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Supersymmetric quantum mechanics for two-dimensional disk

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Abstract. The infinite square well potential in one dimension has a smooth supersymmetric partner potential which is shape invariant. In this paper, we study the generalization of this to two dimensions by constructing the supersymmetric partner of the disk billiard. We find that the property of shape invariance is lost in this case. Nevertheless, the WKB results are significantly improved when SWKB calculations are performed with the square of the superpotential. We also study the effect of inserting a singular flux line through the center of the disk.

Keywords. Supersymmetric quantum mechanics; SWKB, disk billiard; flux lines.

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1. Introduction

During the last three decades, the algebra of supersymmetry has been profitably applied to many non-relativistic quantum mechanical problems [1–8]. In supersymmetric quantum mechanics (SUSYQM), one gets a better understanding of why certain potentials are analytically solvable. All familiar solvable potentials have the property of shape invariance [2,5] which is realized through a relationship between the two supersymmetric partner potentials with a change of parameters related by translation. These (one-dimensional) potentials have the property that their exact eigenvalue spectra may be expressed algebraically in terms of one quantum number. An entirely new class of shape invariant potentials has also been found subsequently in which the parameters are related by scaling [8,10]. These are not discussed in the present context. SUSYQM also generate a large class of isospectral potentials, many of which are reflectionless and turn out to be the multi-solitonic solutions of the nonlinear KdV evolution equation at $t = 0$ [8,11].

For approximate quantum calculations, SUSYQM provides a modification of the semi-classical WKB quantization condition. Supersymmetry-inspired WKB quantization conditions (SWKB) [12–14] reveal several interesting features. Unlike the standard WKB method [9], the leading order SWKB formula yields exact analytic expressions for the energy eigenvalues of all the known shape-invariant potentials for which the change of parameters are of translational type, the higher order correction terms vanishing identically [13,14]. Interestingly, one does not need to invoke the Langer-like modification, such as the replacement of $l(l+1)$ by $(l+1/2)^2$ for spherically symmetric potentials, as it appears naturally in SUSYQM [9,14]. For non-exactly solvable cases, SWKB theory has so far predicted improved energy eigenvalues from the leading order as well as sub-leading orders (in powers of $\hbar$), as compared to those obtained from the corresponding WKB calculations [11,14].

At this point, it may be natural for one to ask whether shape invariance and the exactness of the SWKB quantization formulae remain intact if the potential under study is generalized to higher dimensions. To be more specific, it may be worthwhile to study the supersymmetric structure of the two-dimensional quantum disk billiard, of which the one-dimensional version, the infinite square well potential, has been found to be shape invariant. In two dimensions, the disk billiard does not have a closed algebraic expression for the energy eigenvalues, but these are obtainable from the zeros of the cylindrical Bessel functions for each angular momentum quantum number l. Because of this lack of algebraic solvability, we do not expect the property of shape invariance to persist in the disk. Nevertheless, it is interesting to construct the partner potentials and perform the SWKB calculations in this case, since the usual WKB quantization does not reproduce correctly the quantum-mechanical energy spectrum for the disk [9].

In this paper, we study in some detail the supersymmetric structure associated with the 2D disk (§2). In particular, we examine the supersymmetric partners for each partial wave. These look quite different from the disk, and are not shape invariant. The SWKB lowest-order results, however, are significantly better than the corresponding WKB ones. We also show that the insertion of a semionic flux line through the center of the disk regroups the degeneracies of the partial waves, and restores the shape invariance in the $l=(0,1)$ pair, but not for the others. The disk has been studied in detail from the point of view of the periodic orbit theory, without and with a flux line [17,18]. Unfortunately, because the superpotential is different in each partial wave, we are unable to make a connection of SUSYQM to periodic orbit theory, as may be possible in one dimension [13].

