

## Equations of motion of compact binaries at the third post-Newtonian order

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**Abstract.** The equations of motion of two point masses in harmonic coordinates are derived through the third post-Newtonian (3PN) approximation. The problem of self-field regularization (necessary for removing the divergent self-field of point particles) is dealt with in two separate steps. In the first step the extended Hadamard regularization is applied, resulting in equations of motion which are complete at the 3PN order, except for the occurrence of one and only one unknown parameter. In the second step the dimensional regularization (in  $d$  dimensions) is used as a powerful argument for fixing the value of this parameter, thereby completing the 3-dimensional Hadamard-regularization result. The complete equations of motion and associated energy at the 3PN order are given in the case of circular orbits.

**Keywords.** Gravitational radiation; equations of motion; compact binaries; post-Newtonian.

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### 1. Introduction

The third post-Newtonian approximation of general relativity (in short 3PN)<sup>1</sup> became famous in recent years because of its frightening or, depending on one's state of mind, fascinating intricacy. In particular, the study of this approximation reached a somewhat paroxysmic stage when it was realized that the usual self-field regularization, based on Hadamard's concept of '*partie finie*' [2,3], although having proved to be very efficient up to the 2PN order, fails to provide a complete answer to the problem at the 3PN order. Indeed it seems to inexorably yield the appearance of some numerical coefficients which cannot be determined within the regularization.

Working at such a high approximation level as the 3PN one does not represent a purely academic exercise. The current network of laser-interferometric

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<sup>1</sup>Following the standard practice [1], we refer to  $n$ PN as the terms of the order of  $1/c^{2n}$  in the equations of motion, relatively to the Newtonian acceleration.

gravitational-wave detectors (notably the large-scale ones: LIGO and VIRGO) will soon make possible the study of the inspiral and coalescence of binary systems of neutron stars and black holes. To extract useful information from the gravitational waves, theoretical general-relativistic wave-forms are used as templates in these experiments, and it has been demonstrated that these must be extremely accurate, which means probably as accurate as the 3PN approximation [4–6]. To construct the 3PN templates one needs to control both the binary’s equations of motion, at the 3PN order relative to the Newtonian acceleration, *and* the gravitational radiation field, also consistent at 3PN order but with respect to the famous Einstein quadrupole formula, corresponding to the ‘Newtonian’ order in the wave-form.

In this paper we focus our attention on the problem of motion of a point mass binary system. The undetermined parameters which appear, due to Hadamard self-field regularization, are several, but, in fact, once one has invoked physical arguments to compute some of them, there remains *one and only one* unknown coefficient, the so-called ‘static’ ambiguity parameter  $\omega_s$  in the 3PN Hamiltonian in ADM coordinates [7,8], or, equivalently, the parameter denoted  $\lambda$  in the 3PN equations of motion in harmonic coordinates [9,10]. (We mean by physical arguments the requirement of invariance under global Poincaré transformations, and the demand that the equations of motion should be derivable from a Lagrangian (neglecting the 2.5 PN radiation reaction term).) These parameters are related to each other by [9,11,12]

$$\lambda = -\frac{3}{11}\omega_s - \frac{1987}{3080}. \quad (1)$$

On the other hand, concerning the radiation field, three other parameters,  $\xi$ ,  $\kappa$  and  $\zeta$ , coming from the Hadamard regularization of the 3PN quadrupole moment, appear [13]. There is, however, a single parameter which enters the orbital phase of inspiralling compact binaries, in the form of a linear combination of  $\theta \equiv \xi + 2\kappa + \zeta$  and  $\lambda$ . (Notice that  $\lambda$  enters the radiation field because of time differentiations of the 3PN quadrupole moment and replacement of the accelerations by the 3PN equations of motion.)

The regularization ‘ambiguities’, say  $\omega_s$  or  $\lambda$ , are not real physical ambiguities, which would arise, for instance, from some fundamental failure of the post-Newtonian expansion to approximate the physics of black holes at high order. Simply, they reflect some inconsistency, of mathematical origin, in the Hadamard regularization scheme, when it is applied to the computation of certain integrals at the 3PN order. Alternatively, one can say that this regularization, when ‘literally’ pushed to its maximum (in the way proposed in [14,15]), reveals some ‘incompleteness’ in making physical predictions, which can or cannot be removed by external physical arguments. Fortunately, we shall see that the ambiguity constant (1) can be resolved once one disposes of the appropriate mathematical tools for performing the regularization.

An improved version of the Hadamard regularization, defined in [14,15], is based on: (i) systematic use of ‘partie-finie’ pseudo-functions to represent the functions in the problem which are singular at the location of the particles; (ii) specific distributional derivatives generalizing those of the standard distribution theory [3] in order to differentiate the latter pseudo-functions; (iii) ‘Lorentzian’ way of performing the regularization, defined by the Hadamard partie finie calculated within the

Lorentzian rest frame of the particles. We shall refer to that regularization [14,15] as the ‘extended’ Hadamard (EH) one.

The EH regularization constitutes the first step of a complete calculation of the 3PN equations of motion [14,15]. The second step, aimed at removing the incompleteness  $\lambda$ , consists of going to  $d$ -dimensional space and using complex analytic continuation in  $d$ , in what is known as the *dimensional regularization* (henceforth abbreviated as ‘dimreg’)<sup>2</sup>. For the moment it is not possible to derive the 3PN equations of motion in any  $d$  dimensions, i.e. not necessarily of the form  $d = 3 + \varepsilon$ , where  $\varepsilon \rightarrow 0$ . This is why one still has to rely on the 3-dimensional calculation of the equations of motion by means of EH regularization. This second step (dimensional continuation in  $d$ ) has already been achieved in the context of the 3PN Hamiltonian in ADM coordinates, with result [17]

$$\omega_s = 0. \quad (2)$$

In the present contribution we describe our own application of dimreg (so to say ‘on the top’ of Hadamard’s regularization) to the derivation of the 3PN equations of motion, in the framework of harmonic coordinates, based on recent work in collaboration with Damour and Esposito-Farèse [18].

