PRAMANA — journal of physics

Vol. 66, No. 3
March 2006
pp. 513–519

# The overall phase shift and lens effect calculation using Gaussian boundary conditions and paraxial ray approximation for an end-pumped solid-state laser

H NADGARAN and P ELAHI*
Department of Physics, College of Science, Shiraz University, Shiraz 71454, Iran
*Department of Physics, Payame Noor University, Shiraz 71365-944, Iran
E-mail: nadgaran@susc.ac.ir; phy1pe@yahoo.com; pelahi@spnu.ac.in

MS received 11 April 2005; revised 10 October 2005; accepted 3 December 2005

**Abstract.** In this work, the inhomogeneous equation of heat conduction was exactly solved by applying inhomogeneous boundary conditions for laser crystals of aspect ratio=1 (aspect ratio=radius of the laser rod/length of the laser rod). We have shown that the paraxial ray approximation leads the solution to be a function of $r^2$, that is, the approximation is equivalent to a situation in which a homogeneous pump source is used. The solution was then used to derive expressions for the overall phase shift, focal length of the thermal lens and the end effect induced curvature of the end face. The expressions were then applied to Nd:YAG laser medium. The result shows a meaningful correction of the order of 0.001 cm to the focal length of Nd:YAG rod for 3 W source power and beam waist of 100 $\mu$m.

**Keywords.** Thermal effects; thermal lensing; phase shift.

**PACS No.** **42.55.Xi**

## 1. Introduction

End-pumped solid state lasers have gained considerable attention because of their many applications including laser spectroscopy of atoms and molecules [1–4]. In this respect, a very good beam characteristics is needed for the collection of precise spectroscopy data. Beam quality in turn, can be guaranteed when in-design and construction of such lasers, a deep knowledge of thermally induced effects of the laser material is available. Consideration of thermal effects of such lasers, is done by solving the equation of heat conduction for the laser medium when pumped by homogeneous and inhomogeneous Gaussian profile pump.

Farrukh *et al* [5] presented a generalized analytical solution to the heat conduction equation in cylindrical end-pumped laser rods for both continuous wave and pulsed pumping. Convective and conductive boundary conditions are described in the literature and work of Farrukh is based on the former. In the case of longitudinal pumping, the inhomogeneous pump distribution inside the crystal needs more investigation than the homogeneous one. Tidwell *et al* [6] got the temperature distribution in pure logarithmic form when $z$-term derivative in the heat conduction equation was neglected. MacDonald *et al* [7] derived an equation for thermal induced focal length caused by the thermal effects in the lasing medium.

In this work an inhomogeneous pump source with inhomogeneous Gaussian boundary condition was employed and it is shown that in paraxial ray approximation regime, the behavior of equations are the same for both homogeneous and inhomogeneous pump sources.

## 2. Thermal consideration

A low power end-pumped solid state laser where the periphery of the laser crystal is fixed at temperature, $T_{\mathrm{c}}$, is arranged. Consider the temperature distribution $T(r,z)$ as the solution of the following equation of heat conduction [7]:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = -\frac{S(r)}{K}, \tag{1}$$

where the laser rod is a cylinder of radius $b$ ($0 \le r \le b$) and length $L$. $K$ and $S(r)$ are the thermal conductivity of the lasing medium and heat power density, respectively. The heat source is assumed to have a Gaussian form of

$$S(r) = \frac{2P_{\mathrm{tot}}(1-\eta\frac{\lambda_{\mathrm{p}}}{\lambda_{\mathrm{l}}})}{\pi W_0^2 L}\mathrm{e}^{-(2r^2/W_0^2)}, \tag{2}$$

where $W_0$ is the beam waist and $P_{\mathrm{tot}}, \eta, \lambda_{\mathrm{p}}(\lambda_{\mathrm{l}})$ are the total power of the source, quantum efficiency, and pump wavelength (laser wavelength), respectively. Since we are working with small crystal length the heat is considered to be generated uniformly inside the crystal along the $z$-axis, and so one can neglect the exponential decay term in eq. (2) and it is also assumed that the heat source equation is valid for those pump spot sizes for which the Rayleigh range is much greater than the crystal length. Substituting eq. (2) into eq. (1), we have

