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Phenomenological dynamics: From Navier–Stokes to chiral granular gases

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Abstract. This paper reviews the derivation of equations for slow dynamical processes in a variety of systems, including rotating rigid rotors, crystalline solids, isotropic and nematic elastomers, gels in an isotropic fluid background, and nematic liquid crystals. It presents a recent derivation of the Leslie–Ericksen equations for the dynamics of nematic liquid crystals that clarifies the nature of the nonhydrodynamic modes in these equations. As a final example of the phenomenological approach to slow dynamical processes, it discusses the dynamics of a driven nonequilibrium system: a two-dimensional gas of chiral 'rattlebacks' on a vibrating substrate.

Keywords. Hydrodynamics; broken symmetry; nematic liquid crystals; chirality; granular gas.

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1. Introduction

Many if not most of the dynamical phenomena in condensed-matter systems, like the air currents and the propagation of sound, that we see in everyday life involve collective motions and changes that are slow on the scale of microscopic collision times. They can be described by phenomenological equations for collective variables that are far simpler than the equations describing the microscopic dynamics of all constituent particles. These equations, of which the Navier–Stokes equations for a fluid are the first example, are generally derived from symmetries and conservation laws, and they contain thermodynamic parameters and transport coefficients that either have to be measured or calculated from more microscopic theories. Several methods [1–4] have been developed over the past few decades to derive phenomenological equations governing the slow dynamics of any equilibrium system given its symmetry and conservation laws. The research frontier has shifted to investigations of whether the same techniques that have been so successful in describing equilibrium systems like simple fluids and liquid crystals can be extended with appropriate modification to manifestly nonequilibrium systems like granular gases under shear [5–8], flocking birds [9], or bacterial baths [10]. This article will

4. Nematodynamics

Nematic liquid crystals are uniaxial fluids. They form via spontaneous symmetry breaking from an isotropic fluid phase. Since the direction of uniaxial order, like that of the models of rotors on a lattice discussed in §2, is specified by the Frank unit director $\mathbf{n}$ with two independent degrees of freedom, a nematic liquid crystal has two broken symmetry hydrodynamic variables in addition to the five conserved variables of an isotropic fluid. A nematic liquid crystal thus has $5 + 2 = 7$ hydrodynamic modes, one of which is a heat diffusion mode, which we will ignore. There are two different approaches to the derivation of dynamical equations for a nematic liquid crystal. In the first approach, used by Leslie and Ericksen (LE) in their seminal papers [15–17] on the subject, the nematic liquid crystal is viewed as a system of hard rods, like those in rotor model discussed earlier, that align to form the nematic state and whose centers-of-mass are free to move like atoms in a classical fluid. In this approach, kinetic energy is divided into rotational energy of rods about their centers-of-mass and translational energy of their centers-of-mass. In the second approach, used both in the 'Harvard' derivation [18,19] and in a different form in the book by de Gennes and Prost [20], uniaxial order is not tied specifically to the alignment of objects that have a well-defined rotational energy, and the kinetic energy is viewed as the translational energy of all mass points (including in the case of rigid rods the points on the rods that rotate about their centers-of-mass). The second approach yields purely hydrodynamical equations with exactly seven expected hydrodynamic modes. The first approach necessarily includes non-hydrodynamic modes, which as we shall see were not fully treated in the Leslie–Ericksen papers. In the hydrodynamic limit, however, in which nonhydrodynamic modes are removed, the Leslie–Ericksen approach yields the same hydrodynamic equations as the first approach. We will outline here the derivation of the equations of nematodynamics using both approaches.