## 2. Hadamard regularization of Poisson-like integrals

Let us start by giving some reminders of the way we compute the Hadamard regularization of some potentials having the form of Poisson or Poisson-like integrals. Let  $F(\mathbf{x})$  be a smooth function on  $\mathbb{R}^3$ , except at the value of two singular points  $\mathbf{y}_1$  and  $\mathbf{y}_2$ , around which it admits some Laurent expansions of the type ( $\forall N \in \mathbb{N}$ )

$$F(\mathbf{x}) = \sum_{p_0 \leq p \leq N} r_1^p {}_1f_p(\mathbf{n}_1) + o(r_1^N), \quad (3)$$

where  $r_1 \equiv |\mathbf{x} - \mathbf{y}_1| \rightarrow 0$ , and the  ${}_1f_p(\mathbf{n}_1)$ ’s denote the coefficients of the various powers of  $r_1$ , which are functions of the positions and velocities of the particles, and of the unit direction  $\mathbf{n}_1 \equiv (\mathbf{x} - \mathbf{y}_1)/r_1$  of approach to singularity 1 (we also have the same expansion corresponding to the singularity 2). The powers of  $r_1$  are relative integers,  $p \in \mathbb{Z}$ , bounded from below by some typically negative  $p_0$  depending on the function  $F$ .

We shall discuss the prescription (taken in [14]) to define the ‘value at  $\mathbf{x}' = \mathbf{y}_1$ ’ of the singular Poisson integral  $P(\mathbf{x}')$  of the source function  $F(\mathbf{x})$ . The potential  $P(\mathbf{x}')$  is defined, at any field point  $\mathbf{x}'$  different from the singularities, in the sense of the Hadamard partie-finie (Pf) of an integral, i.e.

$$P(\mathbf{x}') = -\frac{1}{4\pi} \text{Pf}_{s_1, s_2} \int \frac{d^3\mathbf{x}}{|\mathbf{x} - \mathbf{x}'|} F(\mathbf{x}). \quad (4)$$

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<sup>2</sup>Dimreg was invented as a means to preserve the gauge symmetry of perturbative quantum gauge field theories [16].

This ‘partie finie’ involves two constants,  $s_1$  and  $s_2$ , which parametrize some logarithmic terms, and are associated with the characteristics of the regularizing volumes around the two particles, which have been excised from  $\mathbb{R}^3$  in order to define the partie finie by means of the limit, when the size of these volumes tends to zero, of the integral external to the volumes.

The value at  $\mathbf{x}' = \mathbf{y}_1$  of the function  $P(\mathbf{x}')$  is defined by the Hadamard partie finie in the singular limit  $\mathbf{x}' \rightarrow \mathbf{y}_1$ , given as usual by the angular average of the coefficient of the zeroth power of  $r'_1 \equiv |\mathbf{x}' - \mathbf{y}_1|$  when  $r'_1 \rightarrow 0$ . Notice first that  $P(\mathbf{x}')$  does not admit an expansion when  $r'_1 \rightarrow 0$  of the same type as in eq. (3), since it also involves a term proportional to the *logarithm* of  $r'_1$ . Thus we shall have, rather than a power-like expansion,

$$P(\mathbf{x}') = \sum_{p'_0 \leq p \leq N} r'_1{}^p [{}_1g_p(\mathbf{n}'_1) + {}_1h_p(\mathbf{n}'_1) \ln r'_1] + o(r'_1{}^N), \quad (5)$$

where the coefficients  ${}_1g_p$  and  ${}_1h_p$  depend on the angle  $\mathbf{n}'_1$ , and also on the constants  $s_1$  and  $s_2$ , in such a way that when combining together the terms in (5) the constant  $r'_1$  always appears in ‘adimensionalized’ form like in  $\ln(r'_1/s_1)$ . Then we define the Hadamard partie finie at point 1 in the standard way (taking the spherical average of the zeroth-order power of  $r'_1$ ), except that we include the contribution linked to the logarithm of  $r'_1$ , which is possibly present in that coefficient. More precisely, we define

$$(P)_1 \equiv \langle {}_1g_0 \rangle + \langle {}_1h_0 \rangle \ln r'_1, \quad (6)$$

where the brackets denote the angular average, over the solid angle element  $d\Omega(\mathbf{n}'_1)$  on the unit sphere. Let us emphasize that in (6) we have introduced in fact a *new regularization scale* denoted by  $r'_1$ , which can be seen as some ‘small’ but finite cut-off length scale (so that  $\ln r'_1$  in eq. (6) is a finite, but ‘large’ cut-off dependent contribution). To compute the partie finie one must apply the definition (6) to the Poisson integral (4), which involves evaluating correctly the angular integration therein. The result, proved in Theorem 3 of [14], reads

$$(P)_1 = -\frac{1}{4\pi} \text{Pf}_{s_1, s_2} \int \frac{d^3\mathbf{x}}{r_1} F(\mathbf{x}) + \left[ \ln \left( \frac{r'_1}{s_1} \right) - 1 \right] \langle {}_1f_{-2} \rangle. \quad (7)$$

The first term is simply the value of the potential at point 1, namely  $P(\mathbf{y}_1)$ , which would in fact constitute a ‘naïve’ way to implement the regularization, but would not yield 3PN equations of motion compatible with basic physical properties such as energy conservation. The supplementary term makes the partie finie to differ from the naïve guess  $P(\mathbf{y}_1)$  in a way which was found to play a significant role in the computations of [10]. The apparent dependence of the result (7) on the scale  $s_1$  is illusory. The  $s_1$ -dependence of the r.h.s. of eq. (7) cancels between the first and the second terms, so the result depends only on the constants  $r'_1$  and  $s_2$ , and we have a simpler form of (7), as

$$(P)_1 = -\frac{1}{4\pi} \text{Pf}_{r'_1, s_2} \int \frac{d^3\mathbf{x}}{r_1} F(\mathbf{x}) - \langle {}_1f_{-2} \rangle. \quad (8)$$

Similarly the regularization performed at point 2 will depend on  $r'_2$  and  $s_1$ , so that the binary's point-particle dynamics depends on four (*a priori* independent) length scales  $r'_1$ ,  $s_2$  and  $r'_2$ ,  $s_1$ . Because we work at the level of the equations of motion (instead of, say, the Lagrangian), many of the terms we shall need are in the form of the *gradient* of a Poisson potential. For the gradient we have a formula analogous to (8) and given by eq. (5.17a) of [14], namely

$$(\partial_i P)_1 = -\frac{1}{4\pi} \text{Pf}_{s_1, s_2} \int d^3\mathbf{x} \frac{n_1^i}{r_1^2} F(\mathbf{x}) + \ln\left(\frac{r'_1}{s_1}\right) \langle n_{11}^i f_{-1} \rangle \quad (9)$$

$$= -\frac{1}{4\pi} \text{Pf}_{r'_1, s_2} \int d^3\mathbf{x} \frac{n_1^i}{r_1^2} F(\mathbf{x}), \quad (10)$$

where we have taken into account (in the rewriting of the second line) the always correct fact that the constant  $s_1$  cancels out and gets ‘replaced’ by  $r'_1$ .

