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Space curves, anholonomy and nonlinearity

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Abstract. Using classical differential geometry, we discuss the phenomenon of anholonomy that gets associated with a static and a moving curve. We obtain the expressions for the respective geometric phases in the two cases and interpret them. We show that there is a close connection between anholonomy and nonlinearity in a wide class of nonlinear systems.

Keywords. Nonlinear dynamics; differential geometry; space curve; anholonomy; geometric phase.

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1. Introduction

Space curves appear in many diverse branches of science. A trajectory in the phase space of a dynamical system [1], a vortex filament in a fluid [2], a polymer chain such as DNA [3], a twisted optical fiber [4], and a slender elastic rod [5] are obvious examples of *static* space curves. Less obvious an example is the spin vector configuration along a classical magnetic spin chain, where the magnetic moment vector at each point on the chain can be regarded as defining the local tangent to some space curve [6,7]. Clearly, it is possible to have *moving* space curves as well. This happens when a vortex filament, a polymer or an elastic rod is in motion. Again, as one changes some parameters in a dynamical system, a given phase trajectory will shift, in general, thus forming a moving curve in phase space. The time evolution (under some Hamiltonian) of a spin chain also leads to a moving curve. Interestingly, the differential equations associated with such systems turn out to be generically nonlinear. Therefore, the study of static and moving space curves is of considerable interest in many applications of nonlinear science.

Another phenomenon that occurs in diverse contexts is an *anholonomy* [8] (called a holonomy by mathematicians) or a *geometric phase* [9]. This arises when the evolution of a system is such that the value of a quantity in a given state is dependent on the path along which the state has been reached, so that the quantity fails to recover its original value when the parameters on which it depends are varied round a closed path. First introduced by Berry [8] in quantum mechanics as an anholonomic change in the phase of the energy eigenfunction during an adiabatic cyclic evolution of the Hamiltonian in parameter space, the concept has been generalized [10] to include both non-adiabatic and non-cyclic evolutions. Furthermore, it has been recognized [4] that this phase can arise in a purely classical system. Two types of phases can be defined in this regard. The first is the Hannay angle [11] which appears as the direct classical analogue of the Berry phase. It is the anholonomic change in the angle variable conjugate to an action variable during the adiabatic, cyclic evolution in parameter space of a finite-dimensional, integrable, classical Hamiltonian system. The second type of phase is the classical analogue of the Aharonov–Anandan phase [12]. It is purely the effect of the geometry of the evolution of the system in configuration space, and typically appears as the angle of rotation of an appropriately oriented set of axes associated with the evolving system. In contrast to the Hannay angle, the presence of time-dependent external parameters and the feature of integrability are not necessary conditions for this phase to exist. Hence this second type of phase may be expected to occur in more general classical evolutions.

In view of the above, it is pertinent to ask if a static space curve can be associated with a geometric phase. As we shall see, the answer is in the affirmative, and we determine the phase. This, in turn, enables us to find the expression for the anholonomy underlying a moving space curve. We show that there is a close relationship between anholonomy and nonlinearity in a wide class of systems.

2. A static space curve

We start by considering a curve γ embedded in three-dimensional space, generated by the vector $\mathbf{x}(\alpha) = (x_1(\alpha), x_2(\alpha), x_3(\alpha))$, where α is a parameter that could represent time, or any other parameter, depending on the application concerned. The local tangent to the curve, defined as $\mathbf{t}(\alpha) = \mathbf{x}_\alpha$, need not be a unit vector. (Subscripts will stand for the corresponding partial derivatives.) It is more convenient to use the arclength parameter s, defined using the relation $|\mathbf{x}_\alpha| = s_\alpha$. Thus for the space curve $\mathbf{x}(s)$ parametrized by s, the tangent $\mathbf{t} = \mathbf{x}_s$ is clearly a unit vector.