4.1 *The 'Harvard' formalism*

As discussed above, the momentum density $\mathbf{g}$ in this approach arises from all mass points, and the kinetic free energy density is $\frac{1}{2}(g^2/\rho)$. The static free-energy density $f[\mathbf{n},\rho] = f_1[\rho] + f_n[\mathbf{n},\rho]$ (at constant temperature) depends on the director $\mathbf{n}$ and the density ρ. The Frank free-energy density, measuring the energy of director distortions, is

$$f_n = \frac{1}{2}K_1(\nabla\cdot\mathbf{n})^2 + \frac{1}{2}K_2[\mathbf{n}\cdot(\nabla\times\mathbf{n})]^2 + \frac{1}{2}K_3[\mathbf{n}\times(\nabla\times\mathbf{n})]^2, \tag{24}$$

where the splay, twist, and bend elastic constants K_1, K_2, and K_3 in general depend on ρ. The total free energy is thus $F = \int \mathrm{d}^3x\{\frac{1}{2}(g^2/\rho)+f[\mathbf{n},\rho]\}$. The hydrodynamic variables are ρ, $\mathbf{g}$, and $\mathbf{n}$. The Poisson bracket of $\mathbf{n}$ with ρ is zero, and so the only new Poisson bracket we need is $\{g_i(\mathbf{x}), n_j(\mathbf{x}')\} = \lambda_{ijk}\partial_k\delta(\mathbf{x}-\mathbf{x}')$. The reactive coefficient λ_{ijk} satisfies $n_i\lambda_{ijk} = 0$ because $\partial_t n_i$ is perpendicular to n_i. It can be broken up into a symmetric and an anti-symmetric part in jk. The coefficient of the anti-symmetric part is fixed by the requirement that under a uniform rotation of the system at angular velocity $\boldsymbol{\omega} = \nabla \times \mathbf{v}/2, \mathbf{n}$ will rotate with angular velocity $\boldsymbol{\omega}$. The coefficient λ of the symmetric part is undetermined:

$$\lambda_{ijk} = \frac{1}{2}\left(\delta^{\mathrm{T}}_{ij} n_k - \delta^{\mathrm{T}}_{ik} n_j\right) + \frac{1}{2}\lambda\left(\delta^{\mathrm{T}}_{ij} n_k + \delta^{\mathrm{T}}_{ik} n_j\right). \tag{25}$$

The hydrodynamic equations for a nematic liquid crystal are, thus,

$$\partial_t n_i = \lambda_{ijk}\partial_k v_j - \frac{1}{\gamma}\frac{\delta F}{\delta n_i}, \tag{26}$$

$$\partial_t g_i = -\partial_i p + \partial_j\left(\lambda_{kij}\frac{\delta F}{\delta n_k}\right) + \partial_j \sigma'_{ij}, \tag{27}$$

where

$$\sigma'_{ij} = \eta_{ijkl} A_{kl} \tag{28}$$

with $A_{kl} = (\partial_k v_l + \partial_l v_k)/2$ the symmetric part of the spatial derivative of the velocity. η_{ijkl} is a uniaxial viscosity with five independent components. These equations predict two longitudinal sound modes with isotropic velocity but anisotropic damping, two fast predominantly transverse-velocity diffusive modes with frequency $\omega \sim -i(\eta/\rho)q^2$ and two slow predominantly director diffusive modes with $\omega \sim -i(K/\gamma)q^2$. Note that there is no propagating director mode as present in the rotors-on-a-lattice model.

4.2 *Rigid-rod nematodynamics*

If the molecular constituents of a nematic are rigid rods [21], they, like their counterparts in the lattice model of §2, have a spin-angular momentum density $\ell_i(\mathbf{x}) = I_{ij}\Omega_j(\mathbf{x})$ arising from their spinning about their centers-of-mass, where $I_{ij} = I_{||} n_i n_j + I_{\perp}(\delta_{ij} - n_i n_j)$ is the moment of inertia density and $\boldsymbol{\Omega}(\mathbf{x})$ is the spin angular velocity. In addition, they have a conserved center-of-mass momentum with associated momentum density $\mathbf{g}(\mathbf{x})$. The total angular momentum $\mathbf{L} = \int \mathrm{d}^3 x[\mathbf{x} \times \mathbf{g} + \boldsymbol{\ell}]$ is also conserved. The free energy is

$$F = \frac{1}{2}\int \mathrm{d}^d x\left(\frac{g^2}{\rho} + \frac{\ell_{||}^2}{I_{||}} + \frac{\ell_{\perp}^2}{I_{\perp}}\right) + \int \mathrm{d}^d x f[\mathbf{n}, \rho], \tag{29}$$