We must also treat the more general case of potentials in the form of retarded integrals, but since we have to consider (in §3 below) only the *difference* between dimreg and the Hadamard regularization, it will turn out that the first-order retardation (1PN relative order) is sufficient for this purpose. Here we are not interested in radiation-reaction effects, so we shall use the symmetric (half-retarded plus half-advanced) integral. At the 1PN order we thus have to evaluate

$$R(\mathbf{x}') = P(\mathbf{x}') + \frac{1}{2c^2} Q(\mathbf{x}') + \mathcal{O}\left(\frac{1}{c^4}\right), \quad (11)$$

where  $P(\mathbf{x}')$  is given by (4), and  $Q(\mathbf{x}')$  denotes (two times) the double or ‘twice-iterated’ Poisson integral of the second-time derivative, still endowed with a prescription of taking the Hadamard partie finie, namely

$$Q(\mathbf{x}') = -\frac{1}{4\pi} \text{Pf}_{s_1, s_2} \int d^3\mathbf{x} |\mathbf{x} - \mathbf{x}'| \partial_t^2 F(\mathbf{x}). \quad (12)$$

In the case of  $Q(\mathbf{x}')$  the results concerning the partie finie at point 1 were given by eqs (5.16) and (5.17b) of [14],

$$(Q)_1 = -\frac{1}{4\pi} \text{Pf}_{r'_1, s_2} \int d^3\mathbf{x} r_1 \partial_t^2 F(\mathbf{x}) + \frac{1}{2} \langle {}_1k_{-4} \rangle, \quad (13)$$

$$(\partial_i Q)_1 = \frac{1}{4\pi} \text{Pf}_{r'_1, s_2} \int d^3\mathbf{x} n_1^i \partial_t^2 F(\mathbf{x}) + \frac{1}{2} \langle n_{11}^i k_{-3} \rangle, \quad (14)$$

where the  ${}_1k_p$ ’s denote the analogues of the coefficients  ${}_1f_p$ , parametrizing the expansion of  $F$  when  $r'_1 \rightarrow 0$ , but corresponding to the double time-derivative  $\partial_t^2 F$  instead of  $F$ .

There is an important point concerning the treatment of the repeated time derivative  $\partial_t^2 F(\mathbf{x})$  in eq. (14). Since we are discussing about Hadamard-regularized integrals (which excise small balls around both  $\mathbf{y}_1$  and  $\mathbf{y}_2$ ), the value of  $\partial_t^2 F(\mathbf{x})$  can be simply taken in the sense of ordinary functions, i.e., without including

eventual ‘distributional’ contributions proportional to  $\delta(\mathbf{x} - \mathbf{y}_1)$  or  $\delta(\mathbf{x} - \mathbf{y}_2)$  and their derivatives. However, we know that such terms are necessary for the consistency of the calculation (without them, for instance, the calculation would be incorrect already at the 2PN order). In EH regularization, there is a specific prescription for the distributional derivative which is issued from the generalized framework of [14]. In dimreg we shall simply use the standard Schwartz distributional derivatives in  $d$  dimensions. (As it turns out, the Schwartz derivatives yield some ill-defined (formally infinite) expressions in three dimensions – this is why a generalization of the Schwartz distributional derivative defined in [14] was found to be necessary – but the latter expressions are proved to be rigorously zero when computed in  $d$  dimensions.) Therefore, in our computation of the *difference* between dimreg and Hadamard regularization in the next section, we must also include the difference between the different prescriptions for the distributional derivatives in  $d$  and in three dimensions. We refer to [18] for details.

### 3. Difference between the dimensional and Hadamard regularizations

In dimreg the computation of the regularized value of Poisson or Poisson-like integrals is very simple [18]. First of all, the generalization of the function  $F$  in  $d$  dimensions will be some  $F^{(d)}$  when  $r_1 \rightarrow 0$  which admits a more complicated expansion, namely ( $\forall N \in \mathbb{N}$ )

$$F^{(d)}(\mathbf{x}) = \sum_{\substack{p_0 \leq p \leq N \\ q_0 \leq q \leq q_1}} r_1^{p+q\varepsilon} {}_1f_{p,q}^{(\varepsilon)}(\mathbf{n}_1) + o(r_1^N), \quad (15)$$

where  $p$  and  $q$  are relative integers ( $p, q \in \mathbb{Z}$ ), whose values are limited by some  $p_0$ ,  $q_0$  and  $q_1$  as indicated. The expansion (15) differs from the corresponding expansion in three dimensions, as given in eq. (3), by the appearance of integer powers of  $r_1^\varepsilon$  where we denote  $\varepsilon \equiv d - 3$ . The coefficients  ${}_1f_{p,q}^{(\varepsilon)}$  depend on the unit vector  $\mathbf{n}_1$  in  $d$  dimensions, on the positions and coordinate velocities of the particles, and also on the characteristic length scale  $\ell_0$  of dimensional regularization. The latter can be introduced into the formalism by stating that the constant  $G$  in the r.h.s. of the Einstein field equations is related to  $G_N$ , the usual Newton constant in three dimensions, by  $G = G_N l_0^\varepsilon$ . Because  $F^{(d)} \rightarrow F$  when  $d \rightarrow 3$  we necessarily have some constraint on the coefficients  ${}_1f_{p,q}^{(\varepsilon)}$  so that we are in agreement with expansion (3) in this limit.

Consider now the Poisson integral of  $F^{(d)}$ , in  $d$  dimensions, given by the standard Green’s function for the Laplace operator in  $d$  dimensions, namely

$$P^{(d)}(\mathbf{x}') = \Delta^{-1}[F^{(d)}(\mathbf{x})] \equiv -\frac{\tilde{k}}{4\pi} \int \frac{d^d \mathbf{x}}{|\mathbf{x} - \mathbf{x}'|^{d-2}} F^{(d)}(\mathbf{x}), \quad (16)$$

where  $\tilde{k}$  is related to the usual Eulerian  $\Gamma$ -function by<sup>3</sup>

$$\tilde{k} = \frac{\Gamma\left(\frac{d-2}{2}\right)}{\pi^{\frac{d-2}{2}}}. \quad (17)$$

To evaluate the Poisson integral at the singular point  $\mathbf{x}' = \mathbf{y}_1$  is quite easy in dimreg, because the nice properties of analytic continuation simply allow to get  $[P^{(d)}(\mathbf{x}')]_{\mathbf{x}'=\mathbf{y}_1}$  by replacing  $\mathbf{x}'$  by  $\mathbf{y}_1$  into the explicit integral form (16). So, we simply have

$$P^{(d)}(\mathbf{y}_1) = -\frac{\tilde{k}}{4\pi} \int \frac{d^d \mathbf{x}}{r_1^{d-2}} F^{(d)}(\mathbf{x}). \quad (18)$$

