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Bianchi Type-I bulk viscous fluid string dust magnetized cosmological model in general relativity

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Abstract. Bianchi Type-I magnetized bulk viscous fluid string dust cosmological model is investigated. To get a determinate model, we have assumed the conditions $\sigma \propto \theta$ and $\zeta\theta =$ constant where σ is the shear, θ the expansion in the model and ζ the coefficient of bulk viscosity. The behaviour of the model in the presence and absence of magnetic field together with physical and geometrical aspects of the model are also discussed.

Keywords. Bianchi I; bulk viscous; magnetized string cosmological.

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1. Introduction

The general relativistic treatment of strings was initiated by Letelier [1] and Stachel [2]. It is interesting to note that magnetic field present in galactic and inter galactic spaces plays a significant role at cosmological scale. Melvin [3] in the cosmological solution for dust and electromagnetic field suggested that during the evolution of the universe, the matter was in highly ionized state and is smoothly coupled with the field and consequently form a neutral matter as a result of universe expansion. Hence in string dust universe the presence of magnetic field is not unrealistic. Banerjee *et al* [4] investigated an axially symmetric Bianchi Type-I string dust cosmological model in the presence and absence of magnetic field using a supplementary condition $\alpha = a\beta$ where $\alpha = \alpha(t)$, $\beta = \beta(t)$ and a is a constant. The string cosmological models with magnetic field are also investigated by Chakraborty [5] and Tikekar and Patel [6]. Recently Bali and Upadhaya [7] investigated LRS Bianchi Type-I bulk viscous fluid string cosmological model in General Relativity. To get a determinate model, it has been assumed that $\sigma \propto \theta$ where σ is the shear and θ the expansion in the model. Bali and Upadhaya [8] have also investigated Bianchi Type-I magnetized string cosmological model in General Relativity.

In this paper, we have investigated Bianchi Type-I magnetized bulk viscous fluid string dust cosmological model in General Relativity. To get a determinate model, we have assumed the condition $\sigma \propto \theta$ and $\zeta\theta =$ constant where σ is the shear, θ the expansion in the model and ζ the coefficient of bulk viscosity. The behaviour of the model in the presence and absence of magnetic field together with geometrical and physical aspects of the model are also discussed. We consider Bianchi Type-I metric in the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + A^2\mathrm{d}x^2 + B^2\mathrm{d}y^2 + C^2\mathrm{d}z^2 \tag{1.1}$$

where A, B and C are functions of t alone.

The energy momentum T_i^j for string dust is given by

$$T_i^j = \varepsilon v_i v^j - \lambda x_i x^j - \zeta v^\ell_{;\ell}(g_i^j + v_i v^j) + E_i^j \tag{1.2}$$

together with

$$v^i v_i = -x^i x_i = -1 \tag{1.3}$$

and

$$v^i x_i = 0, \tag{1.4}$$

where $\varepsilon = \varepsilon_{\mathrm{p}} + \lambda$ is the rest energy density for a cloud of strings. Here ε_{p} is the particle density, λ the string tension density, v^i the flow velocity vector, x^i the direction of strings and ζ the coefficient of bulk viscosity. Here E_{ij} is the electromagnetic field given by Lichnerowicz [9]

$$E_{ij} = \bar{\mu}\left[|h|^2\left(v_i v_j + \frac{1}{2}g_{ij}\right) - h_i h_j\right], \tag{1.5}$$

where v_i is the flow vector satisfying

$$g_{ij}v^i v^j = -1, \tag{1.6}$$

$\bar{\mu}$ is the magnetic permeability and h_i the magnetic flux vector defined by

$$h_i = \frac{\sqrt{-g}}{2\bar{\mu}}\varepsilon_{ijk\ell}F^{k\ell}v^j, \tag{1.7}$$

where $F_{k\ell}$ is the electromagnetic field tensor and $\varepsilon_{ijk\ell}$ is the Levi Civita tensor density. We assume the coordinates to be comoving so that

