

## Bulk and boundary critical behavior at Lifshitz points

H W DIEHL

Fachbereich Physik, Universität Duisburg-Essen, Campus Essen, D-45117 Essen, Germany  
E-mail: phy300@theo-phys.uni-essen.de

**Abstract.** Lifshitz points are multicritical points at which a disordered phase, a homogeneous ordered phase, and a modulated ordered phase meet. Their bulk universality classes are described by natural generalizations of the standard  $\phi^4$  model. Analyzing these models systematically via modern field-theoretic renormalization group methods has been a long-standing challenge ever since their introduction in the middle of 1970s. We survey the recent progress made in this direction, discussing results obtained via dimensionality expansions, how they compare with Monte Carlo results, and open problems. These advances opened the way towards systematic studies of boundary critical behavior at  $m$ -axial Lifshitz points. The possible boundary critical behavior depends on whether the surface plane is perpendicular to one of the  $m$  modulation axes or parallel to all of them. We show that the semi-infinite field theories representing the corresponding surface universality classes in these two cases of perpendicular and parallel surface orientation differ crucially in their Hamiltonian's boundary terms and the implied boundary conditions, and explain recent results along with our current understanding of this matter.

**Keywords.** Critical behavior; Lifshitz points; field theory; boundary critical behavior; anisotropic scale invariance.

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### 1. Introduction

Lifshitz points (LP) are a particular kind of multicritical points at which a disordered phase meets both a spatially homogeneous ordered phase as well as a modulated ordered one [1–4]. They were introduced in 1975 by Hornreich *et al* [5], though apparently discovered independently by two other groups [6] (cf. ref. [3], p. 59). Their discovery triggered considerable theoretical [7–24] and experimental interest [25,26], which has continued over the years, and after a phase of somewhat reduced intensity, has regained a lot of momentum recently [27–33] – in particular, on the theory side [34–51].

The physics of LP is interesting for a variety of reasons. Let me mention a few.

- (i) A wealth of physically distinct systems exist that are either known to have LP or for which LP have been discussed; this includes such diverse systems as magnets [25], ferroelectrics [31], polymer mixtures [28,38,52], liquid crystals [53], systems undergoing structural phase transitions or domain wall instabilities [54], organic crystals [8,55], and even superconductors [56].
- (ii) The physics of LP embodies many crucial concepts of the modern theory of phase transitions and critical phenomena, and yet has been explored to a much lesser degree than critical behavior at conventional critical points. The best-studied universality classes of bulk critical behavior are the ones for  $d$ -dimensional systems with short-range interactions and an  $n$ -component order parameter field  $\phi$ , represented by the  $O(n)$   $\phi^4$  models. For them, very detailed – and in part impressively accurate – results have been worked out by means of sophisticated renormalization group (RG) approaches [57,58], series expansions [59], and computer simulations [60]; and many of these theoretical predictions have been checked by careful experiments.

By contrast, the application of modern field-theoretic RG approaches to the study of critical behavior at LP is a fairly recent development [41,43,46,51]. The two-loop RG analysis of critical behavior at  $m$ -axial LP in  $d = 4 + \frac{m}{2} - \epsilon$  dimensional systems Shpot and myself [43,46] managed to perform for general values  $0 \leq m \leq d$  has finally yielded the  $\epsilon$  expansions of all critical exponents to second order. The estimates obtained by means of these series expansions for the values of the critical exponents for the scalar uniaxial case  $n = m = 1$  in  $d = 3$  dimensions agree quite well with up-to-date Monte Carlo results [61]. Unfortunately, we are aware of only a few high-temperature series estimates [10,23,24], none of which is very recent. On the experimental side, renewed activity is noticeable. Aside from the recent work on polymer mixtures [27,28], new experiments on magnetic systems have been reported [29,30]. However, so far the latter have not produced results for the critical and cross-over exponents of the  $m = n = 1$  LP point of significantly greater accuracy than achieved in previous studies [25,42].

- (iii) Compared with critical points, LP provide additional challenges. Since they are multicritical points, a further thermodynamic variable besides temperature  $T$  must be fine-tuned to reach them. Furthermore, precise experimental investigations of their critical behavior should include the verification of the expected cross-over scaling forms and is expected to involve the choice of proper nonlinear scaling fields [51,62].

On the theoretical side, progress in analytical RG analyses has been hampered by the substantial technical difficulties one encounters in computations of Feynman diagrams beyond one-loop order. The origins of these problems are two-fold: the anisotropic nature of scale invariance that holds at the LP, which implies that the free propagator does not reduce to a simple power at the LP but involves a scaling function; and the fact that this scaling function in position space turns out to have a rather complicated form in general [43]. The progress made recently [43,46] in handling such field theories could pave the way for systematic investigations of general aspects of anisotropic scale invariance (ASI) in systems with short-range interactions. One important question that has been raised long ago [63,64] but not yet answered in a truly convincing fashion is the following. Scale invariance, in

conjunction with translation and rotation invariance, and short range of interactions, is known to normally imply invariance under a larger symmetry group, namely under conformal transformations [65–67]. Does ASI likewise entail the invariance under additional nontrivial continuous transformations? Henkel has played with this idea for years [64,68]; making concrete propositions for transformations under which two-point correlation functions should be invariant, he has come up with definite predictions for the form of the associated scaling functions, which appear to be consistent with Monte Carlo results [61] for the three-dimensional ANNNI model, yet remain to be carefully checked by analytical calculations [69,70]. The field theories representing the universality classes of critical behavior at  $m$ -axial LP are particularly well-suited for such scrutiny, not least because the parameter  $m$  can be varied.