Similarly, for the twice-iterated Poisson integral, and the relevant gradients of potentials,

$$Q^{(d)}(\mathbf{y}_1) = -\frac{\tilde{k}}{4\pi(4-d)} \int d^d \mathbf{x} r_1^{4-d} \partial_t^2 F^{(d)}(\mathbf{x}), \quad (19)$$

$$\partial_i P^{(d)}(\mathbf{y}_1) = -\frac{\tilde{k}(d-2)}{4\pi} \int d^d \mathbf{x} \frac{n_1^i}{r_1^{d-1}} F^{(d)}(\mathbf{x}), \quad (20)$$

$$\partial_i Q^{(d)}(\mathbf{y}_1) = \frac{\tilde{k}}{4\pi} \int d^d \mathbf{x} n_1^i r_1^{3-d} \partial_t^2 F^{(d)}(\mathbf{x}). \quad (21)$$

The main technical step of our strategy will then consist of computing the *difference* between the  $d$ -dimensional Poisson-type potentials (18)–(21), and their ‘pure Hadamard–Schwartz’ 3-dimensional counterparts, as given by (8). By pure Hadamard–Schwartz (pHS) we mean in some sense the ‘core’ of the Hadamard regularization, i.e. merely based on the usual notion of the *partie finie* of a singular function or a divergent integral, but without the improvements brought about by the EH regularization (see [18] for more details). For instance, the computations of §2 belong to the pHS regularization, but the special treatment of distributional derivatives in three dimensions is specific to the EH regularization. Given the results  $(P)_1$  and  $P^{(d)}(\mathbf{y}_1)$  of the two regularizations, denoting the difference by means of the script letter  $\mathcal{D}$ , we thus pose

$$\mathcal{D}P(1) \equiv P^{(d)}(\mathbf{y}_1) - (P)_1. \quad (22)$$

That is,  $\mathcal{D}P(1)$  is what we shall have to *add* to the pHS result in order to get the correct  $d$ -dimensional result. Note that, in this paper, we shall only compute the

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<sup>3</sup>We have  $\lim_{d \rightarrow 3} \tilde{k} = 1$ . Notice the following connection to the volume of the sphere with  $d-1$  dimensions (i.e., embedded into Euclidean  $d$ -dimensional space):

$$\tilde{k} \Omega_{d-1} = \frac{4\pi}{d-2}.$$

first two terms,  $a_{-1} \varepsilon^{-1} + a_0 + \mathcal{O}(\varepsilon)$ , of the Laurent expansion of  $\mathcal{D}P(1)$  when  $\varepsilon \rightarrow 0$ . (We leave to future work an eventual computation of the  $d$ -dimensional equations of motion as an exact function of the complex number  $d$ .) This is the information we need to fix the value of the parameter  $\lambda$ . As we shall see, the difference  $\mathcal{D}P(1)$  comes exclusively from the contribution of poles  $\propto 1/\varepsilon$  (and their associated finite part) in the  $d$ -dimensional calculation. Here we simply state the result without proof (see [18] for details). We obtain the following closed-form expression for the difference (valid up to the neglect of higher-order terms  $\mathcal{O}(\varepsilon)$ ),

$$\begin{aligned} \mathcal{D}P(1) = & -\frac{1}{\varepsilon(1+\varepsilon)} \sum_{q_0 \leq q \leq q_1} \left( \frac{1}{q} + \varepsilon [\ln r'_1 - 1] \right) \langle {}_1f_{-2,q}^{(\varepsilon)} \rangle \\ & -\frac{1}{\varepsilon(1+\varepsilon)} \sum_{q_0 \leq q \leq q_1} \left( \frac{1}{q+1} + \varepsilon \ln s_2 \right) \\ & \times \sum_{\ell=0}^{+\infty} \frac{(-)^\ell}{\ell!} \partial_L \left( \frac{1}{r_{12}^{1+\varepsilon}} \right) \langle n_2^L {}_2f_{-\ell-3,q}^{(\varepsilon)} \rangle, \end{aligned} \quad (23)$$

which constitutes the basis of all the practical calculations in the work [18]<sup>4</sup>. Here we still use the bracket notation to denote the angular average, but now performed in  $d$  dimensions, i.e. over the solid-angle element  $d\Omega_{d-1}$  associated with the  $(d-1)$ -dimensional sphere. Notice that (23) depends on the two ‘constants’  $\ln r'_1$  and  $\ln s_2$ . Thus  $\ln r'_1$  and  $\ln s_2$  will exactly cancel out the same constants present in the pHS calculation, so that the dimreg acceleration will be finally free of the constants  $r'_1$  and  $s_2$ . Note also that the coefficients  ${}_1f_{p,q}^{(\varepsilon)}$  and  ${}_2f_{p,q}^{(\varepsilon)}$  in  $d$  dimensions depend on the length scale  $\ell_0$  associated with dimreg. Taking this dependence into account one can verify that  $r'_1$  and  $s_2$  in (23) appear only in the combinations  $\ln(r'_1/\ell_0)$  and  $\ln(s_2/\ell_0)$ .

Let us also give the formula for the difference between the *gradients* of potentials, i.e.

$$\mathcal{D}\partial_i P(1) \equiv \partial_i P^{(d)}(\mathbf{y}_1) - (\partial_i P)_1, \quad (24)$$

which is readily obtained by the same method. We have

$$\begin{aligned} \mathcal{D}\partial_i P(1) = & -\frac{1}{\varepsilon} \sum_{q_0 \leq q \leq q_1} \left( \frac{1}{q} + \varepsilon \ln r'_1 \right) \langle n_1^i {}_1f_{-1,q}^{(\varepsilon)} \rangle \\ & -\frac{1}{\varepsilon(1+\varepsilon)} \sum_{q_0 \leq q \leq q_1} \left( \frac{1}{q+1} + \varepsilon \ln s_2 \right) \end{aligned}$$

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<sup>4</sup>With the same notation as in [18] the multipole expansion in  $d$  dimensions reads as

$$r_1^{2-d} = \sum_{\ell=0}^{+\infty} \frac{(-)^\ell}{\ell!} \partial_L \left( \frac{1}{r_{12}^{1+\varepsilon}} \right) r_2^\ell n_2^L$$



$$\times \sum_{\ell=0}^{+\infty} \frac{(-)^{\ell}}{\ell!} \partial_{iL} \left( \frac{1}{r_{12}^{1+\varepsilon}} \right) \langle n_2^L {}_2f_{-\ell-3,q}^{(\varepsilon)} \rangle. \quad (25)$$

