3.5c}$$

$$\begin{aligned}E = &-2e^{\beta'}(\boldsymbol{\phi}_0\cdot\mathbf{u}) - \phi_0^2 e^{\beta'}u + 2A\phi_0 + Bp + 2A(\mathbf{u}\cdot\mathbf{p}) - 2\phi_0(\boldsymbol{\phi}_0\cdot\mathbf{p})\\ &-2\psi_0(\boldsymbol{\phi}_0\cdot\mathbf{q}) - 2\chi_0(\boldsymbol{\phi}_0\cdot\mathbf{r}) + 2p\phi_0(\mathbf{u}\cdot\mathbf{p}) + 2q\phi_0(\mathbf{u}\cdot\mathbf{q})\\ &+2r\phi_0(\mathbf{u}\cdot\mathbf{r}) - 2p(\boldsymbol{\phi}_0\cdot\boldsymbol{\phi}_0) - 2q(\boldsymbol{\phi}_0\cdot\boldsymbol{\psi}_0) - 2r(\boldsymbol{\phi}_0\cdot\boldsymbol{\chi}_0)\\ &-2\phi_0(\boldsymbol{\phi}_0\cdot\mathbf{p}) - 2\psi_0(\boldsymbol{\psi}_0\cdot\mathbf{p}) - 2\chi_0(\boldsymbol{\chi}_0\cdot\mathbf{p}),\end{aligned} \tag{3.6a}$$

$$\begin{aligned}E' = &-2e^{\beta'}(\boldsymbol{\psi}_0\cdot\mathbf{u}) - \psi_0^2 e^{\beta'}u + 2A\psi_0 + Bq + 2A(\mathbf{u}\cdot\mathbf{q}) - 2\phi_0(\boldsymbol{\psi}_0\cdot\mathbf{p})\\ &-2\psi_0(\boldsymbol{\psi}_0\cdot\mathbf{q}) - 2\chi_0(\boldsymbol{\psi}_0\cdot\mathbf{r}) + 2p\psi_0(\mathbf{u}\cdot\mathbf{p}) + 2q\phi_0(\mathbf{u}\cdot\mathbf{q})\\ &+2r\psi_0(\mathbf{u}\cdot\mathbf{r}) - 2p(\boldsymbol{\phi}_0\cdot\boldsymbol{\psi}_0) - 2q(\boldsymbol{\psi}_0\cdot\boldsymbol{\psi}_0) - 2r(\boldsymbol{\psi}_0\cdot\boldsymbol{\chi}_0)\\ &-2\phi_0(\boldsymbol{\phi}_0\cdot\mathbf{q}) - 2\psi_0(\boldsymbol{\psi}_0\cdot\mathbf{q}) - 2\chi_0(\boldsymbol{\chi}_0\cdot\mathbf{q}),\end{aligned} \tag{3.6b}$$

$$\begin{aligned}E'' = &-2e^{\beta'}(\boldsymbol{\chi}_0\cdot\mathbf{u}) - \chi_0^2 e^{\beta'}u + 2A\chi_0 + Br + 2A(\mathbf{u}\cdot\mathbf{r}) - 2\phi_0(\boldsymbol{\chi}_0\cdot\mathbf{p})\\ &-2\psi_0(\boldsymbol{\chi}_0\cdot\mathbf{q}) - 2\chi_0(\boldsymbol{\chi}_0\cdot\mathbf{r}) + 2p\chi_0(\mathbf{u}\cdot\mathbf{p}) + 2q\chi_0(\mathbf{u}\cdot\mathbf{q})\\ &+2r\chi_0(\mathbf{u}\cdot\mathbf{r}) - 2p(\boldsymbol{\phi}_0\cdot\chi_0) - 2q(\boldsymbol{\psi}_0\cdot\boldsymbol{\chi}_0) - 2r(\boldsymbol{\chi}_0\cdot\boldsymbol{\chi}_0)\\ &-2\phi_0(\boldsymbol{\phi}_0\cdot\mathbf{r}) - 2\psi_0(\boldsymbol{\psi}_0\cdot\mathbf{r}) - 2\chi_0(\boldsymbol{\chi}_0\cdot\mathbf{r}),\end{aligned} \tag{3.6c}$$