- (iv) Since LP involve both modulated ordered phases as well as ASI, rich and interesting boundary critical phenomena [71–73] may be expected to occur near them. The systematic investigation of such phenomena, in particular, via field-theoretic RG tools, is still in its infancy [62,74–78].

In this contribution, I will briefly survey the progress made recently in the application of field-theoretic RG methods to bulk and boundary critical phenomena at LP, compare its results with those from other sources such as Monte Carlo simulations, highlight some of the central issues and difficulties, and indicate directions for further research. We begin in the next section by specifying the models, then deal with their bulk critical behavior, before we turn in §3 to the issue of boundary critical behavior.

## 2. Continuum models and bulk critical behavior

### 2.1 Continuum models

Having in mind systems whose microscopic interactions are either short ranged or of a long-range kind that is irrelevant in the RG sense, we consider continuum models with a Hamiltonian of the form

$$\mathcal{H} = \int_{\mathfrak{V}} \mathcal{L}_b(\mathbf{x}) dV + \int_{\mathfrak{B}} \mathcal{L}_1(\mathbf{x}) dA, \quad (1)$$

where  $\mathcal{L}_b(\mathbf{x})$  and  $\mathcal{L}_1(\mathbf{x})$  are functions of the  $n$ -component order parameter  $\phi(\mathbf{x}) = (\phi_a(\mathbf{x}))$  and its derivatives with respect to the coordinates  $(x_\alpha, x_\beta) \equiv \mathbf{x}$ . We index the first  $m$  Cartesian coordinates by  $\alpha$ ; they refer to the  $m$ -dimensional subspace to which the modulation of order is confined. The remaining  $\bar{m} \equiv d - m$  are labeled by  $\beta$ . When we deal with boundary critical behavior, the volume and surface integrals  $\int_{\mathfrak{V}} dV$  and  $\int_{\mathfrak{B}} dA$  extend over the half-space  $\mathbb{R}_+^d = \{\mathbf{x} = (\mathbf{r}, z) | \mathbf{r} \in \mathbb{R}^{d-1}, 0 \leq z < \infty\}$  and the  $z = 0$  hyperplane  $\mathfrak{B}$ , respectively. To investigate bulk critical behavior, we may as well take  $\mathfrak{V} = \mathbb{R}^d$  and forget about the boundary piece in eq. (1), choosing appropriate (periodic) boundary conditions.

Unless stated otherwise, the bulk density is

$$\begin{aligned}\mathcal{L}_b(\mathbf{x}) = & \frac{\hat{\sigma}}{2} \left( \sum_{\alpha=1}^m \partial_{\alpha}^2 \phi \right)^2 + \frac{1}{2} \sum_{\beta=m+1}^d (\partial_{\beta} \phi)^2 + \frac{\hat{\rho}}{2} \sum_{\alpha=1}^m (\partial_{\alpha} \phi)^2 \\ & + \frac{\hat{\tau}}{2} \phi^2 + \frac{\hat{u}}{4!} |\phi|^4\end{aligned}\quad (2)$$

in the sequel. Here  $\partial_{\alpha} \equiv \partial/\partial x_{\alpha}$  and  $\partial_{\beta} \equiv \partial/\partial x_{\beta}$ , and  $\hat{\sigma} > 0$  as well as  $\hat{u} > 0$  is assumed. For the time being we focus on bulk critical behavior. Let us therefore postpone the choice of the boundary density  $\mathcal{L}_1$  to §3. Our selection (2) of  $\mathcal{L}_b$  reflects two tacitly assumed properties:  $O(n)$  invariance, and isotropy in the  $m$ -dimensional  $\alpha$ -subspace of coordinates. An investigation of the effects of spin anisotropies breaking the  $O(n)$  invariance of  $\mathcal{L}_b$  may be found in ref. [18]; they will not be considered here. However, the role of ‘space anisotropies’ reducing the Euclidean invariance in the  $\alpha$ -subspace [51] will be briefly discussed at the end of this section.

From eq. (2) it is easy to understand how a LP can occur. The interaction constants  $\hat{\sigma}, \dots, \hat{u}$  all depend on  $T$  and a second thermodynamic variable, a non-ordering field  $g$  [4] such as pressure (charge-transfer salts [8,55]), a ratio of next-nearest neighbor (NNN) antiferromagnetic and nearest-neighbor (NN) ferromagnetic interactions along an axis (ANNNI model [2]), or a magnetic field component in the subspace orthogonal to the order parameter (the orthorhombic magnetic crystal MnP [22,25,29,42]). Assuming that the coefficient of the  $(\partial_{\beta} \phi)^2$  term does not change sign, we have absorbed it in the amplitude of  $\phi$  so that it becomes  $1/2$ . Landau theory gives a disordered phase for  $\hat{\tau} > 0$  provided  $\hat{\rho} > 0$ , separated from a homogeneous ordered one by the critical line  $\hat{\tau}_c(\hat{\rho} \geq 0) = 0$ . For negative  $\hat{\rho}$ , a continuous transition from the disordered to a modulated ordered phase occurs across the so-called ‘helical section’  $\tau = \hat{\tau}_c(\hat{\rho} < 0)$  of the critical line, which joins the ‘ferromagnetic section’ at the LP  $\hat{\tau} = \hat{\rho} = 0$  (see, e.g. figure 1 of ref. [4]). The other phase boundary emerging from the LP separates the homogeneous ordered from the modulated ordered phase. The transitions across it can be of first or second order; for cases with a scalar order parameter they are generically discontinuous, whereas for specific models with a vector order parameter they turn out to be continuous [79].