Equations (23) and (25) correspond to the difference of Poisson integrals. But we have already discussed that we also need the difference of inverse d'Alembertian integrals at the 1PN order. To express as simply as possible the 1PN-accurate generalizations of eqs (23) and (25), let us define two *functionals*  $\mathcal{H}$  and  $\mathcal{H}_i$  which are such that their actions on any  $d$ -dimensional function  $F^{(d)}$  is given by the r.h.s.'s of eqs (23) and (25), i.e., so that

$$\mathcal{D}P(1) = \mathcal{H}[F^{(d)}], \quad (26)$$

$$\mathcal{D}\partial_i P(1) = \mathcal{H}_i[F^{(d)}]. \quad (27)$$

The difference of 1PN-retarded potentials and gradients of potentials is denoted as

$$\mathcal{D}R(1) \equiv R^{(d)}(\mathbf{y}_1) - (R)_1, \quad (28)$$

$$\mathcal{D}\partial_i R(1) \equiv \partial_i R^{(d)}(\mathbf{y}_1) - (\partial_i R)_1, \quad (29)$$

where in three dimensions the potential  $R(\mathbf{x}')$  is defined by eq. (11) and the regularized values  $(R)_1$  and  $(\partial_i R)_1$  follow from (8), (10), (14), and in  $d$  dimensions  $R^{(d)}(\mathbf{y}_1)$  and  $\partial_i R^{(d)}(\mathbf{y}_1)$  are consequences of eqs (18)–(21). With this notation we now have our result, that the difference in the case of such 1PN-expanded potentials reads in terms of the above defined functionals  $\mathcal{H}$  and  $\mathcal{H}_i$  as

$$\begin{aligned} \mathcal{D}R(1) = \mathcal{H} \left[ F^{(d)} + \frac{r_1^2}{2c^2(4-d)} \partial_t^2 F^{(d)} \right] \\ - \frac{3}{4c^2} \langle {}_1k_{-4} \rangle + \mathcal{O} \left( \frac{1}{c^4} \right), \end{aligned} \quad (30)$$

$$\begin{aligned} \mathcal{D}\partial_i R(1) = \mathcal{H}_i \left[ F^{(d)} - \frac{r_1^2}{2c^2(d-2)} \partial_t^2 F^{(d)} \right] \\ - \frac{1}{4c^2} \langle n_1^i {}_1k_{-3} \rangle + \mathcal{O} \left( \frac{1}{c^4} \right). \end{aligned} \quad (31)$$

These formulae involve some ‘effective’ functions which are to be inserted into the functional brackets of  $\mathcal{H}$  and  $\mathcal{H}_i$ . Beware of the fact that the effective functions are not the same in the cases of a potential and the gradient of that potential. Note the presence, besides the main terms  $\mathcal{H}[\dots]$  and  $\mathcal{H}_i[\dots]$ , of some extra terms, purely of order 1PN, in eqs (30), (31). These terms are the average of some coefficients  ${}_1k_p$  of the powers  $r_1^p$  in the expansion when  $r_1 \rightarrow 0$  of the *second-time derivative* of  $F$ , namely  $\partial_t^2 F$ . They do not seem to admit a simple interpretation. They are important to get the final correct result.

#### 4. Dimensional regularization of the equations of motion

We outline next the way we obtain from the previous computation of the ‘difference’ the 3PN equations of motion in dimreg, and show how they are physically equivalent to the EH-regularized equations of motion. We start from the end results of [10] for the 3PN acceleration of the first particle, say  $\mathbf{a}_1^{\text{BF}}$ , depending on the two arbitrary length scales  $r'_1$  and  $r'_2$  (appearing when regularizing Poisson-like integrals in §2), and on the ‘ambiguity’ parameter  $\lambda$ . Explicitly, we define

$$\mathbf{a}_1^{\text{BF}}[\lambda; r'_1, r'_2] \equiv \text{r.h.s. of eq. (7.16) in ref. [10]}. \quad (32)$$

Here the acceleration is considered as a function of the two masses  $m_1$  and  $m_2$ , the relative distance  $\mathbf{y}_1 - \mathbf{y}_2 \equiv r_{12} \mathbf{n}_{12}$  (where  $\mathbf{n}_{12}$  is the unit vector directed from particle 2 to particle 1), the two coordinate velocities  $\mathbf{v}_1$  and  $\mathbf{v}_2$ , and also, as emphasized in (32), the parameter  $\lambda$  as well as two regularization length scales  $r'_1$  and  $r'_2$ . The latter length scales enter the equations of motion at the 3PN level through the logarithms  $\ln(r_{12}/r'_1)$  and  $\ln(r_{12}/r'_2)$ . They come from the regularization as the field point  $\mathbf{x}'$  tends to  $\mathbf{y}_1$  or  $\mathbf{y}_2$  of Poisson-type integrals (see §2 above). The length scales  $r'_1$ ,  $r'_2$  are ‘pure gauge’ in the sense that they can be removed by the effect induced on the world-lines of a coordinate transformation of the bulk metric [10].

On the other hand, the dimensionless parameter  $\lambda$  entering the final result (32) corresponds to genuine physical effects. It was introduced by requiring that the 3PN equations of motion admit a conserved energy (and more generally be derivable from a Lagrangian). This extra requirement imposed *two relations* between the two length scales  $r'_1$ ,  $r'_2$  and the two other length scales  $s_1$ ,  $s_2$  entering originally into the formalism, namely the constants  $s_1$  and  $s_2$  parametrizing the Hadamard partie finie of a Poisson integral as given by eq. (4) above. Recall that  $s_1$  and  $s_2$  are associated with the characteristics of the two regularizing volumes (notably their shape) around the singularities, which are excised in order to define the Hadamard partie finie of a divergent integral. The latter relations were found to be of the form

$$\ln\left(\frac{r'_2}{s_2}\right) = \frac{159}{308} + \lambda \frac{m_1 + m_2}{m_2} \quad (33)$$

(and  $1 \leftrightarrow 2$ ), where the so-introduced *single* dimensionless parameter  $\lambda$  has been proved to be a purely numerical coefficient (i.e. independent of the two masses). It is often convenient to insert eq. (33) into (32) and to re-express the acceleration of particle 1 in terms of the original regularization length scales entering the Hadamard regularization of  $\mathbf{a}_1$ , which were in fact  $r'_1$  and  $s_2$  (as shown, for instance, in eq. (8)). Thus we can consider alternatively

$$\mathbf{a}_1^{\text{BF}}[r'_1, s_2] \equiv \mathbf{a}_1^{\text{BF}}[\lambda; r'_1, r'_2(s_2, \lambda)], \quad (34)$$

where the regularization constants are subject to the constraints (33) (we can check that the  $\lambda$ -dependence on the r.h.s. of (34) disappears when using eq. (33) to replace  $r'_2$  as a function of  $s_2$  and  $\lambda$ ).