2. SUSYQM for disk

2.1 *Supersymmetric partner potentials*

Consider a particle of mass m confined to a two-dimensional circular domain of radius R with infinitely steep wall as represented by the potential

$$\begin{aligned} V(r) &= 0, \qquad r \le R\,, \\ &= \infty, \qquad r > R\,. \end{aligned} \tag{1}$$

Classically, the particle is reflected at the boundary of the circular billiard. For quantum motion, we consider the two-dimensional stationary Schrödinger equation

$$-\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right]\psi(r,\phi)+V(r)\psi(r,\phi)=E\psi(r,\phi)\ . \tag{2}$$

By substituting

$$\psi(r,\phi)=\sum_{l=-\infty}^{\infty}\frac{u_l(r)}{\sqrt{r}}\mathrm{e}^{il\phi} \tag{3}$$

in eq. (2), we obtain the radial equation

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2u_l(r)}{\mathrm{d}r^2}+\left[\frac{(l^2-1/4)\hbar^2}{2mr^2}+V(r)\right]u_l(r)=E_lu_l(r)\ , \tag{4}$$

in which $k_l=\sqrt{2mE_l/\hbar^2}$ is the wave number. This is a one-dimensional equation for $u_l(r)$ in a fixed partial wave l. Substitution of the disk potential (1), and imposing the boundary condition that the wave function must vanish at $r=R$ discretizes the eigenvalues, which are given by the zeros of the cylindrical Bessel function, $J_{|l|}(k_{l,n}R)=0$. We have introduced an additional quantum number $n=1,2,...$ to specify, for a given l, which zero of the Bessel function is matched at the boundary. It is the nodal quantum number, analogous to the one-dimensional case. The eigenenergies of the disk are given by

$$k_{l,n}=\frac{z_{|l|,n}}{R}\qquad (n=1,\ 2,\ \ldots) \tag{5}$$

$$E_{l,n}=\frac{(\hbar k_{l,n})^2}{2m}=\frac{\hbar^2}{2mR^2}z^2_{|l|,n}. \tag{6}$$

The corresponding eigenfunctions are given by

$$\begin{aligned}u^{(1)}_{l,n}(r)&=N_{l,n}\sqrt{k_{l,n}r}\ J_{|l|}(k_{l,n}r)\\&=N_{l,n}\sqrt{z_{|l|,n}x}\ J_{|l|}(z_{|l|,n}x)\ ,\end{aligned} \tag{7}$$

where $N_{l,n}$ is a normalization constant and $x=r/R$. The superpotential, $W_l(r)$ is given by

$$W_l(r)=-\frac{\hbar}{\sqrt{2m}}\frac{1}{u^{(1)}_{l,1}(r)}\frac{\mathrm{d}u^{(1)}_{l,1}(r)}{\mathrm{d}r}\ , \tag{8}$$

$$=\frac{\hbar}{\sqrt{2m}R}\left[z_{|l|,1}\frac{J_{|l|+1}(z_{|l|,1}x)}{J_{|l|}(z_{|l|,1}x)}-\frac{|l|+1/2}{x}\right]\ . \tag{9}$$

We plot W_0^2 and W_1^2 in figure 1 as a function of x. Note that $W_l^2(x)$ is not symmetric about its center, and that it is tangential to $E=0$. We may now construct the partner potentials $V_l^{(1)},V_l^{(2)}$ as

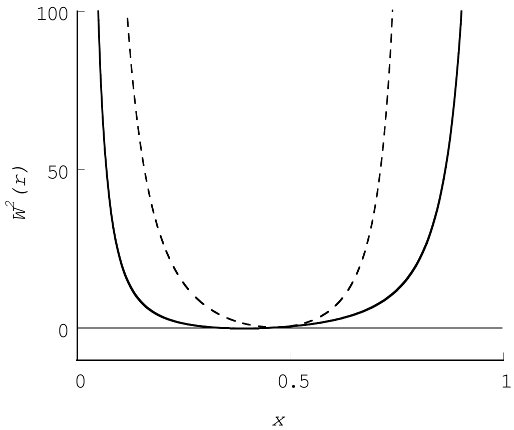

Figure 1. The squared superpotentials $W_l^2(r)$ for $l = 0$ (solid curve) and $l = 1$ (dashed curve).

$$V_l^{(1)}(r) = W_l^2(r) - \frac{\hbar}{\sqrt{2m}} \frac{\mathrm{d}W_l(r)}{\mathrm{d}r}, \tag{10}$$

$$V_l^{(2)}(r) = W_l^2(r) + \frac{\hbar}{\sqrt{2m}} \frac{\mathrm{d}W_l(r)}{\mathrm{d}r}. \tag{11}$$

