

Relativistic compact objects in isotropic coordinates

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Abstract. We present a matrix method for obtaining new classes of exact solutions for Einstein's equations representing static perfect fluid spheres. By means of a matrix transformation, we reduce Einstein's equations to two independent Riccati-type differential equations for which three classes of solutions are obtained. One class of the solutions corresponding to the linear barotropic-type fluid with an equation of state $p = \gamma\rho$ is discussed in detail.

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1. Introduction

Relativistic stellar models have been studied ever since the first solution of Einstein's field equation was obtained by Schwarzschild for the interior of a compact object in hydrostatic equilibrium. The search for the exact solutions is of continuous interest to physicists. The most general exact solution for Vaidya-Tikekar isentropic superdense star was obtained by Gupta and Jasim [1]. Hernandez and Nunez [2] presented a general method for obtaining static anisotropic spherically symmetric solutions satisfying a nonlocal equation of state from known density profiles. Several new exact solutions for anisotropic stars of constant density were presented by Dev and Gleiser [3]. In [4] an algorithm was proposed to generate any number of physically realistic pressure and density profiles for spherically perfect fluid distributions without evaluating integrals. The gravitational field equations for static stellar models with a linear barotropic equation of state and with a polytropic equation of state $p = k\rho^{1+1/n}$ were recast respectively into two complementary three-dimensional regular system of ordinary differential equations on compact state space [5,6]. These systems were analyzed numerically and qualitatively, using the theory of dynamical systems. Schmidt and Homann [7] discussed numerical

solutions of Einstein's field equation describing static spherically symmetric conglomerations of a photon star with an equation of state $\rho = 3p$. Upper limits for the mass-radius ratio were derived for compact general relativistic objects in the presence of a cosmological constant in [8] and for the charged general relativistic fluid spheres in [9]. The effect of a cosmological constant on the structure of a general relativistic object was analyzed by Boehmer [10]. Interior solutions for charged and neutral anisotropic fluid spheres, satisfying all the required physical conditions, were obtained in [11] and [12]. Excellent reviews of the relativistic static fluid spheres were given in Finch and Skea [13], Herrera and Santos [14] and Delgaty and Lake [15].

For a static fluid configuration, Einstein's equations represent an under-determined system of nonlinear ordinary differential equations of the second order. For the special case of a static isotropic perfect fluid, the field equations of Einstein's theory can be reduced to a set of three coupled ordinary differential equations in four unknowns. To obtain a realistic stellar model, one can start with an equation of state. Such input equations of state do not normally allow for closed form solutions. In arriving to exact solutions, one can solve the field equations by making an *ad hoc* assumption for one of the metric functions or for the energy density. Hence the equation of state can be computed from the resulting metric.

The purpose of the present paper is to present a matrix method for obtaining new classes of exact solutions for Einstein's equations representing static perfect fluid spheres. By means of a matrix transformation, we reduce Einstein's equations to two independent Riccati's differential equations for which three classes of solutions are obtained. One class of the solutions corresponding to the linear barotropic-type fluid with an equation of state $p = \gamma\rho$ is discussed in detail.

The present paper is organized as follows. In §2 we reduce the basic equation describing the interior stellar evolution to a matrix equation which is separated into two independent Riccati's differential equations. Three classes of solutions of the gravitational field equations are presented and the stiff solution is discussed in §3. We conclude our results in §4.

2. Field equations and the matrix method

For a relativistic fluid sphere model the line element inside the matter distribution takes the form

$$ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} (dr^2 + r^2 d\Omega^2), \quad (1)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$.

Assuming that the matter content inside the spheres is a perfect fluid, the field equations take the form [16]

$$8\pi p = e^{-\lambda} \left[\frac{(\lambda')^2}{4} + \frac{\lambda'\nu'}{2} + \frac{\lambda' + \nu'}{r} \right], \quad (2)$$

$$8\pi p = e^{-\lambda} \left[\frac{\lambda''}{2} + \frac{\nu''}{2} + \frac{(\nu')^2}{4} + \frac{\lambda' + \nu'}{2r} \right], \quad (3)$$

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$$8\pi\rho = -e^{-\lambda} \left[\lambda'' + \frac{(\lambda'')^2}{4} + \frac{2\lambda'}{r} \right], \quad (4)$$

where a prime denotes d/dr .

