

## Statistical isotropy of the cosmic microwave background

TARUN SOURADEEP and AMIR HAJIAN

Inter-University Centre for Astronomy and Astrophysics, Post Bag 4, Ganeshkhind,  
Pune 411 007, India

**Abstract.** The breakdown of statistical homogeneity and isotropy of cosmic perturbations is a generic feature of ultra-large scale structure of the cosmos, in particular, of non-trivial cosmic topology. The statistical isotropy (SI) of the cosmic microwave background temperature fluctuations (CMB anisotropy) is sensitive to this breakdown on the largest scales comparable to, and even beyond the cosmic horizon. We propose a set of measures,  $\kappa_\ell$  ( $\ell = 1, 2, 3, \dots$ ) which for non-zero values indicate and quantify statistical isotropy violations in a CMB map. We numerically compute the predicted  $\kappa_\ell$  spectra for CMB anisotropy in flat torus universe models. Characteristic signatures of different models in the  $\kappa_\ell$  spectrum are noted.

**Keywords.** Cosmic microwave background; cosmology; observations.

In standard cosmology, the cosmic microwave background (CMB) anisotropy is expected to be statistically isotropic, i.e., statistical expectation values of the temperature fluctuations  $\Delta T(\hat{q})$  are preserved under rotations of the sky. In particular, the angular correlation function  $C(\hat{q}, \hat{q}') \equiv \langle \Delta T(\hat{q}) \Delta T(\hat{q}') \rangle$  is rotationally invariant. In spherical harmonic space, where  $\Delta T(\hat{q}) = \sum_{lm} a_{lm} Y_{lm}(\hat{q})$ , this translates to a diagonal  $\langle a_{lm} a_{l'm'}^* \rangle = C_l \delta_{ll'} \delta_{mm'}$ , where  $C_l$  is the widely used angular power spectrum.

In the absence of statistical isotropy,  $C(\hat{q}, \hat{q}')$  is estimated by a single product  $\Delta T(\hat{q}) \Delta T(\hat{q}')$  and hence is poorly determined from a single realization. Although it is not possible to estimate each element of the full correlation function  $C(\hat{q}, \hat{q}')$ , some measures of statistical anisotropy of the CMB map can be estimated through suitably weighted angular averages of  $\Delta T(\hat{q}) \Delta T(\hat{q}')$ . The angular averaging procedure should be such that the measure involves averaging over sufficient number of independent ‘measurements’, but should ensure that the averaging does not erase all the signature of statistical anisotropy. Another important desirable property is that the measure be independent of the overall orientation of the sky. Based on these considerations, we propose a set of measures  $\kappa_\ell$  of statistical anisotropy given by

$$\kappa_\ell = \int d\Omega \int d\Omega' \left[ \frac{(2\ell + 1)}{8\pi^2} \int d\mathcal{R} \chi_\ell(\mathcal{R}) C(\mathcal{R}\hat{q}, \mathcal{R}\hat{q}') \right]^2, \quad (1)$$