Whereas $V_l^{(1)}$ reduces to just the disk potential with the inclusion of the centrifugal barrier, and an energy shift that yields a zero-energy ground state

$$V_l^{(1)}(r) = \begin{cases} \dfrac{\hbar^2}{2mR^2}\left[\dfrac{l^2 - 1/4}{x^2} - z_{|l|,1}^2\right], & \text{for} \quad r \leq R \\ \infty & \text{for} \quad r > R \end{cases} \tag{12}$$

$$E_{l,n}^{(1)} = E_{l,n} - E_{l,1}, \tag{13}$$

the expression for $V_l^{(2)}$ is more complicated.

$$\begin{aligned} V_l^{(2)}(r) = \frac{\hbar^2}{2mR^2} \Bigg[& 2\left\{ z_{|l|,1} \frac{J_{|l|+1}(z_{|l|,1}x)}{J_{|l|}(z_{|l|,1}x)} \right\}^2 \\ & -4z_{|l|,1} \frac{J_{|l|+1}(z_{|l|,1}x)}{J_{|l|}(z_{|l|,1}x)} \frac{|l| + 1/2}{x} \\ & + \frac{(|l| + 1/2)(|l| + 3/2)}{x^2} + z_{|l|,1}^2 \Bigg]. \end{aligned} \tag{14}$$

Note that the hard-disk nature is incorporated in $V_l^{(2)}$ through the singularity arising from the zero of $J_{|l|}(z_{|l|,1}x)$ at $x = 1$. The eigenfunctions of $V_l^{(2)}$ may be analytically written as

$$u_{l,n}^{(2)}(r) = N_{l,n+1}\sqrt{\frac{z_{|l|,n+1}x}{z_{|l|,n+1}^2 - z_{|l|,n}^2}}$$
$$\times \left[z_{|l|,1}\frac{J_{|l|+1}(z_{|l|,1}x)J_{|l|}(z_{|l|,n+1}x)}{J_{|l|}(z_{|l|,1}x)} - z_{|l|,n+1}J_{|l|+1}(z_{|l|,n+1}x) \right]. \tag{15}$$

In figure 2 we display the partner potentials for the $l = 0$ partial wave. The first few eigenenergies (given by eq. (6)) and the corresponding eigenfunctions, obtained analytically, are shown in figure 3. Note that in two dimensions, for $l = 0$, there is an attractive centrifugal potential that drives $V_0^{(1)}$ negative, although its partner

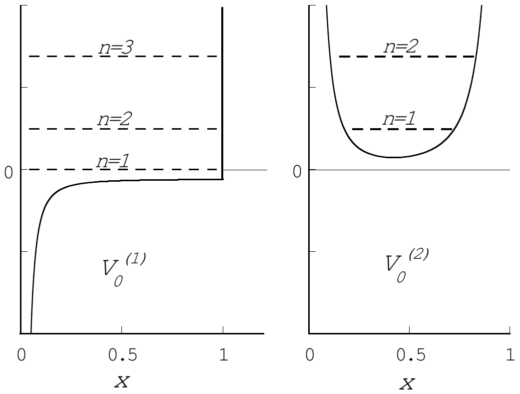

Figure 2. Superpartners with $l = 0$. Corresponding eigenenergies are shown by dashed lines.

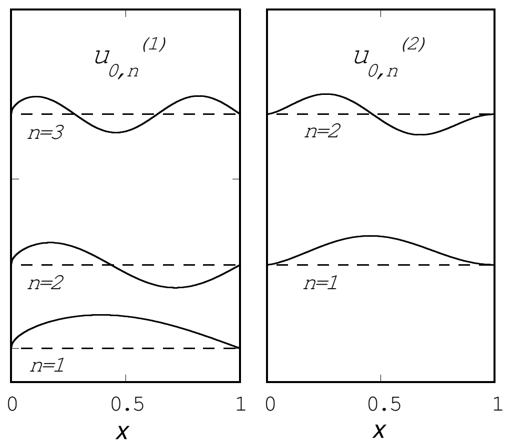

Figure 3. Eigenfunctions for supersystems with $l = 0$. Corresponding eigenenergies are shown by dashed lines.

$V_0^{(2)}$ has no such attraction. The $n = 1$ state of $V_0^{(1)}$ comes at zero energy by construction. Since the wave function $u_{l,n}$ given by eq. (15) looks quite complicated, we also checked numerically that it reproduces the correct eigenvalues. In figures 4 and 5 the partner potentials and their eigenfunctions and eigenenergies for $l = 1$ are similarly shown. Note that now the potential $V_1^{(1)}$ has a repulsive barrier as expected, and its partner is narrower in shape. We see from these diagrams that the superpartners may be very l-dependent.