Let $\mathbf{n}$ and $\mathbf{b}$ denote, respectively, the unit normal and binormal vectors on the curve. The orthogonal right-handed triad $(\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s))$ satisfies the Frenet–Serret (FS) equations [13]

$$\mathbf{t}_s = \kappa\,\mathbf{n}, \quad \mathbf{n}_s = -\kappa\,\mathbf{t} + \tau\,\mathbf{b}, \quad \mathbf{b}_s = -\tau\,\mathbf{n}. \tag{1}$$

The curvature κ and the torsion τ are functions of s that determine the local geometry of the curve. It can be shown that [13]

$$\kappa(s) = |\mathbf{t}_s| \tag{2}$$

and

$$\tau(s) = \mathbf{t}\cdot(\mathbf{t}_s \times \mathbf{t}_{ss})/|\mathbf{t}_s|^2. \tag{3}$$

The curvature measures the departure of a curve from a straight line, while the torsion is a measure of the degree to which it twists out of a plane.

As $\mathbf{t}$ is a unit vector, it can be represented in spherical polar coordinates as

$$\mathbf{t} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta), \tag{4}$$

where θ and φ represent, respectively, the polar angle and the azimuthal angle of $\mathbf{t}$. Using eq. (4) in eqs (2) and (3), we have

$$\kappa = \left(\varphi_s^2 \sin^2\theta + \theta_s^2\right)^{1/2} \tag{5}$$

and

$$\tau = \varphi_s \cos\theta + \frac{\mathrm{d}\chi}{\mathrm{d}s}, \tag{6}$$

where

$$\chi = \tan^{-1}\left(\frac{\varphi_s \sin\theta}{\theta_s}\right). \tag{7}$$

Equations (1) may be rewritten in the unified form $\mathbf{L}_s = \xi \times \mathbf{L}$, where $\mathbf{L}$ stands for any of the vectors $\mathbf{t}$, $\mathbf{n}$, or $\mathbf{b}$, and $\boldsymbol{\xi} \equiv \kappa\mathbf{b} + \tau\mathbf{t}$ is called the Darboux vector. It plays the role of an angular velocity of the FS triad. As one moves on the curve $\mathbf{x}(s)$, the FS triad $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ rotates with an angular velocity $\boldsymbol{\xi}$, with κ and τ representing the magnitudes of the angular velocities around the axes $\mathbf{b}$ and $\mathbf{t}$, respectively.

3. Fermi–Walker parallel transport

We now introduce the concept of the Fermi–Walker parallel transport [14] of any vector $\mathbf{P}$ moved along the curve $\mathbf{x}(s)$. One first defines the Fermi–Walker derivative of $\mathbf{P}$ along the curve as

$$\frac{D\mathbf{P}}{Ds} = \mathbf{P}_s - \kappa\,(\mathbf{b} \times \mathbf{P}). \tag{8}$$

A vector $\mathbf{P}$ is parallel transported according to the Fermi–Walker rule if $D\mathbf{P}/Ds = 0$. To construct a 'non-rotating' frame specified by a unit triad $(\mathbf{t}, \mathbf{U}, \mathbf{V})$, we define two orthonormal unit vectors $(\mathbf{U}, \mathbf{V})$ in the $(\mathbf{n}, \mathbf{b})$ plane according to

$$\mathbf{U} = \mathbf{n}\cos\beta - \mathbf{b}\sin\beta \quad \text{and} \quad \mathbf{V} = \mathbf{n}\sin\beta + \mathbf{b}\cos\beta, \tag{9}$$

and specify β appropriately: it is easily verified that the choice $\beta_s = \tau$ ensures that

$$\frac{D\mathbf{t}}{Ds} = \frac{D\mathbf{V}}{Ds} = \frac{D\mathbf{U}}{Ds} = 0. \tag{10}$$

In contrast to this, $\mathbf{n}$ and $\mathbf{b}$ can be shown to satisfy

$$\frac{D\mathbf{n}}{Ds} = \tau\,\mathbf{b}, \quad \frac{D\mathbf{b}}{Ds} = -\tau\,\mathbf{n}. \tag{11}$$

Hence the ($\mathbf{n}$, $\mathbf{b}$) plane rotates around the $\mathbf{t}$-axis with an angular velocity τ. As one moves from s to $s + \mathrm{d}s$, this plane rotates by an angle $\tau \mathrm{d}s$. Thus a total geometric phase

$$\Phi = \int_\gamma \tau \, \mathrm{d}s \tag{12}$$

develops between the natural frame ($\mathbf{n}$, $\mathbf{b}$) and the non-rotating frame.