By introducing the transformation [17]

$$R = \ln r, \quad (5)$$

eqs (2)–(4) become

$$8\pi p = \frac{e^{-\lambda}}{r^2} \left[\frac{(\dot{\lambda})^2}{4} + \frac{\dot{\lambda}\dot{\nu}}{2} + \dot{\lambda} + \dot{\nu} \right], \quad (6)$$

$$8\pi p = \frac{e^{-\lambda}}{r^2} \left[\frac{\ddot{\lambda} + \ddot{\nu}}{2} + \frac{(\dot{\nu})^2}{4} \right], \quad (7)$$

$$8\pi\rho = -\frac{e^{-\lambda}}{r^2} \left[\ddot{\lambda} + \frac{(\dot{\lambda})^2}{4} + \dot{\lambda} \right], \quad (8)$$

where the dot denotes d/dR . The set of coordinates resulting after the re-scaling of the radial coordinate r by means of the transformation (5) are called isotropic coordinates.

From the isotropic pressure condition, we obtain the following basic equation describing the structure of general relativistic stellar-type objects in isotropic coordinates:

$$\ddot{\lambda} + \ddot{\nu} + \frac{1}{2}[(\dot{\nu})^2 - (\dot{\lambda})^2] - \dot{\lambda}\dot{\nu} - 2(\dot{\nu} + \dot{\lambda}) = 0. \quad (9)$$

By means of the following transformations

$$\lambda = \int u \, dR, \quad \nu = \int w \, dR, \quad (10)$$

and with the use of the two matrices

$$L = [(2\dot{u} - 4u) + (2\dot{w} - 4w)], \quad (11)$$

eq. (9) can be written as a matrix equation in the form

$$L = M^T A M. \quad (12)$$

We introduce now two new variables u_- and u_+ which are obtained by means of a linear transformation, described by the matrix K and applied to the matrix N ,

$$M = KN = \frac{1}{\sqrt{2}} \begin{pmatrix} (1 - \sqrt{2}) m_-^{-1} & (1 + \sqrt{2}) m_+^{-1} \\ m_-^{-1} & m_+^{-1} \end{pmatrix} \begin{pmatrix} u_- \\ u_+ \end{pmatrix}. \quad (13)$$

In the new variables, eq. (12) reduces to the form $L = N^T K^T A K N$. We choose the elements of the matrix K so that

$$K^T AK = \begin{pmatrix} -\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}.$$

As a result of these transformations the right-hand side of the matrix equation (eq. (12)) can be diagonalized, and the resulting equations satisfied by the new unknown functions u_- , u_+ are

$$m_- \frac{du_-}{dR} - 2m_- u_- + u_-^2 = -m_+ \frac{du_+}{dR} + 2m_+ u_+ + u_+^2 = F(R), \quad (14)$$

where $m_{\mp} = \sqrt{2 \mp \sqrt{2}}$ and a solution generating function $F(R)$ is introduced.

3. Three classes of solutions of the static field equations

In the previous section, by means of a matrix transformation, the cross-term $\lambda\nu$ cancelled from the isotropic pressure equation is written in terms of new variables u_- and u_+ . Hence, we have reduced the second-order differential equation (9) to two independent Riccati-type differential equations, for which exact solutions of the field equations describing the stellar structure can be generated by the appropriate choices of the generating function $F(R)$.

Case A

$$F(R) = \beta = \text{constant}.$$

In order to obtain some solutions of the gravitational field equations for our stellar model, we assume the generating function $F(R) = \beta = \text{constant}$. In this case eq. (14) takes the form

$$m_- \frac{du_-}{dR} - 2m_- u_- + u_-^2 = \beta, \quad -m_+ \frac{du_+}{dR} + 2m_+ u_+ + u_+^2 = \beta. \quad (15)$$

Depending on the sign of the parameter β , we obtain two distinct classes of solutions of eq. (15):

(i) $\Delta_{\mp} > 0$

$$u_{\mp} = \frac{\sqrt{\Delta_{\mp}} + 2\varepsilon_{\pm} m_{\mp} + (\sqrt{\Delta_{\mp}} - 2\varepsilon_{\pm} m_{\mp}) \exp\left[\frac{\sqrt{\Delta_{\mp}}}{\varepsilon_{\pm} m_{\mp}} (R_{\mp} - R)\right]}{2 \left[1 - \exp\left[\frac{\sqrt{\Delta_{\mp}}}{\varepsilon_{\pm} m_{\mp}} (R_{\mp} - R)\right]\right]}. \quad (16)$$

(ii) $\Delta_{\mp} < 0$

$$u_{\mp} = \varepsilon_{\pm} m_{\mp} + \frac{\sqrt{-\Delta_{\mp}}}{2} \tan\left[\frac{\sqrt{-\Delta_{\mp}}}{2\varepsilon_{\pm} m_{\mp}} (R_{\mp} - R)\right], \quad (17)$$

where $\Delta_{\mp} = 4(m_{\mp}^2 + \beta)$, $\varepsilon_{\mp} = \mp 1$, R_{\pm} are arbitrary constants of integration.