The strategy followed in [18] consists of *two steps*. The *first step* consists of subtracting all the extra contributions to eq. (32), or equivalently eq. (34), which were specific consequences of the EH regularization defined in [14,15]. As has been

detailed in [18], there are *seven* such extra contributions  $\delta^A \mathbf{a}_1$ ,  $A = 1, \dots, 7$ . Subtracting these contributions boils down to estimating the value of  $\mathbf{a}_1$  that would be obtained by using a ‘pure’ Hadamard regularization, together with Schwartz distributional derivatives, which is what we have already called the ‘pure Hadamard–Schwartz’ (pHS) regularization. Such a pHS acceleration was in fact essentially the result of the first stage of calculation of  $\mathbf{a}_1$ , as reported in the unpublished thesis [19]. It is given by

$$\mathbf{a}_1^{\text{pHS}}[r'_1, s_2] = \mathbf{a}_1^{\text{BF}}[r'_1, s_2] - \sum_{A=1}^7 \delta^A \mathbf{a}_1. \quad (35)$$

The *second step* consists of evaluating the Laurent expansion, in powers of  $\varepsilon = d-3$ , the difference between the dimreg and pHS (3-dimensional) computations of the acceleration  $\mathbf{a}_1$ . As we have seen in §3, this difference makes a contribution only when a term generates a *pole*  $\sim 1/\varepsilon$ , in which case dimreg adds an extra contribution, made of the pole and the finite part associated with the pole (we consistently neglect all terms  $\mathcal{O}(\varepsilon)$ ). One must then be especially wary of combinations of terms whose pole parts finally cancel (‘cancelled poles’) but whose dimensionally regularized finite parts generally do not, and must be evaluated with care. We denote the above-defined difference

$$\mathcal{D}\mathbf{a}_1 = \mathcal{D}\mathbf{a}_1[\varepsilon, \ell_0; r'_1, s_2] \equiv \mathcal{D}\mathbf{a}_1[\varepsilon, \ell_0; \lambda; r'_1, r'_2]. \quad (36)$$

It depends on both the Hadamard regularization scales  $r'_1$  and  $s_2$  (or equivalently on  $\lambda$  and  $r'_1, r'_2$ ) and on the regularizing parameters of dimreg, namely  $\varepsilon$  and the characteristic length  $\ell_0$ . It is made of the sum of all the individual contributions of the Poisson or Poisson-like integrals as computed in §3 above (e.g. eqs (23) and (25)). Finally, our main result will be the explicit computation of the  $\varepsilon$ -expansion of the dimreg acceleration as

$$\mathbf{a}_1^{\text{dimreg}}[\varepsilon, \ell_0] = \mathbf{a}_1^{\text{pHS}}[r'_1, s_2] + \mathcal{D}\mathbf{a}_1[\varepsilon, \ell_0; r'_1, s_2]. \quad (37)$$

With this result in hand, we have proved [18] two theorems.

**Theorem 1.** *The pole part  $\propto 1/\varepsilon$  of the dimreg acceleration (37), as well as of the metric field  $g_{\mu\nu}(x)$  outside the particles, can be re-absorbed (i.e., renormalized away) into some shifts of the two ‘bare’ world-lines:  $\mathbf{y}_a \rightarrow \mathbf{y}_a + \boldsymbol{\xi}_a$ , with, say,  $\boldsymbol{\xi}_a \propto 1/\varepsilon$  (minimal subtraction (MS)), so that the result, expressed in terms of the ‘dressed’ quantities, is finite when  $\varepsilon \rightarrow 0$ .*

The situation in harmonic coordinates is to be contrasted with the calculation in ADM-type coordinates within the Hamiltonian formalism [17], where it was shown that all pole parts directly cancel out in the total 3PN Hamiltonian (no shifts of the world-lines were needed). The central result of the paper is as follows:

**Theorem 2.** *The ‘renormalized’ (finite) dimreg acceleration is physically equivalent to the EH-regularized acceleration (end result of ref. [10]), in the sense that there exist some shift vectors  $\boldsymbol{\xi}_1(\varepsilon, \ell_0; r'_1)$  and  $\boldsymbol{\xi}_2(\varepsilon, \ell_0; r'_2)$ , such that*

$$\mathbf{a}_1^{\text{BF}}[\lambda, r'_1, r'_2] = \lim_{\varepsilon \rightarrow 0} [\mathbf{a}_1^{\text{dimreg}}[\varepsilon, \ell_0] + \delta_{\xi(\varepsilon, \ell_0; r'_1, r'_2)} \mathbf{a}_1], \quad (38)$$

(where  $\delta_{\xi} \mathbf{a}_1$  denotes the effect of the shifts on the acceleration<sup>5</sup>, if and only if the heretofore unknown parameter  $\lambda$  entering the harmonic-coordinates equations of motion takes the value

$$\lambda = -\frac{1987}{3080}. \quad (39)$$

The precise shifts  $\xi_a(\varepsilon)$  needed in Theorem 4. involve not only a pole contribution  $\propto 1/\varepsilon$ , which defines the ‘minimal’ (MS) shifts considered in Theorem 4., but also a finite contribution when  $\varepsilon \rightarrow 0$ . Their explicit expressions read:

$$\xi_1 = \frac{11}{3} \frac{G_N^2 m_1^2}{c^6} \left[ \frac{1}{\varepsilon} - 2 \ln \left( \frac{r'_1 \bar{q}^{1/2}}{\ell_0} \right) - \frac{327}{1540} \right] \mathbf{a}_{N1} \quad \text{and} \quad 1 \leftrightarrow 2, \quad (40)$$

where  $G_N$  is the usual Newton’s constant,  $\mathbf{a}_{N1}$  denotes the acceleration of particle 1 (in  $d$  dimensions) at the Newtonian level, and  $\bar{q} \equiv 4\pi e^C$  depends on the Euler constant  $C = 0.577 \dots$

An alternative way to phrase the result (38)–(39), is to combine eqs (35) and (37) in order to arrive at

$$\lim_{\varepsilon \rightarrow 0} [\mathcal{D}\mathbf{a}_1[\varepsilon, \ell_0; -\frac{1987}{3080}; r'_1, r'_2] + \delta_{\xi(\varepsilon, \ell_0; r'_1, r'_2)} \mathbf{a}_1] = \sum_{A=1}^7 \delta^A \mathbf{a}_1. \quad (41)$$

Under this form one sees that the sum of the additional terms  $\delta^A \mathbf{a}_1$  differs by a mere shift, *if and only if*  $\lambda$  takes the value (39), from the specific contribution  $\mathcal{D}\mathbf{a}_1$ , which comes directly from dimreg. Therefore one can say that, when  $\lambda = -\frac{1987}{3080}$ , the EH regularization [14,15] is in fact (physically) equivalent to dimreg. However the EH regularization is incomplete, both because it is *a priori* unable to determine  $\lambda$ , and also because it necessitates some ‘external’ requirements such as the imposition of the link (33) in order to ensure the existence of a conserved energy – and in fact of the ten first integrals linked to the Poincaré group. By contrast, dimreg succeeds automatically (without extra inputs) in guaranteeing the existence of the ten conserved integrals of the Poincaré group, as already found in ref. [17].