where $\ell_{||} = \mathbf{n} \cdot \boldsymbol{\ell}$ and $\boldsymbol{\ell}_{\perp} = \mathbf{n} \times (\boldsymbol{\ell} \times \mathbf{n})$. There are now $6 + 3 = 9$ independent variables, only six of which are hydrodynamic. We, therefore, expect to find three nonhydrodynamic modes, presumably associated with the relaxation of the spin angular velocity $\boldsymbol{\Omega}$ to the local center-of-mass angular velocity $\boldsymbol{\omega}$. The Poisson bracket between ℓ_i and n_j is $\{\ell_i(\mathbf{x}), n_j(\mathbf{x}')\} = -\epsilon_{ijk} n_k(\mathbf{x})\delta(\mathbf{x} - \mathbf{x}')$. The other new Poisson brackets such as $\{\ell_i(\mathbf{x}), \ell_j(\mathbf{x}')\}$ contribute to the full nonlinear equations but do not contribute to the linearized equations we consider here, and we will ignore them. Their inclusion, however, leads to fully consistent nonlinear equations. The equations of rigid-rod nematodynamics are thus

$$\partial_t n_i = \epsilon_{ijk}\Omega_j n_k + \frac{1}{\gamma} h_i, \tag{30}$$

$$\begin{aligned}\partial_t \ell_i = {} & \epsilon_{ijk} n_j h_k + \nu_{ijkl}\partial_j\partial_l\Omega_k - \Gamma_{ij}(\Omega_j - \omega_j) \\ & - \frac{1}{2}\Gamma^A(\epsilon_{ijl}n_l n_k + \epsilon_{ikl}n_l n_j)A_{jk},\end{aligned} \tag{31}$$

$$\partial_t g_i = -\partial_i p + \partial_j(\sigma'^A_{ij} + \sigma'^S_{ij}), \tag{32}$$

where ν_{ijkl} is the analog of the friction coefficient ν_{ij} in the rotors on a rigid lattice model of §2, the symmetric and anti-symmetric dissipative parts of the stress tensor are, respectively,

$$\begin{aligned}\sigma'^S_{ij} &= \eta_{ijkl}A_{kl}; \\ \sigma'^A_{ij} &= \frac{1}{2}\epsilon_{ijk}\Gamma_{kl}(\Omega_l - \omega_l) + \frac{1}{2}\Gamma^A(n_j A_{ip} n_p - n_i A_{jp} n_p),\end{aligned} \tag{33}$$

and $h_i = -\delta F/\delta n_i$. Note that eq. (30) is the rotational analog of permeation (eq. (19)) that allows $\partial_t \mathbf{n}$ to differ from $\mathbf{\Omega} \times \mathbf{n}$.

If $\mathbf{g}$ is locked to zero, both ω_i and A_{ij} are zero, and the equations for ℓ and $\mathbf{n}$ reduce to those (eqs (10)–(12)) of the model of rigid rotors on a lattice with friction coefficient Γ. When A_{ij} is zero, but ω_i is not, the spin angular velocity will decay to ω rather than zero. This is analogous to the velocity $\dot{\mathbf{u}}$ of the gel in a fluid decaying to the fluid velocity $\mathbf{v}$. When A_{ij} is nonzero, there is an additional dissipative contribution to $\partial_t \ell$ with friction coefficient Γ^A. The anti-symmetric part of the stress tensor is fixed by the requirement that the total angular momentum be conserved.

We can now cast these equations in various forms. $\Omega_{||}$ decouples from the other variables and decays in time $\Gamma_{||}/I_{||}$ to $\omega_{||}$. We will not give further attention to it. The $\nu_{ijkl}\partial_j\partial_l\Omega_k$ term is subdominant compared to the others in eq. (31), and we will now drop it. The equations for $\mathbf{n}$ and $\mathbf{\Omega}_\perp$ can now be written in a compact vector form that makes comparison with the original Leslie–Ericksen equations more straightforward:

$$\partial_t \mathbf{n} = \mathbf{\Omega} \times \mathbf{n} + \frac{1}{\gamma}\mathbf{h}, \tag{34}$$