$$\begin{aligned}F = &\, e^{\beta'}\phi_0 + 2Ap - 2p(\boldsymbol{\phi}_0\cdot\mathbf{p}) - 2q(\boldsymbol{\phi}_0\cdot\mathbf{q}) - 2r(\boldsymbol{\phi}_0\cdot\mathbf{r})\\ &-2p(\boldsymbol{\phi}_0\cdot\mathbf{p}) - 2q(\boldsymbol{\psi}_0\cdot\mathbf{p}) - 2r(\boldsymbol{\chi}_0\cdot\mathbf{p}) - 2\phi_0(\mathbf{p}\cdot\mathbf{p})\\ &-2\psi_0(\mathbf{p}\cdot\mathbf{q}) - 2\chi_0(\mathbf{p}\cdot\mathbf{r}),\end{aligned} \tag{3.7a}$$

$$\begin{aligned}F' = &\, e^{\beta'}\psi_0 + 2Aq - 2p(\boldsymbol{\psi}_0\cdot\mathbf{p}) - 2q(\boldsymbol{\psi}_0\cdot\mathbf{q}) - 2r(\boldsymbol{\psi}_0\cdot\mathbf{r})\\ &-2p(\boldsymbol{\phi}_0\cdot\mathbf{q}) - 2q(\boldsymbol{\psi}_0\cdot\mathbf{q}) - 2r(\boldsymbol{\chi}_0\cdot\mathbf{q}) - 2\phi_0(\mathbf{p}\cdot\mathbf{q})\\ &-2\psi_0(\mathbf{q}\cdot\mathbf{q}) - 2\chi_0(\mathbf{q}\cdot\mathbf{r}),\end{aligned} \tag{3.7b}$$

$$\begin{aligned}F'' = &\, e^{\beta'}\chi_0 + 2Ar - 2p(\boldsymbol{\chi}_0\cdot\mathbf{p}) - 2q(\boldsymbol{\chi}_0\cdot\mathbf{q}) - 2r(\boldsymbol{\chi}_0\cdot\mathbf{r})\\ &-2p(\boldsymbol{\phi}_0\cdot\mathbf{r}) - 2q(\boldsymbol{\psi}_0\cdot\mathbf{r}) - 2r(\boldsymbol{\chi}_0\cdot\mathbf{r}) - 2\phi_0(\mathbf{p}\cdot\mathbf{r})\\ &-2\psi_0(\mathbf{q}\cdot\mathbf{r}) - 2\chi_0(\mathbf{r}\cdot\mathbf{r}),\end{aligned} \tag{3.7c}$$

$$G = p - e^{-\beta'}[2p(\mathbf{p}\cdot\mathbf{p}) + 2q(\mathbf{p}\cdot\mathbf{q}) + 2r(\mathbf{p}\cdot\mathbf{r})], \tag{3.8a}$$

$$G' = q - e^{-\beta'}[2q(\mathbf{q}\cdot\mathbf{q}) + 2p(\mathbf{p}\cdot\mathbf{q}) + 2r(\mathbf{q}\cdot\mathbf{r})], \tag{3.8b}$$

$$G'' = r - e^{-\beta'}[2r(\mathbf{r}\cdot\mathbf{r}) + 2p(\mathbf{p}\cdot\mathbf{r}) + 2q(\mathbf{q}\cdot\mathbf{r})], \tag{3.8c}$$

where $\beta' = f_\pi^2 + p^2 + q^2 + r^2$.