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \beta\mathrm{e}^{-(2r^2/W_0^2)} = 0, \tag{3}$$

where

$$\beta = \frac{2P_{\mathrm{tot}}(1-\eta\frac{\lambda_{\mathrm{p}}}{\lambda_{\mathrm{l}}})}{\pi W_0^2 L K}. \tag{4}$$

The solution to eq. (3) when using the method of separation of variables will then be given as

$$T(r,z) = \sum A_n J_0\left(\frac{\alpha_n}{b}r\right) \mathrm{e}^{-(\alpha_n/b)z} + \frac{1}{8}\beta W_0^2 \times \left[\ln\left(\frac{b}{r}\right)^2 - Ei\left(\frac{2r^2}{W_0^2}\right) + Ei\left(\frac{2b^2}{W_0^2}\right)\right] + T_b, \tag{5}$$

where $J_0(\frac{\alpha_n}{b}r)$ is a first kind Bessel function of order zero with $\alpha_n$s as its roots. $Ei$'s are exponential integrals defined as: $E_i(x) = \int_x^\infty \frac{\mathrm{e}^{-t}}{t}\mathrm{d}t$. $T_\mathrm{b}$ shows the temperature at the surface of the rod. In obtaining the solution $T(r,z)$ in eq. (5), we retained the $z$-derivative term because the radius of the crystal is comparable to its length, that is aspect ratio of 1 was assumed. It is easily seen that as $r$ approaches zero the right-hand side of eq. (5) remains finite.

A stable condition is defined when the heat generated within the rod is totally removed by the coolant of temperature, $T_\mathrm{c}$, that is,

$$\int 2\pi S(r) r\mathrm{d}r\mathrm{d}z = 2\pi bLH(T_\mathrm{b} - T_\mathrm{c}), \tag{6}$$

where $H$ denotes the heat transfer coefficient of the surface. From this condition, one can have

$$T_\mathrm{b} = \frac{P_\mathrm{tot}(1 - \mathrm{e}^{-(2b^2/W_0^2)})(1 - \eta\frac{\lambda_\mathrm{p}}{\lambda_\mathrm{l}})}{2\pi bLH} + T_\mathrm{c}. \tag{7}$$

Moreover,

$$T(r, z = 0) = T_0 \mathrm{e}^{-(2r^2/W_0^2)}, \tag{8}$$

where $T_0$ is the temperature at $r = 0$.

Upon using these boundary conditions, $A_n$'s can be calculated as

$$A_n = \frac{B_n - D_n + E_n - \left[1/8\beta\omega_0^2 Ei(\frac{2b^2}{W_0^2}) + T_\mathrm{b}\right] F_n}{C_n}, \tag{9}$$

where

$$B_n = T_0 \int_0^b J_0\left(\frac{\alpha_n}{b}r\right) \mathrm{e}^{-(2r^2/W_0^2)} r\mathrm{d}r, \tag{10}$$

$$C_n = \frac{b^2}{2}\left[J_1(\alpha_n)\right]^2, \tag{11}$$

$$D_n = 1/8\beta W_0^2 \int_0^b J_0\left(\frac{\alpha_n}{b}r\right) \ln\left(\frac{b}{r}\right)^2 r\mathrm{d}r, \tag{12}$$

$$E_n = 1/8\beta W_0^2 \int_0^b J_0\left(\frac{\alpha_n}{b}r\right) Ei\left(\frac{2r^2}{W_0^2}\right) r\mathrm{d}r, \tag{13}$$