$$v^1 = 0 = v^2 = v^3, \quad v^4 = 1. \tag{1.8}$$

The incident magnetic field is taken along x-axis so that

$$h_1 \neq 0, \quad h_2 = 0 = h_3 = h_4. \tag{1.9}$$

The first set of Maxwell's equation

$$F_{ij;k} + F_{jk;i} + F_{ki;j} = 0 \tag{1.10}$$

leads to

$$F_{23} = \text{constant} = H(\text{say}). \tag{1.11}$$

Here $F_{14} = 0 = F_{24} = F_{34}$ due to assumption of infinite electrical conductivity. The only non-vanishing component of F_{ij} is F_{23}. Hence

$$h_1 = \frac{AH}{\bar{\mu}BC} \tag{1.12}$$

since

$$|h|^2 = h_\ell h^\ell = h_1 h^1 = g^{11}(h_1)^2 \tag{1.13}$$

and

$$|h|^2 = \frac{H^2}{\bar{\mu}^2 B^2 C^2}. \tag{1.14}$$

From eq. (1.5)

$$E_1^1 = -\frac{H^2}{2\bar{\mu}B^2C^2} = -E_2^2 = -E_3^3 = E_4^4. \tag{1.15}$$

Equation (1.2) leads to

$$T_1^1 = -\lambda - \frac{H^2}{2\bar{\mu}B^2C^2} - \zeta v^\ell_{;\ell}, \tag{1.16}$$

$$T_2^2 = \frac{H^2}{2\bar{\mu}B^2C^2} - \zeta v^\ell_{;\ell}, \tag{1.17}$$

$$T_3^3 = \frac{H^2}{2\bar{\mu}B^2C^2} - \zeta v^\ell_{;\ell} \tag{1.18}$$

$$T_4^4 = -\left(\varepsilon + \frac{H^2}{2\bar{\mu}B^2C^2}\right). \tag{1.19}$$

Using $(8\pi G/c^4) = 1$, the Einstein field equation

$$R_i^j - \frac{1}{2}Rg_i^j = -T_i^j \tag{1.20}$$

the line element (1.1) reduces to

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4C_4}{BC} = \lambda + \frac{H^2}{2\bar{\mu}B^2C^2} + \zeta v^\ell_{;\ell}, \tag{1.21}$$

$$\frac{A_{44}}{A} + \frac{C_{44}}{C} + \frac{A_4C_4}{AC} = -\frac{H^2}{2\bar{\mu}B^2C^2} + \zeta v^\ell_{;\ell}, \tag{1.22}$$

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4B_4}{AB} = -\frac{H^2}{2\bar{\mu}B^2C^2} + \zeta v^\ell_{;\ell}, \tag{1.23}$$

$$\frac{A_4B_4}{AB}+\frac{B_4C_4}{BC}+\frac{A_4C_4}{AC}=\varepsilon+\frac{H^2}{2\bar{\mu}B^2C^2}, \tag{1.24}$$

where the suffix 4 after A, B, C denotes ordinary differentiation with respect to t. Equations (1.21)–(1.24) are four equations in five unknowns A, B, C, λ and ε. For the complete determination of these equations, we assume two supplementary conditions: (i) The shear (σ) is proportional to the scalar of expansion θ which leads to

$$A=(BC)^n \tag{1.25}$$

and (ii)

$$\varepsilon=\lambda, \tag{1.26}$$

i.e. the rest energy density is equal to the string tension density.

From eqs (1.24) and (1.23), we have

$$\frac{A_{44}}{A}+\frac{B_{44}}{B}-\frac{B_4C_4}{BC}-\frac{A_4C_4}{AC}=-\frac{H^2}{\bar{\mu}B^2C^2}-\varepsilon+\zeta v^\ell_{;\ell}. \tag{1.27}$$

Equations (1.27) and (1.21), together with condition (1.26) lead to

$$\frac{A_{44}}{A}+\frac{2B_{44}}{B}+\frac{C_{44}}{C}-\frac{A_4C_4}{AC}=-\frac{H^2}{2\bar{\mu}B^2C^2}+2\zeta v^\ell_{;\ell}. \tag{1.28}$$