In order to test whether the partner potentials defined by eqs (10) and (11) preserve the property of shape invariance for the disk problem, we consider the shape invariance condition [8]

$$V_2(r; a_1) = V_1(r, a_2) + R(a_1). \tag{16}$$

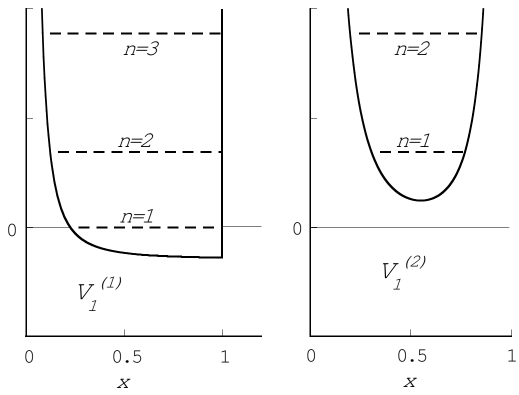

Figure 4. Superpartners with $l = 1$. Corresponding eigenvalues are shown by dashed lines.

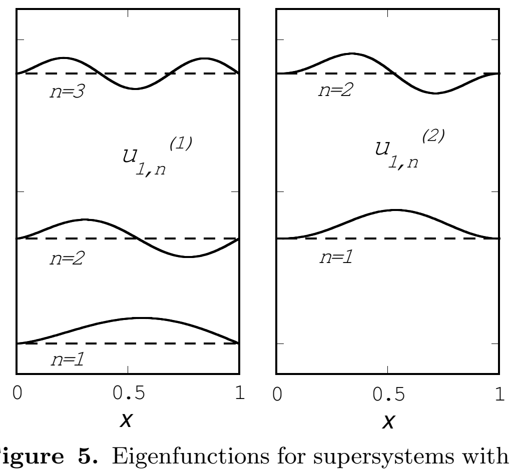

Figure 5. Eigenfunctions for supersystems with $l = 1$. Corresponding eigenvalues are shown by dashed lines.

Here, a_1 is a set of parameters, a_2 is a function of a_1, and the remainder $R(a_1)$ is independent of r. For example, for the one-dimensional square-well, taking $W(x) = -A\cot(\alpha x)$, we have

$$V^{(1)} = A(A-\alpha)\mathrm{cosec}^2(\alpha x) - A^2, \tag{17}$$

$$V^{(2)} = A(A+\alpha)\mathrm{cosec}^2(\alpha x) - A^2. \tag{18}$$

We note that in $V^{(1)}$, by changing $A \to (A+\alpha)$, $\alpha \to \alpha$, we get $V^{(2)}$ plus a remainder R that is independent of x. In the one-dimensional case, therefore, we conclude that the partners are shape invariant. When we implement this test for the disk, we find that even for the $l = 0$ case, we do not obtain shape invariance. Let us rewrite eq. (9) in the form

$$W_l(r) = Ak\frac{J_{l+1}(kr)}{J_l(kr)} - (2l+1)\frac{b}{r} \ , \tag{19}$$

where A and b are constants, and we have suppressed the subscripts for simplicity in writing $z_{|l|,1}x = kr$. Using this superpotential, we may immediately write down the expressions for the superpartners:

$$\begin{aligned} V_l^{(1)} &= A(A-1)k^2\frac{J_{l+1}^2}{J_l^2} - \frac{2Ak}{r}(2b-1)(l+1/2)\frac{J_{l+1}}{J_l} \\ &\quad + \frac{2b}{r^2}(l+1/2)[2b(l+1/2)-1] - Ak^2 \ , \\ V_l^{(2)} &= A(A+1)k^2\frac{J_{l+1}^2}{J_l^2} - \frac{2Ak}{r}(2b+1)(l+1/2)\frac{J_{l+1}}{J_l} \\ &\quad + \frac{2b}{r^2}(l+1/2)[2b(l+1/2)+1] + Ak^2 \ . \end{aligned} \tag{20}$$

We find that the above potentials do not satisfy eq. (16) for any choice of parameters related by translation.