## 2.2 Critical exponents, anisotropic scale invariance

In Landau theory, the helical section  $\hat{\tau}_{hc} \equiv \hat{\tau}_c(\hat{\rho} < 0)$  varies as  $\hat{\tau}_{hc} \sim \hat{\rho}^2$  near the LP. Beyond Landau theory, the LP and the phase boundaries – supposing they still exist – get shifted as a result of fluctuations, and the helical section of the critical line is expected to behave near the LP as

$$\hat{\tau}_{hc}(\delta\hat{\rho}) - \hat{\tau}_{LP} \equiv \delta\hat{\tau}_{hc} \sim |\delta\hat{\rho}|^{1/\varphi} \sim |\delta g|^{1/\varphi}. \quad (3)$$

Here  $\delta\hat{\rho}$  and  $\delta g$  denote deviations of  $\hat{\rho}$  and  $g$  from their values  $\hat{\rho}_{LP}$  and  $g_{LP}$  at the LP. We have introduced the cross-over exponent  $\varphi$ , whose mean-field value  $\varphi_{MF} = 1/2$ , and utilized the fact that  $\delta\hat{\rho} \sim \delta g$  near the LP.

In the modulated ordered phase, the order is modulated with a wave vector  $\mathbf{q}_{\text{mod}}(T, g)$  depending on  $T$  and  $g$ . Since homogeneous order corresponds to  $\mathbf{q}_{\text{mod}} = 0$ ,  $\mathbf{q}_{\text{mod}}$  must also vanish at the LP. Its limiting behavior as the LP is approached along the critical line's helicoidal section  $T = T_{\text{hc}}(g)$  is governed by the wave-vector exponent  $\beta_q$ , defined via

$$q_{\text{mod}}(T_{\text{hc}}, g) \sim |\delta\rho|^{\beta_q} \sim |\delta g|^{\beta_q}. \quad (4)$$

Other important critical exponents characterize the scale invariance at the LP. Let us set  $\hat{\tau} = \hat{\rho} = \hat{u} = 0$  in eq. (2) and transform to momentum ( $\mathbf{q}$ ) space to obtain the two-point bulk vertex function  $\Gamma_{\text{b}}^{(2)}(\mathbf{q})$  in the Ornstein–Zernicke approximation. Its  $\mathbf{q}$ -dependence reads  $\hat{\sigma}(q_\alpha q_\alpha)^2 + q_\beta q_\beta$ , where repeated indices  $\alpha$  and  $\beta$  are to be summed over  $1, \dots, m$  and  $m+1, \dots, d$ , respectively. Beyond this classical approximation one anticipates nontrivial power laws again. Hence one introduces analogs of the usual correlation exponent  $\eta$  by

$$\Gamma_{\text{b}}^{(2)}(\mathbf{q}) \underset{\mathbf{q} \rightarrow \mathbf{0}}{\sim} \begin{cases} q^{2-\eta_{L2}} & \text{for } q_\alpha = 0, \\ q^{4-\eta_{L4}} & \text{for } q_\beta = 0. \end{cases} \quad (5)$$

These relations mean that  $q_\alpha$  scales as  $q_\alpha \sim (q_\beta)^\theta$ , with the ‘anisotropy exponent’

$$\theta = (2 - \eta_{L2})/(4 - \eta_{L4}); \quad (6)$$

in Landau theory it takes the value  $\theta_{\text{MF}} = 1/2$ . Likewise  $x_\alpha \sim (x_\beta)^\theta$ .

To formulate ASI in position space, let us consider a perturbation  $g_{\mathcal{O}} \int_{\mathfrak{V}} \mathcal{O}(\mathbf{x}) dV$  of the fixed-point Hamiltonian associated with the LP, where  $\mathcal{O}(\mathbf{x})$  is a scaling operator with scaling dimension  $\Delta[\mathcal{O}]$ . Let  $y_{\mathcal{O}}$  be the RG eigenexponent of the associated scaling field  $g_{\mathcal{O}}$ , so that  $g_{\mathcal{O}} \rightarrow \bar{g}_{\mathcal{O}}(\ell) = \ell^{-y_{\mathcal{O}}} g_{\mathcal{O}}$  under scale transformations  $x_\beta \rightarrow x_\beta \ell$ . Since the scaling dimension  $\Delta[\mathcal{O}]$  and the eigenexponent  $y_{\mathcal{O}}$  must add up to the scaling dimension of the volume  $V = \int_{\mathfrak{V}} dV$ , which is  $\bar{m} + m\theta$ , we have

$$y_{\mathcal{O}} = \bar{m} + m\theta - \Delta[\mathcal{O}]. \quad (7)$$

The operators  $\mathcal{O}(\mathbf{x})$  satisfy

$$\mathcal{O}(\ell^\theta x_\alpha, \ell x_\beta) = \ell^{-\Delta[\mathcal{O}]} \mathcal{O}(x_\alpha, x_\beta) \quad (8)$$

(ASI) in the long-scale limit  $\ell \rightarrow 0$ .