From eqs (1.25) and (1.28), we have

$$\begin{aligned} n(n-1)\frac{B_4^2}{B^2}+n(n-2)\frac{C_4^2}{C^2}+(2n^2-n)\frac{B_4C_4}{BC}& \\ +(n+2)\frac{B_{44}}{B}+(n+1)\frac{C_{44}}{C}& \\ =-\frac{K}{B^2C^2}+2\zeta v^\ell_{;\ell},& \end{aligned} \tag{1.29}$$

where

$$K=\frac{H^2}{2\bar{\mu}}.$$

From eqs (1.22) and (1.23), we get

$$\frac{C_{44}}{C}-\frac{B_{44}}{B}=\frac{A_4}{A}\left(\frac{B_4}{B}-\frac{C_4}{C}\right). \tag{1.30}$$

Applying eq. (1.25) in (1.30), we get

$$\frac{(CB_4-BC_4)_4}{(CB_4-BC_4)}=-n\frac{(BC)_4}{BC}. \tag{1.31}$$

On integration, it leads to

$$C^2\left(\frac{B}{C}\right)_4 = \frac{L}{(BC)^n}, \tag{1.32}$$

where L is the integration constant.

Thus

$$B^2 = \mu\nu, \quad C^2 = \frac{\mu}{\nu}.$$

Equation (1.32) leads to

$$\frac{\nu_4}{\nu} = \frac{L}{\mu^{n+1}}. \tag{1.33}$$

Equation (1.29) leads to

$$\left(\frac{2n+3}{2}\right)\frac{\mu_{44}}{\mu} + \left(\frac{4n^2-6n-3}{4}\right)\frac{\mu_4^2}{\mu^2} + \frac{1}{4}\frac{\nu_4^2}{\nu^2} + \frac{(n+1)}{2}\frac{\mu_4\nu_4}{\mu\nu} + \frac{1}{2}\frac{\nu_{44}}{\nu}$$
$$= -\frac{K}{\mu^2} + 2\zeta v^{\ell}_{;\ell}, \tag{1.34}$$

where

$$BC = \mu, \quad \frac{B}{C} = \nu.$$

Equations (1.33) and (1.34) lead to

$$\left(\frac{2n+3}{2}\right)\frac{\mu_{44}}{\mu} + \left(\frac{4n^2-6n-3}{4}\right)\frac{\mu_4^2}{\mu^2} + \frac{3}{4}\frac{L^2}{\mu^{2n+2}} = -\frac{K}{\mu^2} + 2\zeta v^{\ell}_{;\ell}. \tag{1.35}$$

In special case, if $L = 0$, eq. (1.35) leads to

$$\left(\frac{2n+3}{2}\right)\frac{\mu_{44}}{\mu} + \left(\frac{4n^2-6n-3}{4}\right)\frac{\mu_4^2}{\mu^2} = -\frac{K}{\mu^2} + 2\zeta v^{\ell}_{;\ell}. \tag{1.36}$$

Applying the condition

$$\zeta\theta = \text{constant}, \tag{1.37}$$

eq. (1.36) leads to

$$\left(\frac{2n+3}{2}\right)\frac{\mu_{44}}{\mu} + \left(\frac{4n^2-6n-3}{4}\right)\frac{\mu_4^2}{\mu^2} - \beta = -\frac{K}{\mu^2} \tag{1.38}$$

which leads to

$$\mu\mu_{44} + \ell\mu_4^2 - \beta\mu^2 = -K, \tag{1.39}$$

where

$$\ell = \frac{4n^2 - 6n - 3}{2(2n+3)}, \quad \beta = \frac{4\zeta v^{\ell}_{;\ell}}{(2n+3)}$$

which leads to

$$2ff' + \frac{2\ell}{\mu} f^2 = 2\beta\mu - \frac{2K}{\mu} \tag{1.40}$$

where

$$\mu_4 = f(\mu) \quad \text{and} \quad f' = \frac{\mathrm{d}f}{\mathrm{d}\mu}.$$

From eq. (1.40), we have

$$\frac{\mathrm{d}}{\mathrm{d}\mu}(f^2) + \frac{2\ell}{\mu} f^2 = 2\beta\mu - \frac{2K}{\mu}. \tag{1.41}$$

Equation (1.41) leads to

$$f^2 = \frac{\beta}{\ell+1}\mu^2 - \frac{K}{\ell} + \frac{P}{\mu^{2\ell}}, \tag{1.42}$$

where P is the constant of integration.