2.2 *SWKB calculation*

Let us first consider the usual WKB calculation for the disk. It is worth pointing out that with the correct Maslov phase factors [16], the lowest order WKB calculation yields exact results for the one-dimensional square-well potential [19]. This, however, is not so for the disk. The leading order WKB quantization formula for the disk is given by [9]

$$(k^2R^2 - l^2)^{1/2} - l\cos^{-1}\left(\frac{l}{kR}\right) = \left(n_r + \frac{3}{4}\right)\pi, \qquad n_r = 0, 1, 2, \dots \ . \tag{21}$$

The Maslov phase factor of 3/4 arises from one smooth and one hard turning point. The SWKB energy eigenvalues for $V^{(1)}$ are given by [12]

Table 1. SWKB results in comparison with the WKB results for $l = 0$ and 1.

n	$\epsilon_{0,n}^{\mathrm{WKB}}$	$\epsilon_{0,n}^{\mathrm{SWKB}}$	$\epsilon_{0,n}^{\mathrm{EXACT}}$	$\epsilon_{1,n}^{\mathrm{WKB}}$	$\epsilon_{1,n}^{\mathrm{SWKB}}$	$\epsilon_{1,n}^{\mathrm{EXACT}}$
1	−0.2315	0.0000	0.0000	−0.2842	0.0000	0.0000
2	24.4425	24.6864	24.6881	34.2760	34.5394	34.5365
3	68.8557	69.1028	69.1038	88.5625	88.8201	88.8175

$$2\sqrt{2m}\int_{r_1}^{r_2} \mathrm{d}r \sqrt{E_{0,n}^{\mathrm{SWKB}} - W_0^2(r)} = 2\pi(n-1)\hbar, \qquad n = 1,\ 2,\ \ldots, \tag{22}$$

where r_1 and r_2 are turning points obtained from

$$E_{0,n}^{\mathrm{SWKB}} - W_0^2(r) = 0. \tag{23}$$

The SWKB quantization condition for $V^{(2)}$ replaces $(n-1)$ by n in the RHS of eq. (22). Even when SWKB is not exact, the SUSY level degeneracy of the partner potentials is preserved. Note from figure 1 that by construction W_l^2 has the lowest eigenvalue at zero energy, where the classical turning points are coincident. In table 1, we present the SWKB results for the $l = 0, 1$ partial waves for the disk. For comparison, the WKB and the exact results are also given. In this table, $\epsilon_{l,n}$ (with the appropriate superscripts) are defined as

$$\frac{\hbar^2}{2m}\epsilon_{l,n} = (E_{l,n} - E_{l,1})\,. \tag{24}$$

The table shows that the largest error in WKB occurs in the ground state. Had we corrected for this error by an energy shift in WKB, the other WKB energies would improve dramatically, but still would not be as good as the SWKB energies. In this respect, the SWKB results for the disk are similar to the much studied one-dimensional potentials [4,7,8,13].

2.3 *Disk with a flux line*

We have seen that in going from the one-dimensional square well to the two-dimensional disk, the property of shape invariance is destroyed. We may ask if shape invariance may be restored by subjecting the particle to a Bohm–Aharonov like gauge field [20]. It is well-known in the literature [21,22] that if one takes the particle to be charged, and couples it to the vector field of a singular flux line, a centrifugal-like term with a coefficient proportional to the flux strength arises in the two-dimensional Schrödinger equation. In two dimensions, it is also legitimate to consider flux lines of fractional strength (in units of $\frac{\hbar c}{|e|}$). A particle with a fractional flux line attached to it is called an anyon. For our one-body problem, the flux line is perpendicular to the plane of the disk, and passes through the center. Its vector potential in polar coordinates is given by

$$A_r = 0, \qquad A_\phi = \frac{\hbar c}{|e|}\,\frac{\alpha}{r}\ , \tag{25}$$

where α is the dimensionless strength of the flux line. The charged particle Hamiltonian in the presence of the vector potential is obtained by changing $\mathbf{p} \rightarrow (\mathbf{p} - \frac{e}{c}\mathbf{A})$. Taking the charge e to be positive, the resulting Hamiltonian of the particle in a potential $V(r)$ is given by

$$H = \frac{p_r^2}{2m} + \frac{\hbar^2}{2mr^2}(L_z - \alpha)^2 + V(r), \tag{26}$$