The total anholonomy Φ can also be interpreted in another way. Substituting eq. (6) in eq. (12) yields $\Phi = 2\pi - \iint_\gamma \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi$. For a closed curve, the second term is just the solid angle subtended by the area enclosed by the closed path $\gamma(s)$ traced out by the tangent indicatrix on the unit sphere S^2. The same result holds good for an open curve as well, since it can always be closed using a geodesic on the sphere [10].

Such a geometric phase Φ is precisely what appears as a rotation in the plane of polarization of light propagated along a helical optical fibre [15] regarded as a static curve. A short calculation shows that it also appears in the spin evolution of an isolated classical spin in a constant magnetic field as a certain solid angle, on identifying the spin vector with the tangent to a space curve. More recently, it has emerged as a useful characterizer of the geometry of phase trajectories in a class of nonlinear dynamical systems [16].

The natural question that arises now is: what happens to the above anholonomy if the space curve has one more degree of freedom – for example, if an isolated spin is replaced by a chain of interacting spins, or if a vortex filament in a fluid moves in a certain way, dictated by kinematics? To answer this, we must extend the idea of Fermi–Walker parallel transport to the case of a moving space curve.

4. General curve evolution equations

Let us now consider a space curve that evolves in time, so that $\mathbf{x} = \mathbf{x}(s, u)$, where we have used u to denote the time in order to avoid any confusion with the tangent vector. In addition to the Frenet–Serret equations (1), where κ and τ are now dependent on both s and u, we must also write down the corresponding equations governing the kinematics of the time evolution of the triad ($\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$). These can be written quite generally in a form similar to the Frenet–Serret set [17]:

$$\mathbf{t}_u = g\,\mathbf{n} + h\,\mathbf{b}, \quad \mathbf{n}_u = -g\,\mathbf{t} + \tau_0\,\mathbf{b}, \quad \mathbf{b}_u = -h\,\mathbf{t} - \tau_0\,\mathbf{n}. \tag{13}$$

The scalars g, h, and τ_0 (which are functions of both s and u), along with appropriate boundary conditions, completely determine the motion of the curve. Note that there is an additional term in the $\mathbf{b}$ direction in the time derivative of $\mathbf{t}$. The reason for the absence of such a term in the space derivative is that one has the freedom to align $\mathbf{t}_s$ in the direction of the normal $\mathbf{n}$. But once this is done, the time derivative can have components along both $\mathbf{n}$ and $\mathbf{b}$ in general.

We may again rewrite eq. (13) in a unified form as $\mathbf{L}_u = \boldsymbol{\eta} \times \mathbf{L}$, where $\mathbf{L}$ stands for any of the vectors $\mathbf{t}$, $\mathbf{n}$, or $\mathbf{b}$, and $\boldsymbol{\eta} = g\,\mathbf{b} - h\,\mathbf{n} + \tau_0\,\mathbf{t}$ is the 'temporal' Darboux vector analogous to the Darboux vector $\boldsymbol{\xi}$ introduced earlier in the spatial case. Once again, we may introduce the concept of Fermi–Walker transport along the 'temporal curve' parametrized by u. From eq. (13) we obtain the following analogues of eqs (8) and (11):

$$\frac{D\mathbf{P}}{Du} = \mathbf{P}_u - (g\,(\mathbf{b} - h\,\mathbf{n}) \times \mathbf{P}) \tag{14}$$

and

$$\frac{D\mathbf{t}}{Du} = 0, \quad \frac{D\mathbf{n}}{Du} = \tau_0\,\mathbf{b}, \quad \frac{D\mathbf{b}}{Du} = -\tau_0\,\mathbf{n}. \tag{15}$$

Hence, as one moves from u to $u + \mathrm{d}u$ (for a fixed s), the $(\mathbf{n}, \mathbf{b})$ plane rotates by an angle $\tau_0\,\mathrm{d}u$.