$$I_\perp \partial_t \mathbf{\Omega}_\perp = \mathbf{n} \times \mathbf{h} + \nu_\perp \nabla^2 \mathbf{\Omega}_\perp - \Gamma_\perp(\mathbf{\Omega}_\perp - \omega_\perp) - \Gamma^A(\mathbf{A} : \mathbf{n}) \times \mathbf{n}, \tag{35}$$

where $(\mathbf{A} : \mathbf{n})_i = A_{ij} n_j$. We can now either use eq. (34) to solve for $\mathbf{\Omega} \times \mathbf{n}$ and use the result in eq. (35) to obtain a dynamical equation for $\mathbf{n}$ that is close to that obtained by Leslie and Ericksen, or we can solve eq. (35) for $\mathbf{\Omega} \times \mathbf{n}$ and use it in eq. (34) for $\mathbf{n}$ to obtain the Harvard equations in the hydrodynamic limit. Let us begin with the first approach:

$$I\left(\mathbf{n} \times \ddot{\mathbf{n}} - \frac{1}{\gamma}\dot{\mathbf{h}}\right) = \alpha \mathbf{n} \times (\mathbf{h} + \gamma_1 \mathbf{A} : \mathbf{n} + \gamma_2 \mathbf{N}), \tag{36}$$

where

$$\alpha = \left(1 + \frac{\Gamma_\perp}{\gamma}\right); \quad \frac{1}{\gamma_1} = \frac{1}{\gamma} + \frac{1}{\Gamma_\perp}; \quad \gamma_2 = -\frac{\Gamma^A}{\Gamma_\perp}\gamma_1 \equiv -\lambda\gamma_1. \tag{37}$$

This reduces identically to the original LE equations [15–17], with an inertial term proportional to $\mathbf{n} \times \ddot{\mathbf{n}}$, when $\gamma \to \infty$. Thus the LE equations are equations for a rigid-rod nematic in which $\boldsymbol{\Omega} \times \mathbf{n}$ is locked to $\partial_t \mathbf{n}$, i.e., in which there is no rotational permeation. LE equations also ignore the possibility of any component of angular momentum parallel to the director: they effectively set $I_{||} = 0$. Note that both the full equations and the LE equations have two nonhydrodynamic modes with characteristic frequencies $\omega \sim -i\Gamma_\perp / I_\perp$ in the $q \to 0$ limit. In the hydrodynamic limit, eq. (37) reduces to $\mathbf{h} = \gamma_1 \mathbf{A} : \mathbf{n} + \gamma_2 \mathbf{N}$ independent of the value of γ. There are two contributions to the coefficient γ_1^{-1}: the first $\Gamma_\perp^{-1}$ arises from the friction torque on a rotating anisotropic particle and the second from rotational permeation. Calculations of γ_1 generally ignore the second contribution [22].

In the second approach, we solve eq. (35) to obtain $\boldsymbol{\Omega} \times \mathbf{n} = (\mathbf{h} + \Gamma_\perp ! \ \times \mathbf{n} + \Gamma^A \mathbf{A} : \mathbf{n})/\Gamma_\perp$ in the hydrodynamic limit. Using this result in eq. (34), we obtain the Harvard equation

$$\partial_t \mathbf{n} = \boldsymbol{\Omega} \times \mathbf{n} + \frac{1}{\gamma}\mathbf{h} = ! \ \times \mathbf{n} + \frac{1}{\gamma_1}\mathbf{h} + \lambda \mathbf{A} : \mathbf{n}. \tag{38}$$

To obtain the complete equations for nematodynamics, we need to treat the stress tensor. We refer the reader to [21] for further details.

5. Nematic elastomers

Nematic elastomers [23] are cross-linked rubber networks that develop nematic order. They thus combine the unusual elastic properties of rubber with the anisotropy of a nematic liquid crystal. At high temperature, there is no nematic order, and the rubber is macroscopically isotropic. When nematic order develops, it spontaneously breaks the isotropic symmetry. In general, the spontaneous breaking of a continuous symmetry brings with it a soft Goldstone mode, which is manifested by a free energy that vanishes in the spatially uniform limit. Thus, for example, the Frank free energy is a manifestation of spontaneous symmetry breaking of a standard nematic liquid crystal. The soft mode of a nematic elastomer is more subtle. The nematic phase of the elastomer is uniaxial. Usually a uniaxial solid is characterized by five independent elastic constants [24]. In an ideal nematic elastomer, coupling between director and strain is such that the elastic constant C_5, measuring the energy of shear strains in planes containing the anisotropy axis, vanishes when the director relaxes to its local value in the presence of strain [25–27]. In practice, it is impossible to produce a monodomain nematic elastomer without the imposition of an external aligning field (e.g. strain) during the cross-linking process [28]. Thus experimental nematic elastomers are at best semi-soft with a small but nonvanishing C_5.