In view of the necessary link (1) provided by the equivalence between the ADM-Hamiltonian and the harmonic-coordinate equations of motion, our result (39) is in perfect agreement with the result  $\omega_s = 0$  obtained in [17]. (One may wonder why the value of  $\lambda$  is a complicated rational fraction while  $\omega_s$  is so simple. This is because  $\omega_s$  was introduced precisely to measure the amount of ambiguities of certain integrals, while, by contrast,  $\lambda$  has been introduced as the only possible

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<sup>5</sup>When working at the level of the equations of motion (not considering the metric outside the world-lines), the effect of shifts can be seen as being induced by a coordinate transformation of the bulk metric as in ref. [10]

unknown constant in the link between the four arbitrary scales  $r'_1, r'_2, s_1, s_2$  (which has *a priori* nothing to do with ambiguities of integrals), in a framework where the use of the EH regularization makes in fact the calculation to be unambiguous.) Besides the confirmation of the value of  $\omega_s$  or  $\lambda$ , this result provides a confirmation of the *consistency* of dimreg, because our explicit calculations are entirely different from the ones of [17]. We use harmonic coordinates (instead of ADM-type ones), and work at the level of the equations of motion (instead of the Hamiltonian), since we use a different form of Einstein's field equations and we solve them by a different iteration scheme. Our result is also in agreement with the recent finding of refs [20,21] (see also [22]), where the 3PN equations of motion are derived in harmonic gauge using a 'surface-integral' approach, aimed at describing *extended* relativistic compact binary systems in the strong-field point particle limit.

## 5. Equations of motion of circular-orbit compact binaries

From a practical point of view, the determination of the value of  $\lambda$  allows one to use the full 3PN accuracy in the analytical computation of the dynamics of the last orbits of binary systems [23,24]. We assume a circular orbit since most inspiralling compact binaries will have been circularized at the time when they enter the frequency bandwidth of the detectors LIGO and VIRGO. In the case of circular orbits – apart from the gradual 2.5PN radiation-reaction inspiral – the quite complicated equations of motion, eq. (7.16) in ref. [10], simplify drastically.

We translate the origin of coordinates to the binary's center-of-mass by imposing that the binary's center-of-mass vector, deduced from the Lagrangian formulation of the 3PN equations of motion, is zero (see e.g. ref. [25]). Then, in the center-of-mass frame, the relative acceleration  $\mathbf{a}_{12} \equiv \mathbf{a}_1 - \mathbf{a}_2$  of two bodies moving on a circular orbit at the 3PN order is given by

$$\mathbf{a}_{12} = -\omega^2 \mathbf{y}_{12} - \frac{32}{5} \frac{G^3 m^3 \nu}{c^5 r_{12}^4} \mathbf{v}_{12} + \mathcal{O}\left(\frac{1}{c^7}\right), \quad (42)$$

where  $\mathbf{y}_{12} \equiv \mathbf{y}_1 - \mathbf{y}_2$  is the relative separation (in harmonic coordinates) and  $\omega$  denotes the angular frequency of the circular motion; the second term in eq. (42), opposite to the velocity  $\mathbf{v}_{12} \equiv \mathbf{v}_1 - \mathbf{v}_2$ , is the 2.5PN radiation reaction force. In (42) we have introduced, in addition to the total mass  $m = m_1 + m_2$ , the symmetric mass ratio

$$\nu \equiv \frac{m_1 m_2}{m^2}, \quad (43)$$

which is generally very useful because of its interesting range of variation  $0 < \nu \leq \frac{1}{4}$ , with  $\nu = \frac{1}{4}$  in the case of equal masses, and  $\nu \rightarrow 0$  in the 'test-mass' limit for one of the bodies. The main content of the 3PN equations (42) is the relation between the frequency  $\omega$  and the orbital separation  $r_{12}$ , that we find to be given by the 3PN-generalized 'Kepler's third law' [9,10]

$$\begin{aligned} \omega_{3\text{PN}}^2 = \frac{Gm}{r_{12}^3} & \left\{ 1 + (-3 + \nu)\gamma + \left( 6 + \frac{41}{4}\nu + \nu^2 \right) \gamma^2 \right. \\ & + \left( -10 + \left[ -\frac{75707}{840} + \frac{41}{64}\pi^2 + 22 \ln \left( \frac{r_{12}}{r'_0} \right) \right] \nu \right. \\ & \left. \left. + \frac{19}{2}\nu^2 + \nu^3 \right) \gamma^3 \right\}, \end{aligned} \quad (44)$$

in which we employ, in order to display the successive post-Newtonian corrections, the post-Newtonian parameter (of the order of  $\mathcal{O}(1/c^2)$ )

$$\gamma = \frac{Gm}{r_{12}c^2}. \quad (45)$$

The acceleration (42)–(45) is entirely specified at the 3PN order except for some unphysical *gauge freedom*, parametrized by the length scale  $r'_0$  appearing in eq. (44), which is nothing but the ‘logarithmic barycenter’ of the two gauge-constants  $r'_1$  and  $r'_2$  entering the end results of [10], i.e.

$$\ln r'_0 = \frac{m_1}{m} \ln r'_1 + \frac{m_2}{m} \ln r'_2. \quad (46)$$

As for the binary’s energy (in the center-of-mass frame), it is readily obtained from the circular-orbit reduction of the conserved energy associated with the 3PN Lagrangian in harmonic coordinates [12]. We find

$$\begin{aligned} E_{3\text{PN}} = -\frac{\mu c^2 \gamma}{2} & \left\{ 1 + \left( -\frac{7}{4} + \frac{1}{4}\nu \right) \gamma + \left( -\frac{7}{8} + \frac{49}{8}\nu + \frac{1}{8}\nu^2 \right) \gamma^2 \right. \\ & + \left( -\frac{235}{64} + \left[ \frac{46031}{2240} - \frac{123}{64}\pi^2 + \frac{22}{3} \ln \left( \frac{r_{12}}{r'_0} \right) \right] \nu \right. \\ & \left. \left. + \frac{27}{32}\nu^2 + \frac{5}{64}\nu^3 \right) \gamma^3 \right\}. \end{aligned} \quad (47)$$