**Table 1.** $A_n$ values for Nd : YAG laser rod.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $\alpha_n$ | 2.4048 | 5.5201 | 8.6537 | 11.7915 | 14.9309 | 18.07 | 21.2116 |
| $A_n$ | 1.8048 | −1.2137 | −0.2736 | −0.3212 | −0.1124 | −0.1054 | −0.0186 |

$$F_n = \int_0^b J_0\left(\frac{\alpha_n}{b}r\right) r\mathrm{d}r. \tag{14}$$

Table 1 shows the values of $A_n$'s for a Nd : YAG laser rod for the following quantities: $P_{\text{tot}} = 3$ W, $b = 0.5$ cm, $L = 0.5$ cm, $H = 1\,\mathrm{W/cm^2 \cdot K}$, $K_{\mathrm{Nd:YAG}} = 0.13\,\mathrm{W/cm \cdot K}$, and $W_0 = 100\ \mu$m. As we are working in the regime of the Rayleigh range being much greater than the crystal length, with the above parameters of Nd : YAG laser rod, the Rayleigh range would be calculated as about 39 mm, whereas the crystal length is taken to be 5 mm.

## 3. Refractive index change

By substituting eq. (5) in [8]:

$$\Delta n(r,z) = [T(r,z) - T(0,z)]\frac{\mathrm{d}n}{\mathrm{d}T} \tag{15}$$

one can obtain the change of the refractive index of the laser rod as

$$\Delta n(r,z) = \left[\sum A_n \left[J_0\left(\frac{\alpha_n}{b}r\right) - 1\right]\mathrm{e}^{-(\alpha_n/b)z} + \frac{1}{8}\beta W_0^2\left[\ln\left(\frac{b}{r}\right)^2 - \ln\left(\frac{b}{W_0^2}\right)^2 - Ei\left(\frac{2r^2}{W_0^2}\right) - \ln 2 - \gamma\right]\right]\frac{\mathrm{d}n}{\mathrm{d}T}, \tag{16}$$

where $\gamma$ is the Euler–Mascheroni constant.

The overall phase shift caused by the changes of the refractive index can be calculated from $\Delta\phi_{\mathrm{T}}(r) = k\int_0^L \Delta(r,z)\mathrm{d}z$, where $k$ is the pump wave number [5]. Using eq. (16), we have

$$\Delta\phi_{\mathrm{T}}(r) = k\left[\sum \frac{bA_n}{\alpha_n}\left[J_0\left(\frac{\alpha_n}{b}r\right) - 1\right](1 - \mathrm{e}^{-(\alpha_n/b)L}) + \frac{L}{8}\beta W_0^2\left[\ln\left(\frac{b}{r}\right)^2 - \ln\left(\frac{b}{W_0^2}\right)^2 - Ei\left(\frac{2r^2}{W_0^2}\right) - \ln 2 - \gamma\right]\right]\frac{\mathrm{d}n}{\mathrm{d}T}. \tag{17}$$

### 3.1 *Paraxial ray approximation (PRA) of the overall phase shift*

Expansions of $J_0(\alpha_n/b)$ about $r$, and $Ei(2r^2/\omega_0^2)$ about $r^2$ are [9]:

$$J_0\left(\frac{\alpha_n}{b}r\right) \simeq 1 - \frac{\alpha_n^2 r^2}{4b^2} + O(r^4), \tag{18}$$

$$Ei\left(\frac{2r^2}{W_0^2}\right) \simeq -\gamma - \ln\left(\frac{2r^2}{W_0^2}\right) + \frac{2r^2}{W_0^2} + O(r^4). \tag{19}$$

Substituting eqs (18) and (19) into (17) gives

$$\Delta\phi_{\mathrm{T}}(r) \simeq -k\left[\sum A_n(1-\mathrm{e}^{-(\alpha_n L/b)})\frac{\alpha_n}{4b} + \frac{L\beta}{4}\right] r^2 \frac{\mathrm{d}n}{\mathrm{d}T}. \tag{20}$$

Equation (20) is the approximate value of the overall phase shift in the PRA regime with $r^2$ as its important term.