The harmonic elastic energy of a nematic elastomer [29] in terms of strain u_{ij} and deviation $\delta\mathbf{n} = \mathbf{n} - \mathbf{n}_0$, where $\mathbf{n}_0$ is along the z direction for the director from a uniform equilibrium value is

$$F = F_u + F_n + F_{u-n}, \tag{39}$$

$$F_u = \int \mathrm{d}^3x \left[\frac{1}{2}C_1 u_{zz}^2 + C_2 u_{zz}u_{aa} + \frac{1}{2}C_3 u_{ii}^2 + C_4 u_{ab}^2 + C_5 u_{az}^2\right], \tag{40}$$

$$F_n = \int \mathrm{d}^3x \left[\frac{1}{2}K_1(\partial_a n_a)^2 + \frac{1}{2}K_2(\epsilon_{ab}\partial_a n_b)^2 + \frac{1}{2}K_3(\partial_z n_b)^2\right], \tag{41}$$

$$F_{u-n} = \int \mathrm{d}^3x \left[\frac{1}{2}D_1 Q_a^2 + D_2 Q_a u_{az}\right], \tag{42}$$

where $a, b = x, y$ labels directions perpendicular to z and $Q_a = n_a - \frac{1}{2}(\partial_z u_a - \partial_a u_z)$. In equilibrium, $Q_a = -D_2 u_{az}/D_2$, and the effective free energy in terms of strain only is [30]

$$F_u^{\mathrm{eff}} = \int \mathrm{d}^3x \left[\frac{1}{2}C_1 u_{zz}^2 + C_2 u_{zz}u_{aa} + \frac{1}{2}C_3 u_{ii}^2 \right. \tag{43}$$

$$\left. + C_4 u_{ab}^2 + C_5^R u_{az}^2 + \frac{1}{2}K_1^R(\partial_a^2 u_z)^2 + \frac{1}{2}K_3^R(\partial_z^2 u_a)^2\right], \tag{44}$$

where $C_5^R = C_5 - D_2^2/(2D_1)$, $K_1^R = [(D_1 + D_2)/(2D_1)]^2 K_1$, and $K_3^R = [(D_1 - D_2)/(2D_1)]^2 K_3$.

The natural dynamic variables for a nematic elastomer are the director $\mathbf{n}$, the displacement variable $\mathbf{u}$, the density ρ and the momentum density $\mathbf{g}$ for a total of $2+3+1+3 = 9$ variables. An elastomer is a tethered solid, so we can replace $\delta\rho/\rho$ by $-\nabla \cdot \mathbf{u}$. This leaves us with eight variables. In the low-frequency hydrodynamic limit, the director will relax to its equilibrium value in the presence of strain, eliminating two more variables and leaving us with six hydrodynamic variables just as in a traditional tethered solid. Thus, the true hydrodynamic equations for a soft or semi-soft elastomer are identical to those for standard tethered solid (eq. (21)) with the free energy replaced by F_u^{eff}. Dynamical equations including the director follow from our previous 'Harvard' derivation of the equations for nematics and for tethered solids. They are [30]

$$\partial_t n_i = \lambda_{ijk}\partial_k v_j - \frac{1}{\gamma}\frac{\delta F}{\delta n_i}, \tag{45}$$

$$\dot{u}_i = \frac{\delta F}{\delta g_i} = \frac{1}{\rho} g_i, \tag{46}$$

$$\partial_t g_i = \partial_j \left(\lambda_{kij}\frac{\delta F}{\delta n_k}\right) - \frac{\delta F}{\delta u_i} + \partial_j \sigma'_{ij}. \tag{47}$$