### 2.3 Field theory and $\epsilon$ expansion

For a conventional critical point it is known that below the upper critical dimension  $d^* = 4$ , where hyperscaling is valid, two independent critical exponents exist in terms of which all critical indices characterizing the leading infra-red singularities can be expressed. They derive from the scaling dimensions of  $\phi$  and the energy density  $\phi^2$ , or equivalently, the RG eigenexponents  $y_h$  and  $y_\tau$ . Furthermore, there are just two metric factors, one associated with each of the corresponding scaling

fields  $h$  and  $\tau$  ('two-scale factor universality' [80]). In the case of  $m$ -axial LP, the upper critical dimension is [5]

$$d^*(m) = 4 + \frac{m}{2}. \quad (9)$$

The easiest way to see this is to determine the dimension  $d = d^*(m)$  below which the Gaussian scaling dimension of  $\hat{u}$  becomes positive; the Ginzburg criterion yields the same result.

In view of the different scaling of  $x_\alpha$  and  $x_\beta$ , and the need to fine-tune an additional variable  $-\hat{\rho}$  or  $g$ , it is natural to expect that four critical exponents will be required to express the bulk critical exponents of the LP for  $d < d^*(m)$ . Of course, some of these might turn out to be trivial, taking on values independent of  $d$  and  $m$ . For example, one might anticipate the anisotropy exponent  $\theta$  to retain its mean-field value for  $d < d^*(m)$ . However, the  $\epsilon$ -expansion results of Shpot and myself [43,46] have revealed that nontrivial  $m$ -dependent contributions to  $\theta$  appear at order  $\epsilon^2$ . The bulk operators  $\mathcal{O}(\mathbf{x})$  from whose scaling dimensions the four independent bulk exponents derive are given in table 1, along with the associated scaling fields and their RG eigenexponents. Each of these four scaling fields involves a nonuniversal metric factor. Hence a four-scale-factor universality applies.

Given a line of upper critical dimensions  $d^*(m)$ , one should be able to expand about any point on it. Although this goal was identified at a very early stage [5], its implementation turned out to be very demanding and took a long time. In [43,46] a two-loop RG analysis was performed in  $d^*(m) - \epsilon$  dimensions for general values of  $m$ . This gave the  $\epsilon$  expansions of the four independent bulk critical exponents  $\eta_{L2}$ ,  $\theta$ ,  $\nu_{L2}$ , and  $\varphi$ , as well as the correction-to-scaling exponent  $\omega_u$ , to  $O(\epsilon^2)$ .

Technically, a massless minimal-subtraction renormalization scheme was employed. To define the ultraviolet (UV) finite renormalized theory, the reparametrizations

$$\begin{aligned} \phi &= Z_\phi^{1/2} \phi_{\text{ren}}, \quad \hat{\sigma} = Z_\sigma \sigma, \quad \hat{u} \hat{\sigma}^{-m/4} F_{m,\epsilon} = \mu^\epsilon Z_u u, \\ \hat{\tau} - \hat{\tau}_{\text{LP}} &= \mu^2 Z_\tau [\tau + A_\tau \rho^2], \quad (\hat{\rho} - \hat{\rho}_{\text{LP}}) \hat{\sigma}^{-1/2} = \mu Z_\rho \rho \end{aligned} \quad (10)$$

were made, where  $\mu$  is a momentum scale, while  $F_{m,\epsilon}$  is a convenient (UV finite) normalization factor whose precise choice need not worry us here. All renormalization factors  $Z_\phi$ ,  $Z_\sigma$ ,  $Z_\tau$ ,  $Z_\rho$ , and  $Z_u$  were computed to  $O(u^2)$ . From the result

**Table 1.** Bulk scaling operators  $\mathcal{O}(\mathbf{x})$ , associated scaling dimensions  $\Delta[\mathcal{O}]$ , bulk scaling fields  $g_\mathcal{O}$ , and their RG eigenexponents  $y_\mathcal{O}$ , giving the four independent bulk critical exponents of the LP.

$\mathcal{O}(\mathbf{x})$	$\Delta[\mathcal{O}]$	$g_\mathcal{O}$	$y_\mathcal{O}$
$\phi$	$(\bar{m} + m\theta - 2 + \eta_{L2})/2$	$\mathbf{h}$	$(\bar{m} + m\theta + 2 - \eta_{L2})/2$
$(\partial_\alpha \partial_\alpha \phi)^2$	$\bar{m} + m\theta - 4\theta + 2$	$\sigma$	$4\theta - 2$
$\phi^2$	$\bar{m} + m\theta - 1/\nu_{L2} = (1 - \alpha_L)/\nu_{L2}$	$\tau$	$1/\nu_{L2}$
$(\partial_\alpha \phi) \partial_\alpha \phi$	$\bar{m} + m\theta - \varphi/\nu_{L2}$	$\rho$	$\varphi/\nu_{L2}$

for  $Z_u$  the RG  $\beta$  function  $\beta_u(u, \epsilon)$  follows to order  $u^3$ ; the other  $Z$ -factors yield RG functions whose values at the nontrivial root  $u^*(m, \epsilon)$  of  $\beta_u$  give the critical exponents. The main consequence of the renormalization function  $A_\tau$  is that the scaling field with the RG eigenexponent  $1/\nu_{L2}$  becomes a linear combination of  $\tau$  and  $\rho^2$  [51,62].