### 3.2 *Lens effect,* $(\mathrm{d}n/\mathrm{d}T)$ *effect (GRadient-INdex, GRIN) effect*

One important effect, the thermal lensing, results from temperature-induced changes in the refractive index of gain medium. Having derived eq. (20), one can equalize it with [7]

$$\Delta\phi_{\mathrm{T}}(r) = -\frac{kr^2}{2f}, \tag{21}$$

where $f$ is the focal length of the thermal lens:

$$\frac{1}{f} = D \simeq \left[\frac{\beta L}{2} + \sum \frac{\alpha_n A_n}{2b}(1-\mathrm{e}^{-(\alpha_n L/b)})\right]\frac{\mathrm{d}n}{\mathrm{d}T} \tag{22}$$

and $D$ is dioptric of the lens. Equation (22) have three distinct terms, the first term is usually present when the $z$-derivative term in the equation of heat conduction is neglected, and this case will occur for a rod of low aspect ratio. Based on retaining the $z$-derivative term, one can obviously see that the last two terms of eq. (22) is just a correction to focal length of the rod with aspect ratio=1.

### 3.3 *Lens effect, end effect*

Elongation of the laser rod will also be taken place. This can simply be found from [6]

$$l(r) = \alpha_{\mathrm{T}} \int_0^{l_0} [T(r,z) - T(0,z)]\,\mathrm{d}z, \tag{23}$$

where $l_0$ is the length of the rod over which elongation occurs and $\alpha_{\mathrm{T}}$ is the thermal expansion coefficient of the rod material. This will lead to

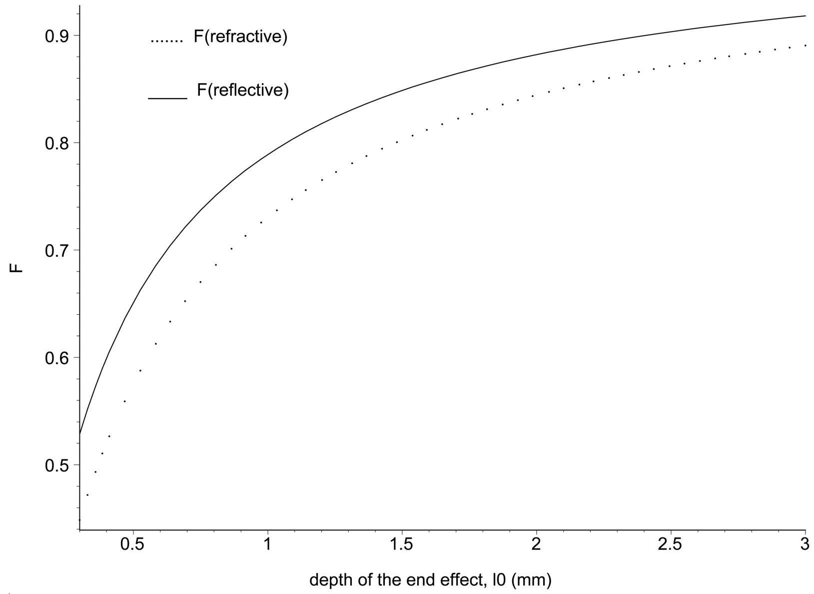


**Figure 1.** Contribution to the thermal lens from the end effect in longitudinally pumped Nd : YAG rod for light reflecting at the end of the rod ($F_{\text{reflective}}$) and light refracting at the end of the rod ($F_{\text{refractive}}$).

$$l(r) = \alpha_{\mathrm{T}}\left[\sum \frac{bA_n}{\alpha_n}\left(J_0\left(\frac{\alpha_n}{b}r\right) - 1\right)(1 - \mathrm{e}^{-(\alpha_n/b)l_0}) + \frac{l_0}{8}\beta W_0^2\left[\ln\left(\frac{b}{r}\right)^2 - \ln\left(\frac{b}{W_0^2}\right)^2 - Ei\left(\frac{2r^2}{W_0^2}\right) - \ln 2 - \gamma\right]\right]. \quad (24)$$