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Case B

$$F(R) = -2m_- u_{0-} + u_{0-}^2 = 2m_+ u_{0+} + u_{0+}^2.$$

Assuming that the solutions of eq. (14) are constants, i.e., $u_{\mp} = u_{0\mp} = \text{constant}$, we can obtain another solution of the field equations for the static stellar model with a linear barotropic equation of state.

By using eq. (11), eq. (10) becomes

$$\lambda = \frac{1}{\sqrt{2}} \int \left[(1 - \sqrt{2}) \frac{u_-}{m_-} + (1 + \sqrt{2}) \frac{u_+}{m_+} \right] dR, \quad (18)$$

$$\nu = \frac{1}{\sqrt{2}} \int \left(\frac{u_-}{m_-} + \frac{u_+}{m_+} \right) dR. \quad (19)$$

The explicit forms of the metric functions e^λ and e^ν representing the relativistic fluid spheres for three distinct classes of solutions are given by the following cases.

Case A(i)

$$e^\lambda = \lambda_0 r^{\frac{1-\sqrt{2}}{2\sqrt{2}} \left(\frac{\sqrt{\Delta_-}}{m_-} + 2 \right) + \frac{1+\sqrt{2}}{2\sqrt{2}} \left(\frac{\sqrt{\Delta_+}}{m_+} - 2 \right)} \left[e^{\frac{\sqrt{\Delta_-}}{m_-} (R_- - \ln r)} - 1 \right]^{\frac{1-\sqrt{2}}{\sqrt{2}}} \\ \times \left[e^{-\frac{\sqrt{\Delta_+}}{m_+} (R_+ - \ln r)} - 1 \right]^{-\frac{1+\sqrt{2}}{\sqrt{2}}}, \quad (20)$$

$$e^\nu = \nu_0 r^{\frac{1}{2\sqrt{2}} \left(\frac{\sqrt{\Delta_+}}{m_+} + \frac{\sqrt{\Delta_-}}{m_-} \right)} \left[e^{\frac{\sqrt{\Delta_-}}{m_-} (R_- - \ln r)} - 1 \right]^{\frac{1}{\sqrt{2}}} \\ \times \left[e^{-\frac{\sqrt{\Delta_+}}{m_+} (R_+ - \ln r)} - 1 \right]^{-\frac{1}{\sqrt{2}}}. \quad (21)$$

Case A(ii)

$$e^\lambda = \lambda_0 r^{-2} \cos^{\frac{1-\sqrt{2}}{\sqrt{2}}} \left[\frac{\sqrt{-\Delta_-}}{2m_-} (R_- - \ln r) \right] \\ \times \cos^{-\frac{1-\sqrt{2}}{\sqrt{2}}} \left[-\frac{\sqrt{-\Delta_+}}{2m_+} (R_+ - \ln r) \right], \quad (22)$$

$$e^\nu = \nu_0 \cos^{\frac{1}{\sqrt{2}}} \left[\frac{\sqrt{-\Delta_-}}{2m_-} (R_- - \ln r) \right] \cos^{-\frac{1}{\sqrt{2}}} \left[-\frac{\sqrt{-\Delta_+}}{2m_+} (R_+ - \ln r) \right]. \quad (23)$$

Case B

$$e^\lambda = \lambda_0 r^{\frac{1}{\sqrt{2}}[(1-\sqrt{2})a+(1+\sqrt{2})b]}, \quad (24)$$

$$e^\nu = \nu_0 r^{\frac{1}{\sqrt{2}}(a+b)}, \quad (25)$$

where we have denoted $a = u_{0-} m_-^{-1}$ and $b = u_{0+} m_+^{-1}$.