What makes calculations beyond one-loop order complicated is that the scaling function  $\Phi_{m,d}(v)$  of the free bulk propagator at the LP,

$$\begin{aligned} G_b(|\mathbf{x}|) &= \int \frac{d^d q}{(2\pi)^d} \frac{\exp(i\mathbf{q} \cdot \mathbf{x})}{q_\beta q_\beta + \tilde{\sigma}(q_\alpha q_\alpha)^2} \\ &= \tilde{\sigma}^{-m/4} (x_\beta x_\beta)^{-1+\epsilon/2} \Phi_{m,d}(v), \quad v \equiv (\tilde{\sigma} x_\beta x_\beta)^{-1/4} \sqrt{x_\alpha x_\alpha}, \end{aligned} \quad (11)$$

is a difference of generalized hypergeometric functions. While, in general, these increase exponentially as  $v \rightarrow \infty$ , their difference has an asymptotic expansion in inverse powers of  $v$  that does not terminate except for special choices of  $(m, d)$ , such as (2, 5) and (6, 7), where it reduces to elementary functions [43,46]. Therefore, the two-loop series coefficients of the renormalization factors could not be computed analytically for general  $m$ . However, they – as well as the implied  $\epsilon$ -expansion coefficients of the critical exponents – could be written in terms of four single integrals  $j_\phi(m)$ ,  $j_\sigma(m)$ ,  $j_\rho(m)$ , and  $j_u(m)$  of the form  $\int_0^\infty dv f(v; m)$ , where  $f(v; m)$  involves  $\Phi_{m,d^*(m)}(v)$ , analogous (related) scaling functions, and powers of  $v$  [46]. For  $m = 0, 2, 6, 8$ , these integrals could be computed analytically; for other values of  $m$  they had to be determined by numerical means.

The resulting  $\epsilon$  expansions of the critical exponents  $\lambda = \nu_{L2}, \dots, \varphi$  and the correction-to-scaling exponent  $\omega_u$  take the form

$$\lambda(n, m, d) = \lambda_{MF} + \lambda_1(n) \epsilon + \lambda_2(n, m) \epsilon^2 + O(\epsilon^3). \quad (12)$$

Note that  $\lambda_1(n)$  is independent of  $m$ , so that the  $m$ -dependence starts at order  $\epsilon^2$ . This means that the coefficients  $\lambda_1(n)$  coincide with their  $m = 0$  counterparts for the standard  $\phi^4$  model for all exponents that remain meaningful when  $m = 0$ . (Recall that exponents such as  $\eta_{L4}$ ,  $\varphi$ , and  $\theta$  are not needed in the isotropic case  $m = 0$ .)

The result (12) allows several interesting checks. First, if we substitute the analytically known  $m = 0$  values of the integrals  $j_i(m)$  into it, choosing  $\lambda = \eta_{L2}$ ,  $\nu_{L2}$ , and  $\omega_u$ , then the familiar expansions to  $O(\epsilon^2)$  of the standard exponents  $\eta$ ,  $\nu$ , and  $\omega_u$  of the  $\phi^4$  model are recovered. A second check concerns the special cases  $m = 2$  and  $m = 6$ . Owing to enormous simplifications, the two-loop RG analysis can be performed fully analytically. The results one obtains in this fashion are fully consistent with what one gets upon insertion of the analytically known values of  $j_i(2)$  and  $j_i(6)$  into the two-loop expressions for general  $m$ . Third, considering the case of the isotropic LP [5,48], one can set  $d = m = 8 - \epsilon_8$  in eq. (12) and expand to second order in  $\epsilon_8 = 2\epsilon$ . The limiting values  $j_i(8-)$  are again known analytically [46,48]. Considering exponents that remain meaningful in the isotropic case  $m = d$ , such as  $\eta_{L4}$  or  $\nu_{L4} = \theta \nu_{L2}$ , we can derive their expansions in  $\epsilon_8$  to  $O(\epsilon_8^2)$  from eq. (12). The results agree with those obtained via a direct analysis of the isotropic model with  $m = d$  in  $8 - \epsilon_8$  dimensions [5,48].

A cautionary remark is appropriate here. As a candidate for an experimental system with an isotropic Lifshitz point, ternary mixtures of A and B homopolymers and AB diblock copolymers have been studied both experimentally [28] and theoretically [52]. In their case, modulated order occurs in the lamellar phase. While self-consistent field theory predicts the transition from the disordered to the lamellar phase to be continuous [52], theoretical arguments in favor of a first-order transition have been presented [81]. This would mean that there is actually no isotropic LP. According to some experiments (see the discussion in §7 of [52]), the Lifshitz point found in mean-field (MF) theories gets apparently destroyed. Unfortunately, recent Monte Carlo simulations [52] were not able to decide whether the transition between the disordered and lamellar phases is of first order or continuous. However, they yielded modifications of the MF phase diagram similar to those seen in experiments – in particular, no LP. If fluctuations indeed preclude the appearance of an isotropic LP, then the  $\epsilon$  expansions of the critical exponents for that case are mainly of academic interest. Nevertheless, their consistency with the results for general  $m$  is very gratifying from a mathematical point of view.