To find the solution, we take $P = 0$. Equation (1.42) leads to

$$\mu = \sqrt{\frac{K\alpha}{\beta\ell}} \cosh\left(\sqrt{\frac{\beta}{\alpha}} t + N\right), \tag{1.43}$$

where $\ell + 1 = \alpha$ and N is the integration constant.

Since $L = 0$, from eq. (1.33)

$$\nu = b \text{ (constant)}. \tag{1.44}$$

Therefore metric (1.1) reduces to

$$\begin{aligned} \mathrm{d}s^2 = &-\mathrm{d}t^2 + \left(\frac{K\alpha}{\ell\beta}\right)^n \cosh^{2n}\left(\sqrt{\frac{\beta}{\alpha}} t + N\right) \mathrm{d}x^2 \\ &+ b\sqrt{\frac{K\alpha}{\ell\beta}} \cosh\left(\sqrt{\frac{\beta}{\alpha}} t + N\right) \mathrm{d}y^2 \\ &+ \frac{1}{b}\sqrt{\frac{K\alpha}{\ell\beta}} \cosh\left(\sqrt{\frac{\beta}{\alpha}} t + N\right) \mathrm{d}z^2. \end{aligned} \tag{1.45}$$

On applying transformation

$$\begin{aligned} \alpha &= \mathrm{e}^{\beta} - 1, \\ \ell &= a \sin K, \\ t &= \sin \frac{K\tau}{K}, \end{aligned}$$

metric (1.45) reduces to

$$
\begin{aligned}
\mathrm{d}s^2 = &-\cos^2 K\tau \mathrm{d}\tau^2 + \left(\frac{K(\mathrm{e}^\beta - 1)}{\beta a \sin K}\right)^n \\
&\times \cosh^{2n}\left(\sqrt{\frac{\beta}{\mathrm{e}^\beta - 1}}\frac{\sin K\tau}{K} + N\right)\mathrm{d}X^2 \\
&+ \left(\frac{K(\mathrm{e}^\beta - 1)}{\beta a \sin K}\right)^{1/2} \cosh\left(\sqrt{\frac{\beta}{\mathrm{e}^\beta - 1}}\frac{\sin K\tau}{K} + N\right)\mathrm{d}Y^2 \\
&+ \left(\frac{K(\mathrm{e}^\beta - 1)}{\beta a \sin K}\right)^{1/2} \cosh\left(\sqrt{\frac{\beta}{\mathrm{e}^\beta - 1}}\frac{\sin K\tau}{K} + N\right)\mathrm{d}Z^2.
\end{aligned} \tag{1.46}
$$

In the absence of magnetic field, i.e. $K = 0$, metric (1.46) reduces to

$$
\begin{aligned}
\mathrm{d}s^2 = &-\mathrm{d}\tau^2 + \left(\frac{(\mathrm{e}^\beta - 1)}{a\beta}\right)^n \cosh^{2n}\left(\sqrt{\frac{\beta}{\mathrm{e}^\beta - 1}}\tau + N\right)\mathrm{d}X^2 \\
&+ \left(\frac{(\mathrm{e}^\beta - 1)}{a\beta}\right)^{1/2} \cosh\left(\sqrt{\frac{\beta}{\mathrm{e}^\beta - 1}}\tau + N\right)\mathrm{d}Y^2 \\
&+ \left(\frac{(\mathrm{e}^\beta - 1)}{a\beta}\right)^{1/2} \cosh\left(\sqrt{\frac{\beta}{\mathrm{e}^\beta - 1}}\tau + N\right)\mathrm{d}Z^2.
\end{aligned} \tag{1.47}
$$