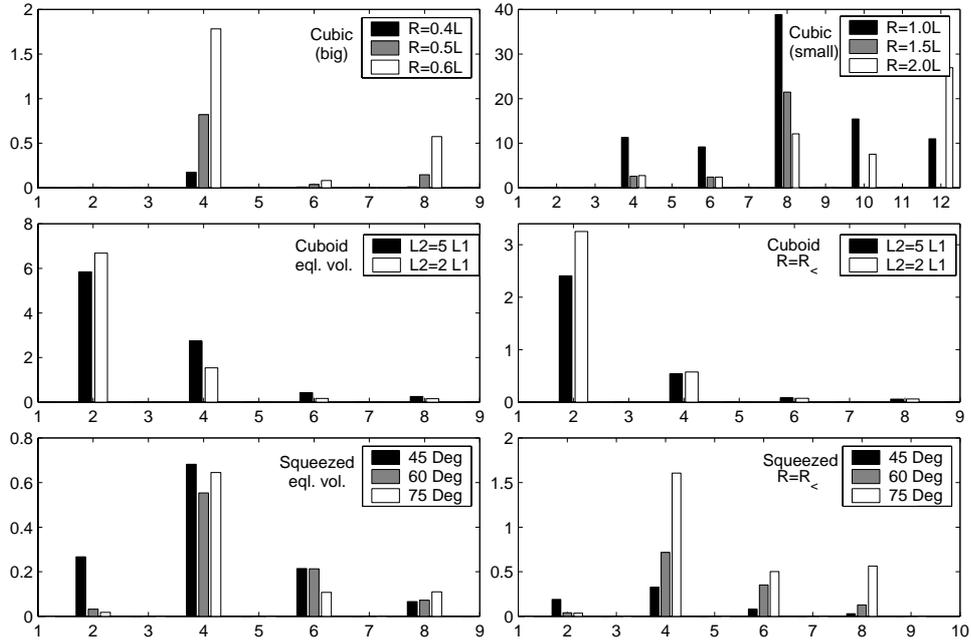
where  $C(\mathcal{R}\hat{q}, \mathcal{R}\hat{q}')$  is the two-point correlation between  $\mathcal{R}\hat{q}$  and  $\mathcal{R}\hat{q}'$  obtained by rotating  $\hat{q}$  and  $\hat{q}'$  by an element  $\mathcal{R}$  of the rotation group [1]. The measures  $\kappa_\ell$  involve angular average of the correlation weighed by the characteristic function of the rotation group  $\chi_\ell(\mathcal{R}) = \sum_M D_{MM}^\ell(\mathcal{R})$ , where  $D_{MM}^\ell$  are the Wigner  $D$ -functions [2]. When SI holds  $C(\mathcal{R}\hat{q}, \mathcal{R}\hat{q}') = C(\hat{q}, \hat{q}')$  is invariant under rotation, and eq. (1) gives  $\kappa_\ell = \kappa_0 \delta_{\ell 0}$  due to the orthonormality of  $\chi_\ell(\mathcal{R})$ . Hence, non-zero  $\kappa_\ell$  for  $\ell > 0$  measure violation of statistical isotropy.

The measure  $\kappa_\ell$  has a clear interpretation in harmonic space. The two-point correlation  $C(\hat{q}, \hat{q}')$  can be expanded in terms of the orthonormal set of bipolar spherical harmonics whose coefficients  $A_{ll'}^{\ell M}$  are related to ‘angular momentum’ sum over the covariances  $\langle a_{lm} a_{l'm'}^* \rangle$  as  $A_{ll'}^{\ell M} = \sum_{mm'} \langle a_{lm} a_{l'm'}^* \rangle (-1)^{m'} \mathfrak{C}_{lm' - m'}^{\ell \mathfrak{M}}$ , where  $\mathfrak{C}_{lm' - m'}^{\ell \mathfrak{M}}$  are Clebsch–Gordan coefficients [2]. The estimation of  $\kappa_\ell$  from a CMB map is discussed in [1].

The detection of statistical isotropy (SI) violations can have exciting and far-reaching implications for cosmology. The realization that the universe with the same local geometry has many different choices of global topology has been a theoretical curiosity as old as modern cosmology. Motivations for cosmic topology and their consequences have been extensively studied [3]. CMB anisotropy measurements have brought cosmic topology from the realm of theoretical possibility to within the grasp of observations [3,4]. A generic consequence of cosmic topology is the breaking of statistical isotropy in characteristic patterns determined by the photon geodesic structure of the manifold. Global isotropy of space is violated in all multi-connected models (except  $S^3/Z_2$ ). In cosmology, the Dirichlet domain (DD) constructed around the observer represents the universe as ‘seen’ by the observer. The SI breakdown is apparent in the principal axes present in the shape of the DD constructed with the observer located at the base-point [5].

In this paper we compute and study the  $\kappa_\ell$  spectrum of SI violation arising in flat (Euclidean) simple torus models with a cubic, cuboidal and more generally, parallelepiped (squeezed) fundamental domain. The CMB anisotropy in torus spaces has been well-studied [3]. We can relate the  $\kappa_\ell$  spectrum to the principal directions normal to pair of faces of the DD, their relative orientation and the relative importance given by the distance to the faces along them. Along the most dominant axes, the distance is minimum, and equals the in-radius,  $R_<$ , the radii of largest sphere fully enclosed within the DD [4].

The compact spaces with Euclidean geometry (zero curvature) have been completely classified. In three dimensions, there are known to be six possible topologies that lead to orientable spaces [3,6]. The simple flat torus,  $\mathcal{M} = T^3$ , is obtained by identifying the universal cover  $\mathcal{M}^u = \mathcal{E}^3$  under a discrete group of translations along three non-degenerate axes,  $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3$ :  $\mathbf{s}_i \rightarrow \mathbf{s}_i + \mathbf{n}L_i$ , where  $L_i$  is the identification length of the torus along  $s_i$  and  $\mathbf{n}$  is a vector with integer components. In the most general form, the fundamental domain (FD) is a parallelepiped defined by three sides  $L_i$  and the three angles  $\alpha_i$  between the axes (we call it squeezed torus). If  $\mathbf{s}_i$  are orthogonal then one gets cuboidal FD, which for equal  $L_i$  reduces to the cubic torus. The cuboid and squeezed spaces which can be obtained by a linear coordinate transformation  $\mathcal{L}$  on cubic torus can have distinctly different global symmetry [6a].