The series-expansion results for general  $m$  can be, and were, used in particular to obtain approximate values for the critical exponents of the uniaxial LP with  $n = 1$  at  $d = 3$  [46]. Both experiments on MnP [25] as well as Monte Carlo calculations for the ANNNI [20,61] model provide clear evidence for the existence of such a LP. Recent field-theoretic estimates are  $\nu_{L2} \simeq 0.75$ ,  $\beta_L \equiv \nu_{L2} \Delta[\phi] \simeq 0.22$ ,  $\theta = \nu_{L4}/\nu_{L2} \simeq 0.47$ ,  $\varphi \simeq 0.68$ ,  $\alpha_L \simeq 0.16$ , and  $\gamma_L \simeq 1.4$  [46]. The agreement with current Monte Carlo results, which gave  $\alpha_L = 0.18 \pm 0.03$ ,  $\beta_L = 0.235 \pm 0.005$ , and  $\gamma_L = 1.36 \pm 0.03$ , is fairly good. For more detailed comparisons covering also other cases, experimental work, and further theoretical estimates, the reader is referred to refs [46,61,73].

## 2.4 Space anisotropies

A natural generalization of the ANNNI model is the biaxial NNN Ising (BNNNI) model, which has competing NN and NNN interactions along two cubic axes rather than along a single one. In  $d$  dimensions, even  $m$ -axial variants of the latter, ‘ $m$ NNNI models’ with  $m \leq d$ , can be considered. The continuum models onto which they map upon coarse graining generically have fourth-order derivative terms *breaking isotropy* in the  $\alpha$ -subspace. Their symmetry may be cubic or – if we consider similar microscopic systems involving other crystal lattices – even weaker. Hence, whenever  $m > 1$ , the bulk density (2) should be supplemented by anisotropic contributions of the form

$$\mathcal{L}_b^{(w)} = \frac{\hat{\sigma}}{2} \hat{w}_i T_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}^{(i)} (\partial_{\alpha_1} \partial_{\alpha_2} \phi) \partial_{\alpha_3} \partial_{\alpha_4} \phi = \frac{\hat{\sigma}}{2} \hat{w} \sum_{\alpha=1}^m (\partial_{\alpha}^2 \phi)^2 + \dots, \quad (13)$$

where all tensors  $T_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}^{(i)}$  permitted by symmetry must be included. The  $\hat{w}_i$  are dimensionless interaction constants. For cubic symmetry, only the first term on the far right remains.

The effects of such space anisotropies were investigated in ref. [51]. A new renormalization factor  $Z_{w_i}$  is required for each independent anisotropy, and these as



well as the previously introduced renormalization functions [eq. (10)] now depend on  $u$  and the renormalized anisotropies  $w_i$ . Specifically, the cross-over exponent  $\varphi_2$  associated with the cubic anisotropy  $\hat{w}$  was computed to  $O(\epsilon^2)$ . For  $m = 2$  and  $m = 6$ , the  $O(\epsilon^2)$  coefficient could be determined analytically, for other values of  $m$  expressed in terms of another (numerically computable) single integral. It turned out to be small, but positive. For example, for  $m = 2$  and  $n = 1$  its evaluation at  $\epsilon = 2$  yielded the  $d = 3$  estimate  $\varphi_2 = 1/81 \simeq 0.012$ . Thus the isotropic fixed point  $w_i = 0$  is unstable, at least for small  $\epsilon$ . Whenever such anisotropy is present, the previously found universality classes should not apply. Unfortunately, no new stable fixed point could be found. A detailed clarification of the behavior for  $w_i \neq 0$  remains a challenge. It would be interesting to investigate the role of such anisotropies in Monte Carlo simulations of suitably designed three-dimensional models (e.g., the BNNI model), albeit deviations from the  $w_i = 0$  universality classes may be difficult to measure because of the smallness of  $\varphi_2$ .

### 3. Boundary critical behavior at LP

The study of boundary critical behavior at LP started with Gumbs' work based on Landau theory [74], in which  $z$  was taken to be an  $\alpha$ -direction. Later, considerably more detailed MF analyses [75,76] and Monte Carlo calculations [77] of semi-infinite ANNI models with perpendicular ( $z = \alpha$ -direction) and parallel ( $z = \beta$ -direction) surface orientations were performed. So far, detailed field-theoretic RG studies were made only for the case of parallel surface orientation [62,78].

Let me emphasize that the two primary types of surface orientations ( $\parallel$  or  $\perp$ ) correspond to substantially distinct cases. This can be seen from the following observations: First,  $z$  scales differently, namely, as  $\ell^{-1}$  and  $\ell^{-\theta}$ , respectively. This has an immediate consequence. Consider a perturbation  $g_{\mathcal{O}^\mathfrak{B}} \int_{\mathfrak{B}} \mathcal{O}^\mathfrak{B}(\mathbf{r}) dA$ , where  $\mathcal{O}^\mathfrak{B}(\mathbf{r})$  is a boundary operator with scaling dimension  $\Delta[\mathcal{O}^\mathfrak{B}]$  and hence has the ASI property (8). The analogs of eq. (7) for the RG eigenexponent  $y_{\mathcal{O}^\mathfrak{B}}$  of  $g_{\mathcal{O}^\mathfrak{B}}$  differ depending on the surface orientation:

$$y_{\mathcal{O}^\mathfrak{B}} = \bar{m} + m\theta - \Delta_{\parallel,\perp}[z] - \Delta[\mathcal{O}^\mathfrak{B}], \quad \Delta_{\parallel}[z] = 1, \Delta_{\perp} = \theta. \quad (14)$$

Second, owing to the different engineering dimensions  $[z] = \mu^{-1}$  and  $[z] = \hat{\sigma}^{1/4} \mu^{-1/2}$ , power counting considerations to estimate the relevance or irrelevance of contributions to the surface density  $\mathcal{L}_1$  differ. Third, since Ginzburg–Landau theory yields differential equations for the order parameter of second ( $\parallel$ ) or fourth ( $\perp$ ) order in  $\partial_z$ , either a single or else two boundary conditions are needed at  $z = 0$  and  $z = \infty$ .