In the absence of viscosity, i.e., $\beta = 0$, the metric (1.46) reduces to

$$
\begin{aligned}
\mathrm{d}s^2 = &-\cos^2 K\tau \mathrm{d}\tau^2 + \left(\frac{K}{a \sin K}\right)^n \cosh^{2n}\left(\frac{\sin K\tau}{K} + N\right)\mathrm{d}X^2 \\
&+ \left(\frac{K}{a \sin K}\right)^{1/2} \cosh\left(\frac{\sin K\tau}{K} + N\right)\mathrm{d}Y^2 \\
&+ \left(\frac{K}{a \sin K}\right)^{1/2} \cosh\left(\frac{\sin K\tau}{K} + N\right)\mathrm{d}Z^2.
\end{aligned} \tag{1.48}
$$

2. Some physical and geometrical features

The density (ε) for model (1.46) is given by

$$
\begin{aligned}
\lambda = \varepsilon = &\left(\frac{4n+1}{4}\right)\sqrt{\frac{\beta}{\mathrm{e}^\beta - 1}} \tanh\left(\sqrt{\frac{\beta}{\mathrm{e}^\beta - 1}}\frac{\sin K\tau}{K} + N\right) \\
&- \frac{\beta a \sin K}{(\mathrm{e}^\beta - 1)} \mathrm{sech}^2\left(\sqrt{\frac{\beta}{\mathrm{e}^\beta - 1}}\frac{\sin K\tau}{K} + N\right).
\end{aligned} \tag{2.1}
$$

The scalar of expansion (θ) and the expression of shear tensor (σ_i^j) for the space-time (1.46) are given by

$$\theta = (n+1)\sqrt{\frac{\beta}{e^{\beta}-1}}\tanh\left(\sqrt{\frac{\beta}{e^{\beta}-1}}\frac{\sin K\tau}{K} + N\right), \tag{2.2}$$

$$\sigma_1^1 = \left(\frac{2n-1}{3}\right)\sqrt{\frac{\beta}{e^{\beta}-1}}\tanh\left(\sqrt{\frac{\beta}{e^{\beta}-1}}\frac{\sin K\tau}{K} + N\right), \tag{2.3}$$

$$\sigma_2^2 = \sigma_3^3 = \left(\frac{1-2n}{6}\right)\sqrt{\frac{\beta}{e^{\beta}-1}}\tanh\left(\sqrt{\frac{\beta}{e^{\beta}-1}}\frac{\sin K\tau}{K} + N\right). \tag{2.4}$$

Thus

$$\sigma^2 = \frac{1}{2}(\sigma_{ij}\sigma^{ij}),$$

$$\sigma^2 = \frac{(2n-1)^2}{12}\left(\frac{\beta}{e^{\beta}-1}\right)\tanh^2\left(\sqrt{\frac{\beta}{e^{\beta}-1}}\frac{\sin K\tau}{K} + N\right). \tag{2.5}$$

Spatial volume

$$R^3 = \left(\frac{K(e^{\beta}-1)}{\beta a \sin K}\right)^{(n+2)/2}\cosh^{n+1}\left(\sqrt{\frac{\beta}{e^{\beta}-1}}\frac{\sin K\tau}{K} + N\right). \tag{2.6}$$

3. Discussion

The expansion in the model (1.46) increases as $\left(\sqrt{\frac{\beta}{e^{\beta}-1}}\frac{\sin K\tau}{K} + N\right)$ increases, i.e. it represents expanding universe. From (2.1), we find that $\varepsilon > 0$. For $\tau = 0$ we find ε as a finite quantity. The spatial volume tends to infinity as $\left(\sqrt{\frac{\beta}{e^{\beta}-1}}\frac{\sin K\tau}{K} + N\right)$ tends to infinity. Since $\lim_{\tau\to\infty}\frac{\sigma}{\theta} \neq 0$, the model does not approach iostropy for large values of τ. For $n = \frac{1}{2}$, model (1.46) represents an isotropic universe.