This expression is that of a physical observable  $E$ . However it depends on the choice of a coordinate system, because it involves the post-Newtonian parameter  $\gamma$  defined from the harmonic-coordinate separation  $r_{12}$ . But the *numerical* value of  $E$  should not depend on the choice of a coordinate system, so  $E$  must admit a frame-invariant expression, the same in all coordinate systems. To find it we re-express  $E$  with the help of a frequency-related parameter  $x$  instead of the separation-related parameter  $\gamma$  (this is always a good thing to do). We define  $x$  to be, like for  $\gamma$ , of the order of  $\mathcal{O}(1/c^2)$  by posing

$$x = \left( \frac{Gm\omega}{c^3} \right)^{2/3}. \quad (48)$$

Then we readily obtain the expression of  $\gamma$  in terms of  $x$  at 3PN order,

$$\begin{aligned} \gamma_{3\text{PN}} = x & \left\{ 1 + \left(1 - \frac{\nu}{3}\right) x + \left(1 - \frac{65}{12}\nu\right) x^2 \right. \\ & + \left(1 + \left[-\frac{2203}{2520} - \frac{41}{192}\pi^2 - \frac{22}{3}\ln\left(\frac{r_{12}}{r'_0}\right)\right] \nu \right. \\ & \left. \left. + \frac{229}{36}\nu^2 + \frac{1}{81}\nu^3\right) x^3 \right\}, \end{aligned} \quad (49)$$

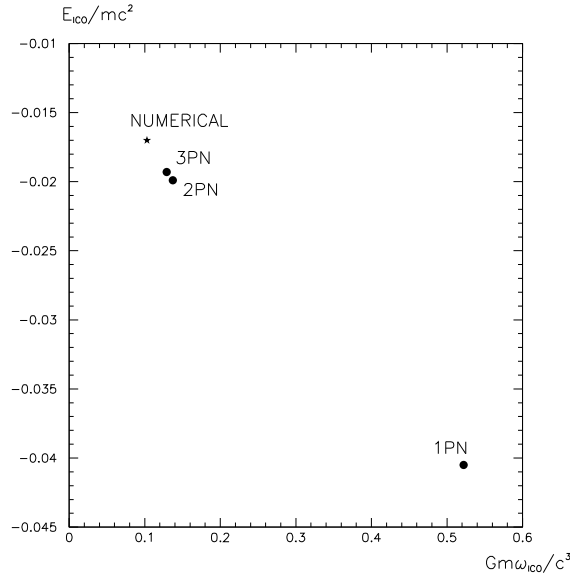
that we substitute back into eq. (47), making all appropriate post-Newtonian re-expansions. As a result we discover that the logarithms together with their associated gauge constant  $r'_0$  have cancelled out. Therefore our final result is

$$\begin{aligned} E_{3\text{PN}} = -\frac{\mu c^2 x}{2} & \left\{ 1 + \left(-\frac{3}{4} - \frac{1}{12}\nu\right) x + \left(-\frac{27}{8} + \frac{19}{8}\nu - \frac{1}{24}\nu^2\right) x^2 \right. \\ & + \left(-\frac{675}{64} + \left[\frac{34445}{576} - \frac{205}{96}\pi^2\right] \nu \right. \\ & \left. \left. - \frac{155}{96}\nu^2 - \frac{35}{5184}\nu^3\right) x^3 \right\}. \end{aligned} \quad (50)$$

In the test-mass limit  $\nu \rightarrow 0$ , we recover the energy of a particle with mass  $\mu = m\nu$  in a Schwarzschild background of mass  $m$ , i.e.  $E_{\text{test}} = \mu c^2 [(1 - 2x)(1 - 3x)^{-1/2} - 1]$ , when developed to 3PN order. Of course, the subtleties we have discussed, linked with the self-field regularization, disappear in the test-mass limit, but, interestingly enough, they affect only the term proportional to  $\nu$  in the 3PN coefficient of eq. (50); the terms proportional to  $\nu^2$  and  $\nu^3$  in eq. (50) have been found to be ‘complete’ in EH regularization.

Finally let us compute the innermost circular orbit (ICO) of point-particle binaries through 3PN order, following [24]. The ICO is defined as the minimum, when it exists, of the binary’s circular-orbit energy function (50). In particular, we do not define the ICO as a point of dynamical (general-relativistic) instability (See §6 of [25] for a discussion of the dynamical instability in the post-Newtonian framework.) In figure 1, we plot  $E_{\text{ICO}}$  versus  $\omega_{\text{ICO}}$  in the case of equal masses ( $\nu = \frac{1}{4}$ ), and compare the values with the recent finding of numerical relativity, obtained by means of a sequence of quasi-equilibrium configurations under the assumptions of helical symmetry and conformal flatness [26,27]. As we can see, the 2PN and 3PN points are rather close to each other and to the numerical value. However, the 1PN approximation is clearly not precise enough, but this is not very surprising in the highly relativistic regime of the ICO where the orbital velocity reaches  $v/c \sim (Gm\omega_{\text{ICO}}/c^3)^{1/3} \sim 0.5$ . A striking fact from figure 1 is that the post-Newtonian series seems to ‘converge well’, but actually the series could be only asymptotic (hence divergent), and, of course, still give excellent results provided that the series is truncated near some optimal order of approximation.

Our conclusions, therefore, are that (1) the post-Newtonian approximation is likely to be valid and quite accurate in the regime of the ICO (in the equal-mass case) and (2) it is in good agreement with the result of numerical relativity. Note that the conclusion (1) contradicts some earlier prejudices about the slow convergence of the post-Newtonian approximation (see e.g. ref. [4]). Furthermore, our computations are based on the standard post-Newtonian expansion, without using



**Figure 1.** Results for  $E_{\text{ICO}}$  in terms of  $\omega_{\text{ICO}}$  for equal-mass binaries ( $\nu = \frac{1}{4}$ ). The asterisk marks the result calculated by numerical relativity.

any resummation techniques such as Padé approximants and/or effective one-body method. For recent comparisons of the post-Newtonian and numerical calculations in the regime of the ICO, including finite-size effects appropriate to neutron-star binaries, see [28,29].

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