To bring the problem into focus, let me recall that in the  $m = 0$  case of the standard semi-infinite  $\phi^4$  model it is sufficient to choose  $\mathcal{L}_1 = \frac{1}{2} \hat{c} \phi^2$ , unless terms breaking the  $O(n)$  symmetry are permitted [72] (which will be avoided here). On the basis of power counting alone, one might think that the symmetry-allowed monomial  $\phi \partial_n \phi$  (where  $\partial_n$  means derivative along the inner normal), should be included as well. But this is redundant because of the boundary condition  $\partial_n \phi = \hat{c} \phi$ , which as usual follows from the boundary part of the classical equation  $\delta \mathcal{H} = 0$  and holds beyond Landau theory inside of averages.

The surface enhancement variable  $\mathring{c}$  determines the type of surface transition that occurs at bulk criticality: Depending on whether its deviation  $\delta\mathring{c} = \mathring{c} - \mathring{c}_{\text{sp}}$  from a special value  $\mathring{c}_{\text{sp}}$  satisfies  $\delta\mathring{c} > 0$ ,  $\delta\mathring{c} = 0$  or  $\delta\mathring{c} < 0$  an ordinary, special or extraordinary transition occurs [71,72], provided the dimension of the surface,  $d - 1$ , exceeds the value below which a long-range ordered surface phase in the presence of a disordered bulk is not possible (i.e., if  $d > 2$  and  $d > 3$  in the Ising and  $n > 1$  cases, respectively).

What modifications occur in the  $m > 0$  LP case? They are easy to understand if the surface orientation is parallel: An additional derivative term must be included in  $\mathcal{L}_1$ , which thus becomes [62]

$$\mathcal{L}_1^{\parallel}(\mathbf{x}) = \frac{\mathring{c}}{2} \phi^2 + \frac{\mathring{\lambda}}{2} \sum_{\alpha=1}^m (\partial_{\alpha} \phi)^2. \quad (15)$$

Since  $[\mathring{c}] = [\mathring{\sigma}^{1/2} \partial_{\alpha}^2] = \mu$ , the variable  $\mathring{\lambda} \sigma^{-1/2}$  is dimensionless. The implied boundary condition reads  $(\partial_n - \mathring{\lambda} \partial_{\alpha} \partial_{\alpha}) \phi = \mathring{c} \phi$ ; it can be employed to conclude that contributions to  $\mathcal{L}_1^{\parallel}$  of the form  $\phi \partial_n \phi$  and  $(\partial_{\alpha} \phi) \partial_n \partial_{\alpha} \phi$  are redundant. By contrast, the inclusion of the term  $\propto \mathring{\lambda}$  is necessary: Not only is it required to absorb UV singularities of the theory, but it would be generated under the RG if originally absent. This can be seen as follows: In order to renormalize the model defined by eqs (1), (2), and (15), we must complement the reparametrizations (10) by

$$\begin{aligned} \phi^{\mathfrak{B}} &= (Z_{\phi} Z_1)^{1/2} \phi_{\text{ren}}^{\mathfrak{B}}, \\ \mathring{\lambda} \mathring{\sigma}^{-1/2} &= \lambda + P_{\lambda}(u, \lambda, \epsilon), \\ \mathring{c} - \mathring{c}_{\text{sp}} &= \mu Z_c [c + A_c(u, \lambda, \epsilon) \rho]. \end{aligned} \quad (16)$$

Here the surface renormalization factors  $Z_1$  and  $Z_c$  depend on  $u$  and  $\lambda$ , just as  $P_{\lambda}$  and  $A_c$ . At  $O(u^2)$ ,  $P_{\lambda}$  does not vanish for  $\lambda = 0$ , so a nonzero  $\mathring{\lambda}$  gets indeed generated. Furthermore, there are no RG fixed points at  $\lambda = 0$  on the hyperplane  $u = u^*$  (see figure 2 of ref. [62]). The fixed points associated with the ordinary, special, and extraordinary transitions turn out to be located at a nontrivial  $\lambda$ -value  $\lambda_+^* = \lambda_0(m) + O(\epsilon)$  and  $c = c_{\text{ord}}^* \equiv \infty$ ,  $c_{\text{sp}}^* \equiv 0$ , and  $c_{\text{ex}}^* \equiv -\infty$ , respectively.