5. Anholonomy of a moving curve

In order to find the anholonomy, we use the concept of Fermi–Walker transport described in §3 to determine the angle of rotation of the $(\mathbf{n}, \mathbf{b})$ plane as we move from a point $a = (s, u)$ to the point $c = (s + \mathrm{d}s, u + \mathrm{d}u)$, but using two different paths. Path 1 goes from a to c via $b = (s + \mathrm{d}s, u)$, while path 2 goes from a to c via $d = (s, u + \mathrm{d}u)$. Clearly, if the angle of rotation is different for the two paths connecting a and c, then there is an underlying geometric phase or anholonomy arising due to the path dependence. It can be verified that the rotation angles for paths 1 and 2 are given respectively by [17]

$$\Phi_1 = \tau(s, u)\,\mathrm{d}s + \tau_0(s + \mathrm{d}s, u)\,\mathrm{d}u \tag{16}$$

and

$$\Phi_2 = \tau_0(s, u)\,\mathrm{d}u + \tau(s, u + \mathrm{d}u)\,\mathrm{d}s. \tag{17}$$

On using Taylor expansions in $\mathrm{d}s$ and $\mathrm{d}u$ and retaining terms up to second order, we find that the phase difference $\delta\Gamma = (\Phi_1 - \Phi_2)$ is given by

$$\delta\Gamma = ((\tau_0)_s - \tau_u)\,\mathrm{d}s\,\mathrm{d}u \equiv \rho(s, u)\,\mathrm{d}s\,\mathrm{d}u. \tag{18}$$

In other words, if we start with a given direction of the vector $\mathbf{n}$ (or $\mathbf{b}$) at the point a, and execute the closed circuit $abcda$, $\mathbf{n}$ does not come back to its original direction. Instead, it rotates around the direction of $\mathbf{t}$ by the angle $\delta\Gamma$. This implies a local anholonomy density or geometric phase density $\rho(s, u)$.

Relationship of the anholonomy with the Pontryagin index

The expression in eq. (18) has another interesting interpretation, in terms of the Pontryagin index that classifies maps $S^2 \to S^2$. We first note that, for non-stretching curves, the FS triad satisfies the compatibility conditions $\mathbf{t}_{us} =$

$\mathbf{t}_{su}, \mathbf{n}_{us} = \mathbf{n}_{su}$, and $\mathbf{b}_{us} = \mathbf{b}_{su}$. With straightforward manipulations, these conditions can be shown to lead to the relations

$$(\tau_u - (\tau_0)_s) = \kappa\, h, \quad (\kappa_u - g_s) = -\tau\, h, \quad \tau_0 = (h_s + \tau\, g)/\kappa. \tag{19}$$

Using the first of the three relations above in eq. (18), we get

$$\rho(s,u) = -\kappa\, h = \mathbf{t}\cdot(\mathbf{t}_u \times \mathbf{t}_s), \tag{20}$$

where we have used eqs (1) and (13) to write the last equality. It is immediately seen that the anholonomy density $\rho(s\,,\,u)$ is just the density associated with the Pontryagin index in certain cases: if $\mathbf{t}(s\,,\,u)$ is such that it takes on the same value at the boundaries at infinity, then the total anholonomy or geometric phase becomes

$$\Gamma = \int \rho(s,u)\,\mathrm{d}s\,\mathrm{d}u = 4\pi n, \tag{21}$$

where n is the Pontryagin index of the map, i.e., the number of times S^2 wraps around S^2.

