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The superposition method in seeking the solitary wave solutions to the KdV–Burgers equation

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Abstract. In this paper, starting from the careful analysis on the characteristics of the Burgers equation and the KdV equation as well as the KdV–Burgers equation, the superposition method is put forward for constructing the solitary wave solutions of the KdV–Burgers equation from those of the Burgers equation and the KdV equation. The solitary wave solutions for the KdV–Burgers equation are presented successfully by means of this method.

Keywords. KdV–Burgers equation; superposition method; solitary wave solution.

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1. Introduction

As more and more problems in the branches of modern mathematical physics and other interdisciplinary sciences are described in terms of suitable nonlinear models, directly exploring explicit and exact solutions (in particular, the solitary wave solutions) of nonlinear partial differential equations (NPDEs for short) plays a very important role in the nonlinear science, especially in nonlinear physics science. In recent years, quite a few simple and direct approaches have been developed to seek the explicit and exact solutions (particularly the solitary wave solutions) of NPDEs. Among these are the function transformation method [1,2], the homogeneous balance method [3,4], the hyperbolic tangent function expansion method [5,6], the trial function method [7,8], the auxiliary ordinary differential equation [9,10], the sine–cosine method [11], the Jacobi elliptic function expansion method [12,13], and so on. Unfortunately, not all these approaches are universally applicable for solving all kinds of NPDEs directly. As a result, it is still a very significant task to search for various powerful and efficient approaches to solve NPDEs.

In this paper, by carefully analyzing the characteristics of the Burgers equation and KdV equation as well as KdV–Burgers equation, we present a superposition method which is generally acknowledged to simply work for linear equations to construct the solitary wave solutions to the KdV–Burgers equation from those of the Burgers equation and KdV equation and apply it to finding successfully the solitary wave solutions of the KdV–Burgers equation.

2. The superposition approach for solving the KdV–Burgers equation

The well-known Burgers equation and KdV equation as well as KdV–Burgers equation respectively are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \alpha\frac{\partial^2 u}{\partial x^2} = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} = 0, \tag{2}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \alpha\frac{\partial^2 u}{\partial x^2} + \frac{\partial^3 u}{\partial x^3} = 0, \tag{3}$$

where α and β are arbitrary constants with $\alpha\beta \neq 0$.

The above three equations are probably the most popular nonlinear evolution equations of physical interest, which not only arise from realistic physical phenomena, but can also be widely used in many physically significant fields such as plasma physics, fluid dynamics, crystal lattice theory, nonlinear circuit theory and astrophysics [14–18]. In the present paper, our primary interest is to investigate the solitary wave solutions of the KdV–Burgers equation which occurs in many different physical contexts as a nonlinear model equation incorporating the effects of dispersion and dissipation as well as nonlinearity and which was applied by Liu [18,19] to model the inverse energy cascade and intermittent turbulence where a dispersion effect is taken into consideration by utilizing the superposition method. To start with, let us carefully analyze the characteristics of the Burgers equation and the KdV equation as well as the KdV–Burgers equation. From eqs (1)–(3), it is not difficult to observe that they are all of the same nonlinear term $u(\partial u/\partial x)$ and of distinct linear terms and that the linear terms of eq. (3) $(-\alpha(\partial^2 u/\partial x^2)+\beta(\partial^3 u/\partial x^3))$ is exactly equal to the sum of those of eqs (1) and (2). For this reason, we may construct the exact solutions of eq. (3) through the linear superposition of the solutions to eqs (1) and (2), that is to say, we can presume that eq. (3) has the following formal solution:

$$u = au_{\mathrm{B}} + bu_{\mathrm{K}} + c, \tag{4}$$

where a, b and c are constants to be determined later, and u_{B} is the solution of the Burgers equation (1), and u_{K} the solution of the KdV equation (2). We call eq. (4) as the linear superposition formula to find the solutions of the KdV–Burgers equation (3). In what follows, let us look for the solitary wave solutions to eq. (3) according to the superposition formula (4).

From ref. [20], we are told that the Burgers equation (1) has the solution of the following form

$$u_{\mathrm{B}} = -\alpha k \left(1 + \tanh \frac{1}{2}\zeta\right) \tag{5}$$

and the KdV equation (2) has the solution of the following form

$$u_{\mathrm{K}} = 12\beta k^2 \mathrm{sech}^2\zeta, \tag{6}$$

where

$$\zeta = kx - \omega t \tag{7}$$

in which k and ω are the wave number and the angular frequency, respectively.

In view of eqs (5)–(7), and considering the superposition formula (4), we suppose that the KdV–Burgers equation (3) has the following formal solution

$$u = a(1 + \tanh d\xi) + b \sec h^2 d\xi + c, \tag{8}$$

where d is a constant to be determined later. Here it should be noted that we have inserted a controlling parameter d in eq. (8) when utilizing the superposition formula (4) in that the solutions to the Burgers equation (1) and the KdV equation (2) may admit different k and ω.