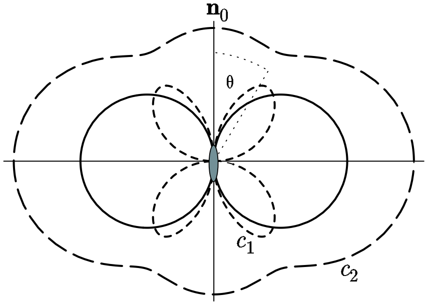

Figure 1. Schematic polar plot (arbitrary units) of the sound velocities c_1, c_2, and c_t of a nematic elastomer. The c_2-mode is predominantly longitudinal with a nonvanishing velocity at all angles, the c_1-mode is like that of a smectic liquid crystal, and the c_t-mode is like a columnar mode.

These equations have also been derived using a modification of the Leslie–Ericksen approach [23]. In the hydrodynamic limit they reduce to

$$\rho \ddot{u}_i = -\frac{\delta F_u^{\mathrm{eff}}}{\delta u_i} + \eta_{ijkl} \partial_j \partial_l v_k. \tag{48}$$

In the soft case in which $C_5^{\prime R}$ is zero, these equations yield six sound modes, two with a velocity c_2 that is nonzero for all directions of propagation, two 'smectic-A' modes with a velocity c_1 that vanishes for propagation direction both along z and perpendicular to z, and two 'columnar' modes with velocity c_t that vanishes for propagation direction along z. This mode structure is identical to that of a columnar liquid crystal [20]. The three sound velocities are plotted in figure 1. Along the symmetry directions where the sound velocities are zero, there are slow and fast diffusive modes similar to those of a nematic liquid crystal with respective frequencies $\omega_s = -i(K/\eta_5)q^2$ and $\omega_f = -i(\eta_5/\rho)q^2$.

6. A chiral granular gas

So far, we have considered only equilibrium systems. Some of the general methods we have developed can however be applied to driven nonequilibrium systems. In this section we will develop equations for the dynamics of a two-dimensional granular gas of chiral objects on a vibrating substrate that keeps them in a state of constant agitation [11]. This system is similar in many ways to a liquid crystal in its isotropic state. The chiral objects comprising the gas are inspired by rattlebacks, which are toys that can be purchased in most science museums. They spin in a preferred direction on a hard substrate; if they are spun in the opposite direction, they will wobble and then eventually reverse their direction of spin to the preferred direction. The Russian rattleback provides insight into the properties necessary to produce a preferred direction of rotation. It is shaped like a boat with a symmetric hull and a flat deck on which there are two turtles placed symmetrically at either end.

The orientation of the turtles can be changed. If the turtles face each other at the center of the boat, the rattleback will rotate in either direction; if they face in opposite directions perpendicular to the long axis of the boat, the rattleback will spin preferentially in the direction of the turtle's heads. The second configuration is chiral, whereas the first, without a preferred spin direction, is not.

These observations suggest that essentially any chiral object might have a preferred spin direction. They inspired experiments [11] carried out in Jerry Gollub's lab at Haverford in which first a single bent-wire, depicted schematically in figure 2, was vibrated on a vertically oscillating platform and then a collection of 400 nearly identical bent-wires were vibrated on the same oscillating platform. The single wires spun about their centers-of-mass with an angular frequency of order 1 Hz and a direction that changed as the sign of their chirality changed. Individual wires in the collection of wires, which formed a two-dimensional chiral granular gas, spun at a slower rate about their centers-of-mass, and their centers-of-mass rotated about the center of the platform at a rate that increase rapidly with distance r from the center.