Anholonomy density as the magnitude of a non-closure vector

The anholonomy density $\delta\Gamma$ can be viewed in yet another way. Determine the value of a unit vector $\mathbf{P}$ (where $\mathbf{P}$ stands for $\mathbf{t}$, $\mathbf{n}$, or $\mathbf{b}$) when it is Fermi–Walker transported along the path abc (path 1) and the path adc (path 2), respectively. (These paths have been defined in the beginning of this section.) These values are given by

$$\begin{aligned}\mathbf{P}_1(s+\mathrm{d}s, u+\mathrm{d}u) &= \mathbf{P}(s+\mathrm{d}s, u) + \frac{D\mathbf{P}}{Du}(s+\mathrm{d}s, u)\,\mathrm{d}u,\\ \mathbf{P}_2(s+\mathrm{d}s, u+\mathrm{d}u) &= \mathbf{P}(s, u+\mathrm{d}u) + \frac{D\mathbf{P}}{Ds}(s, u+\mathrm{d}u)\,\mathrm{d}s.\end{aligned} \tag{22}$$

Using Taylor expansion and retaining terms up to second order, we get

$$\delta\mathbf{P} = (\mathbf{P}_1 - \mathbf{P}_2) = \frac{D^2\mathbf{P}}{Ds\,Du} - \frac{D^2\mathbf{P}}{Du\,Ds}. \tag{23}$$

A non-vanishing $\delta\mathbf{P}$ means that $\mathbf{P}$ does not regain its original value when it is Fermi–Walker parallel transported along a closed path $abcda$ in configuration space. The anholonomy or geometric phase is simply the 'non-closure' vector $\delta\mathbf{P}$ associated with the $\mathbf{P}$-indicatrix on the unit sphere. Thus the incompatibility of Fermi–Walker cross derivatives is a measure of non-closure.

On the other hand, we can show that this incompatibility is related to the anholonomy density $\rho(s,u)$. Using eqs (11) and (15), we obtain

$$\delta\mathbf{t} = \frac{D^2\mathbf{t}}{Ds\,Du} - \frac{D^2\mathbf{t}}{Du\,Ds} = 0, \tag{24}$$

$$\delta \mathbf{n} = \frac{D^2\mathbf{n}}{Ds\,Du} - \frac{D^2\mathbf{n}}{Du\,Ds} = ((\tau_0)_s - \tau_u)\mathbf{b} = -\kappa h \mathbf{b} = \rho \mathbf{b}, \tag{25}$$

and

$$\delta \mathbf{b} = \frac{D^2\mathbf{b}}{Ds\,Du} - \frac{D^2\mathbf{b}}{Du\,Ds} = -((\tau_0)_s - \tau_u)\,\mathbf{n} = \kappa h \mathbf{n} = -\rho \mathbf{n}. \tag{26}$$

Thus the anholonomy density $\rho(s,u)$ is a measure of the non-closure of the normal indicatrix (or binormal indicatrix) on the surface S^2 of the unit sphere, when $\mathbf{n}$ (or $\mathbf{b}$) is Fermi–Walker transported along the infinitesimal closed circuit *abcda* in (s,u) space.

6. Connection between anholonomy and nonlinearity

In this section we present examples of two nonlinear systems that exhibit a close connection between the anholonomy of moving curves and nonlinearity.

A. *Landau–Lifshitz equation*

Spin vector dynamics in a classical Heisenberg ferromagnetic chain in the continuum limit is described by the Landau–Lifshitz equation [18]

$$\mathbf{S}_u = \mathbf{S} \times \mathbf{S}_{ss}. \tag{27}$$

Identifying $\mathbf{S}$ with the unit tangent to a curve, we see that the Landau–Lifshitz equation describes a moving space curve whose tangent vector satisfies

$$\mathbf{t}_u = \mathbf{t} \times \mathbf{t}_{ss}. \tag{28}$$

Imposing the moving curve compatibility conditions $\mathbf{P}_{su} = \mathbf{P}_{us}$ (where $\mathbf{P}$ stands for $\mathbf{t}$, $\mathbf{n}$ or $\mathbf{b}$), it can be shown [19] that eq. (28) takes the form of two coupled nonlinear partial differential equations for κ and τ, respectively. In terms of the Hasimoto function [2]

$$\psi = \kappa \exp\left\{ i \int^s \tau \, \mathrm{d}s \right\}, \tag{29}$$

these coupled equations can be combined and mapped to the nonlinear Schrödinger equation (NLSE) for ψ:

$$i\,\psi_u + \psi_{ss} + \frac{1}{2}|\psi|^2\,\psi = 0. \tag{30}$$

We consider the anholonomy density for the system. Using eqs (2), (20), (28) and (29), a short calculation shows that the anholonomy density of the moving curve underlying the nonlinear evolution equation (30) is just

$$\rho(s,\,u) = \mathbf{t} \cdot [\mathbf{t}_s \times (\mathbf{t} \times \mathbf{t}_{ss})] = \frac{1}{2}(\kappa^2)_s = \frac{1}{2}(|\psi|^2)_s\,. \tag{31}$$

Integrating over s, we get

$$\int \rho(s,u)\,\mathrm{d}s = \Gamma_u = \frac{1}{2}|\psi|^2. \tag{32}$$

This immediately shows that the coefficient of ψ in the nonlinear term of the NLSE for ψ (eq. (30)) is just the time derivative of the total anholonomy of the moving curve associated with the NLSE. This establishes a close connection between nonlinearity and anholonomy in the continuous ferromagnetic chain, which is an inherently nonlinear system.

B. *Belavin–Polyakov equation*

It has been shown recently [20] that a sector of the effective low-energy dynamics of the classical isotropic antiferromagnetic chain is described by the Belavin–Polyakov equation [21]

$$\mathbf{t}_u = \mathbf{t} \times \mathbf{t}_s, \quad \mathbf{t}^2 = 1, \tag{33}$$

where $\mathbf{t}(s,u)$ stands for the 'staggered' spin field on the chain. The appearance of the first-order spatial derivative on the right-hand side is noteworthy, in contrast to the second derivative which appears in the ferromagnetic case (eq. (28)). This leads to several dissimilarities in the dynamics of the two systems.

At a given instant of time u, the unit vector $\mathbf{t}(s,u)$ may be associated with the unit tangent $\mathbf{t}$ of a space curve. This curve evolves in time. Then, using eq. (1) in eq. (33), imposing compatibility conditions and following the same procedure as before, we obtain [22]

$$iq_u + q_s + q\int^s |q|^2 \mathrm{d}s = 0, \tag{34}$$

where the complex function q is once again a Hasimoto function, defined analogous to ψ as in eq. (29).

We compute the anholonomy density in this case with the help of eqs (33) and (20) to get

$$\rho(s,\, u) = \mathbf{t}\cdot(\mathbf{t}_s \times (\mathbf{t}\times\mathbf{t}_s)) = \kappa^2 = |q|^2, \tag{35}$$

where we have used eqs (2) and (29) and replaced ψ by q. Therefore $\Gamma = \int^s |q|^2\,\mathrm{d}s\,\mathrm{d}u$. Hence, as in the previous case, the coefficient of q in the nonlinear term of the Lamb equation (eq. (34)) is just the time derivative of the total anholonomy of the moving curve associated with it. We thus have a connection between nonlinearity and anholonomy in the continuous antiferromagnetic chain as well.

7. Conclusions

We have shown how the concept of anholonomy gets closely linked to nonlinearity. While we have worked out two examples explicitly for the sake of illustration, some general results indicating this connection have also been obtained [23], by starting with the general curve evolution equations (13) and imposing the compatibility conditions. For certain choices of g and h as functionals of κ and τ, these lead to integrable, soliton-bearing equations [24]. It is also possible to determine [23] the total anholonomy Γ explicitly for a sub-class of general AKNS-type [25] soliton equations. As in the two examples above, a term related to the time-derivative of this geometric phase (or anholonomy) appears in the partial differential equation for the Hasimoto function, for this wide class of moving curves as well. We also find that for a particular kind of evolution of the moving curve for which the associated partial differential equation becomes linear, the anholonomy density vanishes identically. This suggests that nonlinearity is essential for the existence of a non-trivial anholonomy. In the light of these results, it would be interesting to study possible anholonomy–nonlinearity relationships in non-integrable systems as well.

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