Substituting eq. (8) into eq. (3), and with the aid of the computerized symbolic computation of the powerful *Mathematica*, we obtain

$$\begin{aligned}
&a^2dk + abdk + acdk + 2bd^2k^2\alpha - 2ad^3k^3\beta - ad\omega \\
&+(a^2dk - 2abdk - 2bcdk - 2b^2dk + 2ad^2k^2\alpha + 16bd^3k^3\beta + 2bd\omega)\tanh d\xi \\
&+(-a^2dk - 4abdk - acdk - 8bd^2k^2\alpha + 8ad^3k^3\beta + ad\omega)\tanh^2 d\xi \\
&+(-a^2dk + 2abdk + 2bcdk + 4b^2dk \\
&-2ad^2k^2\alpha - 40bd^3k^3\beta - 2bd\omega)\tanh^3 d\xi \\
&+(3abdk + 6bd^2k^2\alpha - 6ad^3k^3\beta)\tanh^4 d\xi \\
&+(-2b^2dk + 24bd^3k^3\beta)\tanh^5 d\xi = 0.
\end{aligned} \tag{9}$$

Setting the coefficients of $\tanh^j d\zeta$ $(j = 0, 1, 2, \ldots, 5)$ in eq. (9) to zero gives rise to a set of over-determined algebraic equations with respect to the unknown variables a, b, c and d as follows:

$$a^2dk + abdk + acdk + 2bd^2k^2\alpha - 2ad^3k^3\beta - ad\omega = 0, \tag{10}$$

$$a^2dk - 2abdk - 2bcdk - 2b^2dk + 2ad^2k^2\alpha + 16bd^3k^3\beta + 2bd\omega = 0, \tag{11}$$

$$-a^2dk - 4abdk - acdk - 8bd^2k^2\alpha + 8ad^3k^3\beta + ad\omega = 0, \tag{12}$$

$$-a^2dk + 2abdk + 2bcdk + 4b^2dk - 2ad^2k^2\alpha - 40bd^3k^3\beta - 2bd\omega = 0, \tag{13}$$

$$3abdk + 6bd^2k^2\alpha - 6ad^3k^3\beta = 0, \tag{14}$$

$$-2b^2dk + 24bd^3k^3\beta = 0. \tag{15}$$

Solving the above system of algebraic equations by using *Mathematica*, we have the following results:

$$a = \pm\frac{6\alpha^2}{25\beta}, \quad b = \frac{3\alpha^2}{25\beta}, \quad c = \frac{\omega}{k} \mp \frac{6\alpha^2}{25\beta}, \quad d = \mp\frac{\alpha}{10k\beta}, \tag{16}$$

where k and ω are arbitrary constants.

Inserting eq. (16) into eq. (8) and taking into account eq. (7), we get the solitary wave solution to the KdV–Burgers equation (3) as follows:

$$u = \frac{\omega}{k} + \frac{3\alpha^2}{25\beta} - \frac{6\alpha^2}{25\beta}\tanh\left(\frac{\alpha}{10k\beta}\zeta\right) - \frac{3\alpha^2}{25\beta}\tanh^2\left(\frac{\alpha}{10k\beta}\zeta\right). \tag{17}$$

With the help of the following two equalities:

$$\tanh\frac{x}{2} = \frac{\sinh x}{1+\cosh x} \tag{18}$$

and

$$\sinh^2 x = \cosh^2 x - 1, \tag{19}$$

eq. (17) can be rewritten as

$$u = \frac{\omega}{k} - \frac{6\alpha^2}{25\beta}\ \frac{\sinh\left(\frac{\alpha}{5k\beta}\zeta\right)}{1+\cosh\left(\frac{\alpha}{5k\beta}\zeta\right)} + \frac{6\alpha^2}{25\beta}\ \frac{1}{1+\cosh\left(\frac{\alpha}{5k\beta}\zeta\right)}. \tag{20}$$

From ref. [21], we can see that the Burgers equation (1) admits the following solution:

$$u_{\mathrm{B}} = -\alpha k\left(1+\coth\frac{1}{2}\zeta\right) \tag{21}$$

and the KdV equation (2) admits the following solution

$$u_{\mathrm{K}} = -12\beta k^2 csch^2\zeta. \tag{22}$$

According to the superposition formula (4) and using the same procedure as above, we may find the following singular traveling solutions for the KdV–Burgers equation (3) as

$$u = \frac{\omega}{k} + \frac{3\alpha^2}{25\beta} - \frac{6\alpha^2}{25\beta}\coth\left(\frac{\alpha}{10k\beta}\zeta\right) - \frac{3\alpha^2}{25\beta}\coth^2\left(\frac{\alpha}{10k\beta}\zeta\right). \tag{23}$$

Making use of the identity (19) and the following identity

$$\coth\frac{x}{2} = \frac{\sinh x}{\cosh x - 1}, \tag{24}$$

eq. (23) can be rewritten as

$$u = \frac{\omega}{k} - \frac{6\alpha^2}{25\beta}\frac{\sinh\left(\frac{\alpha}{5k\beta}\zeta\right)}{\cosh\left(\frac{\alpha}{5k\beta}\zeta\right)-1} - \frac{6\alpha^2}{25\beta}\frac{1}{\coth\left(\frac{\alpha}{5k\beta}\zeta\right)-1}. \tag{25}$$

Obviously, the solutions (17) and (23) are equivalent or similar to those given in refs [22–24], and the solutions (20) and (25) are also analogous to those obtained in ref. [25].

3. Conclusions

By analyzing carefully the characteristics of the Burgers equation and the KdV equation as well as the KdV–Burgers equation, we bring forward a superposition method to construct the solitary wave solution to the KdV–Burgers equation from those of the Burgers equation and the KdV equation.

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