Having found $l(r)$, the radius of curvature of the surface given by $R^{-1} = -(\mathrm{d}^2 l/\mathrm{d}r^2)$ will be

$$\frac{1}{R} = \alpha_{\mathrm{T}}\left[\sum A_n(1 - \mathrm{e}^{-(\alpha_n l_0/b)})\frac{\alpha_n}{2b} + \frac{l_0\beta}{2}\right]. \quad (25)$$

The curvature will be accompanied by a focusing power depending on whether the end face of the rod is coated to serve as a resonator mirror or an external mirror is used. In the latter case, the light is transmitted through the curved end face such that the radius of curvature, $R$, gives a lens of focal length of [7]

$$f_{\text{refractive}} = \frac{R}{2(n-1)}. \quad (26)$$

The reflective focal length can also be found from [7]

$$f_{\text{reflective}} = \frac{R}{2n}. \quad (27)$$

It is then possible to derive an analytical expression for the fraction $F = \frac{D_{\mathrm{E}}}{D_{\mathrm{E}}+D_{\mathrm{T}}}$ [7] where $D_{\mathrm{E}} = \frac{1}{f_{\text{refractive(reflective)}}}$ and $D_{\mathrm{T}} = \frac{1}{f}$.

Figure 1 shows the fraction, $F$, against $l_0$ for Nd : YAG crystal pumped by 3 W Gaussian diode laser. This figure clearly shows that the deeper the heat penetration into the rod, the more pronounced is the end effect seen in the rod.

## 4. Discussion and conclusion

The analytical solution to inhomogeneous equation of heat conduction in an end-pumped solid-state laser is presented. In this regard, a Gaussian pump profile together with its Gaussian boundary condition was used. Using paraxial ray approximation, we found that the functional form of the overall phase shift in the inhomogeneous pumping case is the same as the homogeneous one. According to eq. (22), it can be shown that the focal length of a Nd : YAG laser rod pumping by a Gaussian source of $P_{\text{tot}} = 3$ W with $W_0 = 100$ $\mu$m, will be 5.5917 cm. The correction to this focal length arising from the last two terms of eq. (22) is $2.9 \times 10^{-3}$ cm. The other quantities used in obtaining such a result are $K = 0.13\,\text{W/cm} \cdot \text{K}$, $L = 0.5$ cm, $H = 1.0\,\text{W/cm}^2 \cdot \text{K}$, $K_{\text{Nd : YAG}} = 0.13\,\text{W/cm} \cdot \text{K}$. Although this correction seems to be negligible for 3 W pump power and 100 $\mu$m spot size, one should notice that it will take a considerable amount when one employs a high power pump source and an optimized spot size as far as the Rayleigh range remains much greater than the crystal length.

## References

[1] H Nadgaran and P Elahi, The analysis of temperature distribution in an end-pumped solid-state laser for spectroscopic applications, *Proc. 18th Colloquium on High Resolution Molecular Spectroscopy* edited by H Berger (Dijon, France, September, 2003) p. L15

[2] R S Abbott and P J King, *Rev. Sci. Instrum.* **72**, 1346 (2001)

[3] A Diening, P E A Mobert and G Huber, *J. Appl. Phys.* **84**, 5900 (1998)

[4] K P Petrov, R F Curl and F K Tittel, *Appl. Phys.* **B66**, 531 (1998)

[5] U O Farrukh, A M Buoncristiani and C E Byvik, *IEEE J. Quantum Electron.* **24**, 2253 (1988)

[6] S C Tidwell *et al*, *IEEE J. Quantum Electron.* **34**, 366 (1998)

[7] M P MacDonald, Th Graf, J E Balmer and H P Weber, *Optics Commun.* **178**, 383 (2000)

[8] W Koechner, *Solid state laser engineering*, 5th edn (Springer-Verlag, Berlin, Heidelberg, 1999) p. 408

[9] M Abramawitz and I A Stegun (eds), *Handbook of mathamatical functions* (Dover, New York, 1965) p. 227