**Figure 1.** The  $\kappa_\ell$  spectra for flat tori models are plotted. The top row panels are for cubic tori spaces. The left panel shows spaces of volume,  $V_{\mathcal{M}}$ , larger than the volume  $V_*$  contained in the sphere of last scattering (SLS) with  $V_{\mathcal{M}}/V_* = 3.7, 1.9, 1.1$ , respectively. The right panel shows small spaces with  $V_{\mathcal{M}}/V_* = 0.24, 0.07, 0.03$ , respectively. Note that  $\kappa_2 = 0$  for cubic tori. The middle panels consider cuboid tori with 1 : 5 and 1 : 2 ratio of identification lengths. The bottom panels show  $\kappa_\ell$  for equal-sided squeezed tori with  $\alpha = 45^\circ, 60^\circ$  and  $75^\circ$ . In the middle and bottom rows, the right panels show the case when radius of SLS,  $R_* = R_<$  the in-radius of the space. Here, the SLS just touches its nearest images which is at the threshold where CMB anisotropy is multiply imaged for larger  $R_*$ . The cases in the left panels of lower two rows have  $V_{\mathcal{M}}/V_* = 1$  and are at the divide between large and small spaces.

We restrict attention to the case where CMB anisotropy arises entirely at the sphere of last scattering (SLS) of radius  $R_*$ . Invoking the method of images, the CMB correlation pattern on the SLS is known to be dictated by the distribution of nearest ‘images’ of the SLS on the universal cover [4]. The correlations are distorted even when the SLS and its images do not intersect ( $R_* < R_<$ ). When SLS intersects its images the CMB sky is multiply imaged in characteristic correlation pattern of pairs of circles [7].

We compute  $\mathbf{C}(\hat{\mathbf{q}}, \hat{\mathbf{q}}')$  for CMB anisotropy in torus space using regularized method of images [4]. Figure 1 plots the predicted  $\kappa_\ell$  spectrum for a number of cubic, cuboidal and squeezed torus spaces. We note the following interesting results:

- (i)  $\kappa_\ell = 0$  for odd  $\ell$  for all torus models. This does not hold for compact space of non-zero curvature, e.g., compact hyperbolic spaces.

- (ii) For cubic torus  $\kappa_2 = 0$ .  $\kappa_2$  is non-zero for cuboidal and squeezed torus. This is a clear signature of non-cubic torus where the DD differs from the FD and has more than three principal axes.
- (iii) For equal-sided squeezed torus,  $\kappa_4$ , decreases as  $\alpha$  decreases from  $90^\circ$  to  $60^\circ$  as  $R_<$  increases. For  $\alpha < 60^\circ$  sharply increases with decreasing  $\alpha$  as  $R_<$  decreases sharply.
- (iv)  $\kappa_2 = 0$  increases monotonically as  $\alpha$  decreases from  $90^\circ$ .
- (v) The peak of  $\kappa_\ell$  shifts to larger  $\ell$  for small spaces.

The results can be understood using the leading order terms of the correlation function in a torus where  $\kappa_\ell$  can be calculated analytically [5].

Preferred directions and statistically anisotropic CMB anisotropy have been discussed in the literature [8]. When CMB anisotropy is multiply imaged, the  $\kappa_\ell$  spectrum corresponds to a correlation pattern of matched pairs of circles [7]. The generic features of  $\kappa_\ell$  spectrum are related to the symmetries of correlation pattern. But  $\kappa_\ell$  are sensitive to SI violation even when CMB is not multiply imaged. Moreover,  $\kappa_\ell$  have an advantage of being insensitive to the overall orientation of the correlation features. The  $A_{\ell}^{\ell M}$  signature, which was not discussed here, contains more details of the SI violation [8]. The estimation of  $\kappa_\ell$  from a CMB map is described [1]. Before ascribing the detected breakdown of statistical anisotropy to cosmological or astrophysical effects, one must carefully account for and model out other mundane sources of statistical anisotropy in real data, such as, incomplete and non-uniform sky coverage, beam anisotropy, foreground residuals and statistically anisotropic noise. These observational artifacts will be discussed in future publications.

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