Before continuing our account of the available results for this parallel case, let us briefly discuss how to choose  $\mathcal{L}_1$  when the surface orientation is perpendicular. Clearly, the two monomials included in eq. (15) should be expected here as well, although different couplings ought to be associated with  $(\partial_{\alpha} \phi)^2$  for  $\alpha = 1$  ( $z$ -direction) and  $\alpha \geq 2$ . As long as terms breaking the  $O(n)$  symmetry can be ruled out, the choice

$$\mathcal{L}_1^{\perp} = \frac{\mathring{c}_{\perp}}{2} \phi^2 + \frac{\mathring{\lambda}_{\parallel}}{2} \sum_{\alpha=2}^m (\partial_{\alpha} \phi)^2 + \mathring{b} \phi \partial_n \phi + \frac{\mathring{\lambda}_{\perp}}{2} (\partial_n \phi)^2 \quad (17)$$

should be sufficient. From the vanishing of the contributions  $\int_{\mathfrak{B}} \dots \delta \partial_n \phi$  and  $\int_{\mathfrak{B}} \dots \delta \phi$  to  $\delta \mathcal{H}$  two boundary conditions on  $\mathfrak{B}$  are found, namely

$$\left[ \hat{\sigma} \partial_n^3 + (\hat{b} - \hat{\rho}) \partial_n + \hat{c}_\perp - \hat{\lambda}_\parallel \sum_{\alpha=2}^m \partial_\alpha^2 \right] \phi = 0, \quad \left[ -\hat{\sigma} \partial_n^2 + \hat{\lambda}_\perp + \hat{b} \right] \phi = 0. \quad (18)$$

They tell us that the monomials  $\phi \partial_n^2 \phi$ ,  $(\partial_n \phi) \partial_n^2 \phi$ , and  $\phi \partial_n^3 \phi$  (which are potentially dangerous for  $\epsilon \geq 0$  according to power counting) are redundant. A detailed RG analysis of the model with the bulk and surface densities (2) and (17) remains to be done.

In the case of parallel surface orientation, it is possible to investigate the ordinary transition without retaining the dependence on  $\lambda$  and  $c$  [62]: In the limit  $c \rightarrow c_{\text{ord}}^* = \infty$  a Dirichlet boundary condition applies and the dependence on  $\lambda$  drops out (resides only in metric factors). Hence one can set  $\hat{c} = \infty$  and  $\hat{\lambda} = 0$ , choosing from the outset Dirichlet boundary conditions for the bare theory. The critical exponent  $\beta_1$  of the surface order parameter  $\phi^{\text{B}}(\mathbf{r}) = \phi(\mathbf{r}, 0)$  follows via the boundary operator expansion

$$\phi(\mathbf{r}, z) \underset{z \rightarrow 0}{\approx} C(z) \partial_n \phi(\mathbf{r}), \quad C(z) \sim z^{\Delta[\partial_n \phi] - \Delta[\phi]}, \quad (19)$$

giving  $\beta_1^{\text{ord}}/\nu_{L2} = \Delta[\partial_n \phi]$ . Hence one must study multi-point cumulants involving arbitrary number of fields  $\phi$  and boundary operators  $\partial_n \phi$ . This strategy was followed in ref. [62] and utilized to determine the critical index  $\beta_1^{\text{ord}}$  to  $O(\epsilon^2)$  for general  $0 \leq m \leq 6$ . The  $\epsilon^2$  term involves a further single integral  $j_1(m)$ , which again could be computed analytically for  $m = 0, 2, 6$ , though only numerically for other values. All other surface exponents of the ordinary transition can be expressed in terms of a single one, e.g.  $\beta_1^{\text{ord}}$  and four independent bulk indices. The form (12) of the  $\epsilon$  expansion, with  $m$ -independent  $O(\epsilon)$  terms, also applies to these surface exponents. Furthermore, for  $m \rightarrow 0$  their expansions to  $O(\epsilon^2)$  turn into the known ones [72,82] of the standard semi-infinite  $\phi^4$  model. The  $d = 3$  estimates one obtains from these  $\epsilon$  expansions in the uniaxial scalar case  $m = n = 1$  (e.g.,  $\beta_1^{\text{ord}} \simeq 0.68 \dots 0.7$ ) agree reasonably well with recent Monte Carlo results for the ANNNI model [77], which gave  $\beta_1^{\text{ord}} = 0.687(5)$ .

The special transition is harder to analyze because the  $\lambda$ -dependence must be retained, though  $c$  can be set to its fixed-point value  $c_{\text{sp}}^* = 0$ . A recent one-loop analysis [78] showed that  $\beta_1^{\text{sp}}$  agrees with the bulk exponent  $\beta_L$  to  $O(\epsilon)$  and that the cross-over exponent  $\Phi$  associated with  $c$  becomes  $m$ -dependent already at  $O(\epsilon)$ . According to recent Monte Carlo results [61,77],  $\beta_1^{\text{sp}} = 0.23(1)$  and  $\beta_L = 0.238 \pm 0.005$ . Thus the difference  $\beta_1 - \beta_L$  seems to be small indeed.

Returning briefly to the case of perpendicular surface orientation, let me conclude with a – hopefully educated – guess concerning the ordinary transition. I expect that the asymptotic behavior at this transition is described by a theory that obeys the boundary conditions  $\phi^{\text{B}} = \partial_n \phi = 0$ . The critical exponent  $\beta_1$  in this case should follow from the boundary operator expansion

$$\phi(\mathbf{r}, z) \underset{z \rightarrow 0}{\approx} C_\perp(z) \partial_n^2 \phi(\mathbf{r}), \quad C_\perp(z) \sim z^{(\Delta[\partial_n^2 \phi] - \Delta[\phi])/\theta}, \quad (20)$$

and be given by  $\beta_1^{\text{ord}}/\nu_{L2} = \Delta[\partial_n^2 \phi]$ .

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