For Case B, the physical quantities, the energy density ρ and the fluid pressure p are expressed in the following form:

$$8\pi\rho = \lambda_0^{-1} \left\{ \left(\frac{5}{8} - \frac{1}{\sqrt{2}} \right) a^2 + \left(-1 + \sqrt{2} + \frac{b}{4} \right) a + \left[1 + \sqrt{2} + \left(\frac{5}{8} + \frac{1}{\sqrt{2}} \right) b \right] b \right\} r^{-\left\{ \frac{1}{\sqrt{2}}[(1-\sqrt{2})a+(1+\sqrt{2})b] \right\} - 2}, \quad (26)$$

$$p = \frac{-8a + 5a^2 + 8b + 2ab + 5b^2 - 4\sqrt{2}(a-b-2)(a+b)}{8a - 3a^2 - 8b + 2ab - 3b^2 + 2\sqrt{2}(a-b-2)(a+b)} \rho = \gamma\rho, \quad (27)$$

where $0 \leq \gamma \leq 1$.

Therefore we have obtained an exact solution of the field equations for the perfect fluid sphere in isotropic coordinate with a linear barotropic type equation of state.

By choosing the parameters $u_{0+} = 0, u_{0-} = 2m_-$ or $u_{0-} = 0, u_{0+} = -2m_+$, the metric functions given by eqs (24) and (25) generate the line element

$$ds^2 = r^{\pm\sqrt{2}} dt^2 - r^{\pm\sqrt{2}-2} (dr^2 + r^2 d\Omega^2), \quad (28)$$

with the equation of state $p = \rho$ (stiff or Zeldovich equation of state) for the dense matter.

It can be shown that the + and - signs are equivalent through the transformation $r \rightarrow r^{-1}$. Therefore, by using the matrix method we have rediscovered this exact solution for the stiff fluid matter that was obtained by Haggag and Hajj-Boutros [17].

To overcome the difficulty of infinite energy density at the center of the sphere, it is assumed that the matter distribution has a core of radius r_0 and constant density ρ_0 which is surrounded by the fluid with pressure equal to energy density. In view of eqs (24) and (25), by choosing $b = a$ we obtain $e^\lambda \sim e^\nu r^{\frac{1}{\sqrt{2}}(b-a)}$, or equivalently, $u_{0\mp} = -2\sqrt{4 \mp 2\sqrt{2}}$. Therefore the interior metric is conformally flat satisfying the zero condition of the Weyl tensor.

By means of the transformation $r^{\frac{a}{\sqrt{2}}+1} \rightarrow r$, the conformally flat metric becomes

$$ds^2 = r^{\frac{2a}{a+\sqrt{2}}} dt^2 - \frac{2}{(a+\sqrt{2})^2} dr^2 - r^2 d\Omega^2. \quad (29)$$

In Schwarzschild coordinates Tolman [18] presented this solution over sixty years ago. However, Gurses and Gursej [19] proved that the Schwarzschild interior metric for a non-charged sphere is the *unique* solution for the Einstein's gravitational field equations which is static and conformally flat.

Surprisingly, the line element (29) contradicts the results suggested by Gurses and Gursej [19].

4. Discussions and final remarks

In the present paper we have studied general relativistic stellar models described by the line element (1), and have presented two new classes of solutions, given by eqs (20)–(23), of the field equations representing the interior structure of the compact objects in the framework of general relativity. The class of the solution corresponding to the stiff fluid has been discussed in detail. From the physical point of view, the mathematical solutions must satisfy certain physical requirements to render them physically meaningful. The following conditions or requirements have been generally accepted [14].

- (a) The energy density ρ and the pressure p should be positive and finite.
- (b) Their gradients $d\rho/dr$ and dp/dr should be negative.
- (c) The speed of sound should be less than the speed of light.
- (d) A physically reasonable energy–momentum tensor must obey a trace condition, $\rho \geq 3p$.
- (e) The interior metric should be joined continuously with the exterior Schwarzschild metric.
- (f) The pressure p should vanish at the vacuum boundary of the sphere.
- (g) The structures should be stable under radial pulsations.

Not many exact solutions for relativistic fluid spheres are known in isotropic coordinates. Hence, the main purpose of this paper is to present the matrix method for obtaining the exact solutions of the field equations for the compact objects in isotropic coordinates. Equation (14) is a first-order differential equation in two unknowns u_- and u_+ . To obtain its solutions a solution generating function $F(R)$ has been introduced in eq. (14). Because of the mathematical structure of the master eq. (14), hosts of solutions can be found by the appropriate choice of the generating function $F(R)$ or by making an *ad-hoc* assumption for one of the variables u_- or u_+ .

In order to have physically realistic stellar models, one needs to test the solutions obtained via this method to ensure that they satisfy all the physical conditions (a)–(g).

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