Now, if one has $G = 0, G' = 0, G'' = 0$ then one can say that eqs in (3.1) represent auto-Backlund transformation [18,19] between two pairs of solutions of (1.4) given by (ϕ, ψ, χ) and (p, q, r) subject to the condition

$$Pu^{-5} + Qu^{-4} + Cu^{-3} + Eu^{-2} + Fu^{-1} + G = 0 \tag{3.9a}$$

$$P'u^{-5} + Q'u^{-4} + C'u^{-3} + E'u^{-2} + F'u^{-1} + G' = 0 \tag{3.9b}$$

$$P''u^{-5} + Q''u^{-4} + C''u^{-3} + E''u^{-2} + F''u^{-1} + G'' = 0 \tag{3.9c}$$

where P, Q, C, E, F, G, etc are given by (3.3) to (3.8).

It would have been nice if auto-Backlund transformation between two sets of nontrivial solution could be shown. At this stage the complicacy of the system did not allow us to achieve that goal. However, in the following section we have shown the auto-Backlund transformation between a set of trivial solutions ($p = 0, q = 0, r = 0$) and the nontrivial solutions (ϕ, ψ, χ) reported in the Introduction.

4. Rediscovery of solutions reported in the Introduction

Normally an overdetermined system is obtained by equating the coefficients of u^{-j} in (3.2) separately to zero. However, at the time of rediscovering previous solutions with $p = 0, q = 0, r = 0$ it is found that the act of equating the coefficients of u^{-j} separately to zero imposes a very strong condition which cannot be satisfied. Therefore, we have kept (3.9) as such and made $p = 0, q = 0, r = 0$ so that $G = 0, G' = 0, G'' = 0$ are automatically satisfied.

4.1 *Solutions reported in (1.5)*

Here the solutions can be obtained from (3.1), (3.8), ($p = 0, q = 0, r = 0$) and the assumption

$$(\phi_0/u) = a(w), \quad (\psi_0/u) = m(w), \quad (\chi_0/u) = n(w), \tag{4.1}$$

where a, m, n are functions of $w, w = w(x^1, x^2, x^3, x^4)$ and we get

$$\phi = a(w), \quad \psi = m(w), \quad \chi = n(w) \tag{4.2}$$

which is the same as (1.5).

4.2 *Solutions reported in (1.6)*

Here the solutions can be obtained from (3.1), (3.8), ($p = 0, q = 0, r = 0$) and the assumption

$$(\phi_0/u) = a'(w'), \quad (\psi_0/u) = m'(w'), \quad (\chi_0/u) = n'(w'), \tag{4.3}$$

where a', m', n' are functions of $w', w' = w'(x^1, x^2, x^3 - x^4)$ and we get

$$\phi = a'(x^1, x^2, x^3 - x^4) \tag{4.4a}$$
$$\psi = m'(x^1, x^2, x^3 - x^4) \tag{4.4b}$$
$$\chi = n'(x^1, x^2, x^3 - x^4) \tag{4.4c}$$

which is the same as (1.6).

4.3 *Solutions reported in (1.7)*

Here the solutions can be obtained from (3.1), (3.8), ($p = 0, q = 0, r = 0$) and the assumption

$$(\phi_0/u) = a''(\tau, \sigma), \quad (\psi_0/u) = m''(\tau, \sigma), \quad (\chi_0/u) = n''(\tau, \sigma) \tag{4.5}$$

where a'', m'', n'' are functions of (τ, σ). $\tau = \tau(x^1, x^2), \sigma = \sigma(x^3, x^4)$, and we get

$$\phi = a''(\tau, \sigma) \tag{4.6a}$$
$$\psi = m''(\tau, \sigma) \tag{4.6b}$$
$$\chi = n''(\tau, \sigma) \tag{4.6c}$$

which is the same as (1.7).

5. Summary

The field equations for Charap's chiral invariant model of the pion dynamics pass the Painlevé test for complete integrability in the sense of Weiss *et al.* The truncation procedure of the same analysis leads to auto-Backlund transformation between two pairs of solutions. With the help of this transformation nontrivial solutions have been rediscovered. However, only the transformation between a set of trivial solutions and another set of nontrivial solutions could be demonstrated. The transformation between two sets of nontrivial solutions remains to be demonstrated.

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