where $p_r^2 = -\hbar^2(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r})$ and $\hbar L_z$ is the angular momentum operator. The net effect of the flux line is to split the degeneracy of the states with angular momentum quantum numbers l and $-l$, since in all equations $|l|$ is replaced by $|l-\alpha|$. The result is specially interesting for $\alpha = 1/2$ (semion), since a little algebra will show that the cylindrical Bessel functions in the wave functions are now replaced by the spherical ones, as in three dimensions. In particular, for $l = 0$, and $l = 1$ (but not for $l = -1$) eq. (12) shows that the semionic flux line exactly cancels the $-\frac{1}{4r^2}$-term, restoring shape invariance. Without the flux line, there was degeneracy in levels with $l = \pm 1, \pm 2, ...,$ and $l = 0$ levels had no degeneracy. With the semionic flux line, levels with $l = (0,1), (-1,2), (-2,3)...$ become degenerate in pairs. The radial parts of the degenerate states are identical, but the angular parts are orthogonal. Since the pair (0,1) has now the wave function $u(kr) = \sqrt{(kr)}J_{1/2}(kr) = \sin(kr)$, W^2 has the same form as in the one-dimensional square well, but in the radial coordinate $r > 0$. More generally, the wave functions are given by

$$u_{l,n}^{(1)}(r) = N_{l,n} \times \begin{cases} (z_{|l|,n}x)j_{|l|}(z_{|l|,n}x), & l \leq 0 \\ (z_{l-1,n}x)j_{l-1}(z_{l-1,n}x), & l \geq 1 \end{cases}, \tag{27}$$

where the notation is the same as before. These wave functions are of the same form as in three dimensions, except that the radial coordinate is planar, and the degeneracy is 2 and not $(2l+1)$. We note from the above equation that pair-wise the wave functions have the same spherical Bessel function of order $\tilde{l} = |l|$, where l is the angular momentum of the one in the pair with the negative (or zero) l. For example, the pair $(0,1)$ has $\tilde{l} = 0$, the pair $(-1,2)$ has $\tilde{l} = 1$, etc. Thus the superpotential for a pair may be denoted as $W_{\tilde{l}}(r)$, and calculated as before. In figure 6 we display the superpartners for $\tilde{l} = 0$, and their spectra. The SWKB results are exact for this case, but not for $\tilde{l} > 0$. The results of the SWKB calculations are displayed in table 2 for $\tilde{l} = 0, 1$ and 2. SWKB is exact for $\tilde{l} = 0$, and the discrepancy in the fourth decimal place is due to numerical inaccuracy.

Finally, it is to be noted that it is not in every case that generalizing to two or higher dimensions destroys shape invariance. Harmonic oscillator and Coulomb

Table 2. SWKB calculations in comparison with the analytic values for $\tilde{l} = 0$, 1 and 2.

n	$\epsilon_{0,n}^{\mathrm{SWKB}}$	$\epsilon_{0,n}^{\mathrm{EXACT}}$	$\epsilon_{1,n}^{\mathrm{SWKB}}$	$\epsilon_{1,n}^{\mathrm{EXACT}}$	$\epsilon_{2,n}^{\mathrm{SWKB}}$	$\epsilon_{2,n}^{\mathrm{EXACT}}$
1	0	0	0	0	0	0
2	29.6089	29.6088	39.4950	39.4888	49.5151	49.5018
3	78.9571	78.9568	98.7148	98.7091	118.6498	118.6374

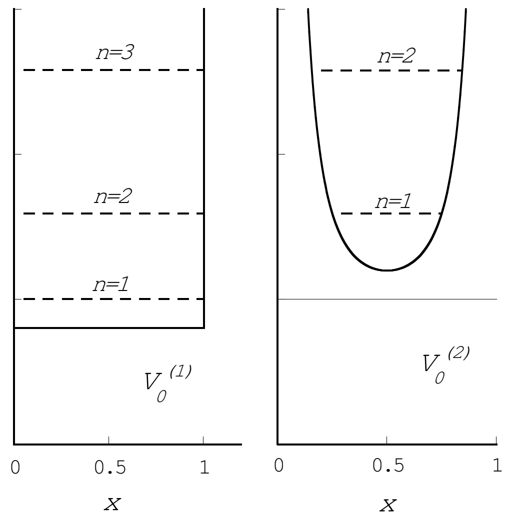

Figure 6. The superpartners of the two-dimensional disk in the magnetic field with $\alpha = 1/2$. The dashed lines represent the energy eigenvalues.

potentials in two and three dimensions still have shape invariant partners. In both these cases, the energy eigenvalues are expressible in terms of a single principal quantum number n, and not separately on n_r and l.

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