A phenomenological theory for the collective behavior of the chiral granular gas can be derived using the principles outlined in this article. The relevant dynamical variables are the momentum density $\mathbf{g} = \rho\mathbf{v}$ and the spin angular momentum density ` $= I\Omega$, where as before I is the moment of inertia and Ω is the spin rotation frequency. The vibration of the platform provides a continuous input of energy that endows the gas with an average kinetic energy (or granular temperature), which we take to be constant, and a pressure p. The platform exerts a friction force $-\Gamma^v\mathbf{v}$ to the gas that breaks Galilean invariance. As in a nematic liquid crystal, there is a friction force favoring $\Omega = \omega$, and there is a complementary term in the stress tensor to enforce conservation of total angular momentum when there is no friction with the platform. There is of course, friction with the platform, and this gives rise to a dissipative torque $-\Gamma^\Omega\Omega$ that slows the spin to zero in the absence of any external torques. The unique property of the chiral gas is that the vibrating platform imparts an average spin to the particle. We model this by adding an

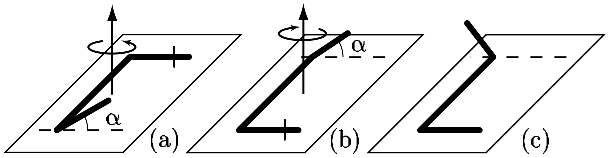

Figure 2. Schematic representations of bent-wire chiral objects made from a central stem and two arms. (**a**) Chiral wire with angle $\alpha > 0$, defined as the angle by which the second arm rises out of the plane formed by the stem and first arm (marked with a vertical line). (**b**) Mirror image of (a), formed by reflection through a plane perpendicular to the central stem, for which the angle α is defined to be negative. In both cases, the curved arrows indicate the direction of spin under vibration. (**c**) Configuration used in the gas experiments with $\alpha = -3\pi/4$. In general the spin direction is such that the raised arm moves in the direction of its projection on to the plane of the vibrating surface so that the bent-wire in (c) rotates in the counterclockwise direction.

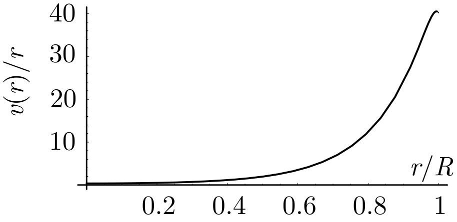

Figure 3. Plot of $v(r)/r$ calculated from eqs (49) and (50).

external torque τ to the equation of motion for τ. The resulting equations for $\mathbf{g}$ and ℓ are

$$\partial_t l = -\partial_j(lv_j) - \Gamma^{\Omega}\Omega - \Gamma(\Omega - \omega) + \nu_{\Omega}\nabla^2\Omega + \tau, \tag{49}$$

$$\partial_t g_i = -\partial_j(g_i v_j) - \partial_i p + \eta\nabla^2 v_i - \Gamma^v v_i + \frac{1}{2}\epsilon_{ij}\partial_j\Gamma(\Omega - \omega), \tag{50}$$

where ϵ_{ij} is the anti-symmetric tensor in two dimensions.

To make contact with the Haverford experiments, we seek solutions to eqs (49) and (50) in which $\Omega = \Omega(r)$ depends only on the distance r from the origin at the center of platform, and $\mathbf{v} = v(r)\mathbf{e}_\phi$ points in the azimuthal direction $\mathbf{e}$ perpendicular to the radius vector $\mathbf{r}$ from the origin. To find a solution, we need to specify boundary conditions. Two boundary conditions are that Ω and ω are nonsingular at the origin. A third condition follows from the observation that the particles slide along the outer wall with their long axes parallel to it. This implies that the spin rate about the particles' centers-of-mass is equal to the angular velocity of the particles about the sample center: $\Omega(R) = v(R)/R$. Finally, a friction force on the particles proportional to the tangential velocity at the outer wall with a friction coefficient Γ_S imposes the boundary condition $\sigma_{\phi r} = -\Gamma_S v$ on the mixed tangential (ϕ) – radial (r) component of the stress tensor. This condition is equivalent to the condition $\partial_r v = -l^{-1} v$ on the tangential velocity at $r = R$, where $l^{-1} = [4\Gamma_S - (\Gamma/R)]/(4\eta + \Gamma)$ is an inverse slipping length [31]. A solution to eqs (49) and (50) for $v(r)/r$ with these boundary conditions along with experimentally observed average values in three bins is shown in figure 3. The theory does predict that the velocity increases significantly with r in agreement with experiments. The experiments and theory together clearly establish that the local spin angular momentum induced by the vibrating substrate gets converted to center-of-mass angular momentum about the origin.

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