In the absence of magnetic field, i.e., when $K = 0$ then the energy density (ε), string tension (λ), the expansion θ, the components of shear tensor and the spatial volume (R^3) in the presence of bulk viscosity are given by

$$\varepsilon = \lambda = \frac{(4n+1)}{4}\left(\frac{\beta}{e^{\beta}-1}\right)\tanh\left(\sqrt{\frac{\beta}{e^{\beta}-1}}\tau + N\right), \tag{3.1}$$

$$\varepsilon = (n+1)\sqrt{\frac{\beta}{e^{\beta}-1}}\tanh\left(\sqrt{\frac{\beta}{e^{\beta}-1}}\tau + N\right), \tag{3.2}$$

$$\sigma_1^1 = \left(\frac{2n-1}{3}\right)\sqrt{\frac{\beta}{e^{\beta}-1}}\tanh\left(\sqrt{\frac{\beta}{e^{\beta}-1}}\tau + N\right), \tag{3.3}$$

$$\sigma_2^2 = \sigma_3^3 = \left(\frac{1-2n}{6}\right)\sqrt{\frac{\beta}{e^{\beta}-1}}\tanh\left(\sqrt{\frac{\beta}{e^{\beta}-1}}\tau + N\right). \tag{3.4}$$

Thus

$$\sigma^2 = \frac{(2n-1)^2}{12}\left(\frac{\beta}{e^{\beta}-1}\right)\tanh^2\left(\sqrt{\frac{\beta}{e^{\beta}-1}}\tau + N\right) \tag{3.5}$$

and

$$R^3 = \left(\frac{(e^{\beta}-1)}{a\beta}\right)^{(2+n)/2}\cosh^{n+1}\left(\sqrt{\frac{\beta}{e^{\beta}-1}}\tau + N\right). \tag{3.6}$$

In the absence of viscosity, i.e., $\beta = 0$ then the above quantities in the presence of magnetic field are given by

$$\varepsilon = \left(\frac{4n+1}{4}\right)\tanh\left(\frac{\sin K\tau}{K} + N\right) - a\sin K \operatorname{sech}^2\left(\frac{\sin K\tau}{K} + N\right), \tag{3.7}$$

$$\theta = (n+1)\tanh\left(\frac{\sin K\tau}{K} + N\right), \tag{3.8}$$

$$\sigma_1^1 = \left(\frac{2n-1}{3}\right)\tanh\left(\frac{\sin K\tau}{K} + N\right), \tag{3.9}$$

$$\sigma_2^2 = \sigma_3^3 = \frac{1}{3}\left(\frac{1-2n}{2}\right)\tanh\left(\frac{\sin K\tau}{K} + N\right). \tag{3.10}$$

Thus

$$\sigma^2 = \frac{(2n-1)^2}{12}\tanh^2\left(\frac{\sin K\tau}{K} + N\right) \tag{3.11}$$

and

$$R^3 = \left(\frac{K}{a\sin K}\right)^{(2+n)/2}\cosh^{n+1}\left(\frac{\sin K\tau}{K} + N\right) \tag{3.12}$$

when $K \to 0$ and $\beta \to 0$ then these quantities are given by

$$\varepsilon = \lambda = \left(\frac{4n+1}{4}\right)\tanh(\tau + N), \tag{3.13}$$

$$\theta = (n+1)\tanh(\tau + N), \tag{3.14}$$

$$\sigma_1^1 = \left(\frac{2n-1}{3}\right)\tanh(\tau + N), \tag{3.15}$$

$$\sigma_2^2 = \sigma_3^3 = \left(\frac{1-2n}{6}\right)\tanh(\tau+N), \tag{3.16}$$

$$\sigma^2 = \frac{(2n-1)^2}{12}\tanh^2(\tau+N) \tag{3.17}$$

and

$$R^3 = a^{2/2+n}\cosh^{n+1}(\tau+N). \tag{3.18}$$

In the absence of magnetic field and bulk viscosity, the model represents an expanding universe and $\varepsilon > 0$. The spatial volume increases as τ increases and it is infinity when $\tau = \infty$. Since $\lim_{\tau\to\infty}(\sigma/\theta) \neq 0$, the model does not approach isotropy for large values of τ.

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